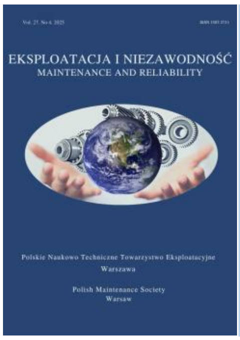

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The optimal joint preventive maintenance strategy for the equipment under two-dimensional extended warranty

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Highlights

- Propose a J-PM strategy performed by manufacturers and users within TEW period.
- A product's maintenance cost-availability ratio model within TEW period is established.
- Combine genetic algorithms and pattern-seeking methods to solve the model.
- Sensitivity analysis was conducted on the model in terms of usage rate distribution.
- Derive a product's two-dimensional failure rate function under the J-PM strategy.

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Abstract

In pursuit of the optimal maintenance cost-availability ratio for large engineering equipment during a two-dimensional extended warranty (TEW) period, this paper puts forward a joint preventive maintenance (J-PM) strategy executed by both users and manufacturers. By deriving the dynamic evolution of the product's two-dimensional failure rate function, a maintenance cost-availability ratio model is established. Subsequently, pattern-seeking methods and genetic algorithms are utilized to solve the cost-availability ratio model, yielding the optimal combination of decision variables. A comparison is made between the proposed J-PM strategy and preventive maintenance (PM) strategy executed solely by users or manufacturers. The results prove that the J-PM strategy can reduce the maintenance cost-availability ratio more effectively and achieve better maintenance outcomes, thereby validating the superiority of the J-PM strategy.

Keywords

two-dimensional extended warranty (TEW), joint preventive maintenance (J-PM), cost-availability ratio model, genetic algorithms and pattern-seeking methods.

1. Introduction

A. Research background and significance

With the rapid advancement of technology, the level of informatization and integration in current large engineering equipment is increasingly high[1]. Relying solely on the maintenance efforts of users often fails to achieve the best results. Manufacturers, on the other hand, possess strong technical advantages and maintenance expertise[2]. Collaborative maintenance efforts between manufacturers and users can increase the efficiency of equipment repairs in a warranty period[3]. Users can also enhance their autonomous maintenance capabilities for engineering equipment with the assistance of manufacturers, this strategy for maintenance has been implemented within the field of engineering[4].

Most products come with an initial warranty period upon purchase. Should the product malfunction or become damaged during this initial warranty period, the manufacturer offers free repair services[5]. Given that manufacturers possess higher levels of technical expertise and capability in repairs, and can

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provide these services at no cost, they are usually fully responsible for the maintenance work during the product's initial warranty period[6]. However, the duration of the initial warranty is often quite limited, and the full service life of the product is not completely utilized. Therefore, continuing to extend the warranty for the product after the initial period ends retains important value and significance[7]. Within the extended warranty period, how to sufficiently leverage the manufacturer's repair technical advantages while also cultivating the user's independent maintenance capabilities is an issue that requires exploration and resolution.

In contrast to ordinary products, users of large engineering equipment often need to develop their own maintenance capabilities, because they must have the ability to execute independent maintenance in emergency[8]. For instances, in the industrial production, enterprises need to repair equipment malfunctions to restore production activities as soon as possible[9]. If an enterprise cannot handle maintenance issues on its own and must wait for the manufacturer's repair team, this will inevitably lead to production delays, resulting in significant economic losses for the enterprise[10]. The occurrence of malfunctions in large engineering equipment can cause severe property losses to users, hence warranty services generally encompass preemptive PM actions aimed at diminishing or averting failure incidents[11]. During the extended warranty period, PM activities executed jointly by manufacturers and users not only help users gradually develop the ability for autonomous repair but also fully utilize the manufacturer's technological advantages[12]. Therefore, the J-PM of large engineering equipment by users and manufacturers is an effective and rational strategy.

Currently, in the field of PM, models for large engineering equipment that employ multiple maintenance strategies and adapt to different maintenance environments are being extensively studied. Wang et al.[13] designed an optimal PM strategy for heterogeneous systems with competitive failures, utilizing semi-Markov decision processes. Zhou et al.[14] developed a PM model that simultaneously considers the interplay between random external shocks and continuous internal degradation of equipment. Su et al.[15] proposed that imperfect preventive maintenance is carried out during a Advance planning period and modeled using a mixed failure rate model and quasi renewal coefficient. Einabadi et al.[16] investigated the synchronization of preventive and predictive maintenance strategies for systems with multiple components, confirming that their proposed methods can significantly reduce maintenance costs. Peng et al.[17] made different PM plans for parallel systems characterized by two distinct failure modes, followed by case analysis. Lots of PM strategies have been proposed, but they are almost all executed independently not jointly, unable to simultaneously utilize the manufacturer's technological advantages and enhance the user's ability for self maintenance.

Besides that, in the field of extended warranty decision-making, various models and methods have been researched and discussed. For example, Li et al.[18] believed that extended warranty can improve consumer satisfaction and analyzed the warranty decision procedure from the perspective of platform. Wang et al.[19] approached the issue from the supplier's perspective, studying the design of customized extended warranty menus. Gupta et al.[20] designed a model that considers the premium growth rate and the premium cap, and drew conclusions after analyzing experimental data. Mitra[21] considered that customers are sensitive to the price of product extended warranties and established an optimal warranty pricing model for products during the extended warranty period. Su et al.[22] put forward one-dimensional extended warranty policy and two-dimensional non-renewing extended warranty policy. Salmasinia et al.[23] recognized the continuity of usage variables and proposed an optimal warranty service strategy for products during the extended warranty period. However, most of these studies focus on the cost and pricing of product extended warranties, with little consideration given to the availability and cost-availability ratio of product extended warranty. There is almost no research on the J-PM of products during their TEW period.

As can be seen from the above, it is evident that the J-PM strategy during TEW period and the optimal cost-availability ratio model have not yet been considered. To fill this research gap and assist manufacturers and users in making more informed warranty decisions, this paper proposes the implementation of J-PM strategy during the product's TEW period, aimed at enhancing product availability and decreasing the warranty cost-availability ratio.

B. Overview of the research

This paper primarily focuses on the J-PM modeling and optimization of the engineering equipment during TEW period. Both user and manufacturer perform a specific number of regular PM tasks to enhance product availability and reduce warranty cost-availability ratio. The paper first establishes cost and availability models for the TEW, and then generates a cost-availability model for the TEW. With the decision objective of making the cost-availability ratio minimum in TEW period, considering the two-dimensional PM intervals, and the frequency and moment of PM performed by manufacturer as decision variables. By combining genetic algorithm and pattern-seeking method, the model is solved, leading to the optimal J-PM plan in the TEW stage.

Through case analysis, the J-PM strategy is compared with the independent PM strategy, and answer the question: Whether the J-PM strategy have an advantage in reducing the cost-availability ratio of the product's TEW?

The remaining sections of this paper is structured as bellow: Section 2 provides the presuppositions of model and defines the J-PM strategy. Section 3 derives the evolution law of product failure rates under the preventive maintenance strategy. Section 4 establishes the extended warranty cost, availability, and cost-availability ratio model considering J-PM. Section 5 develops and discusses the optimization problem and solution algorithm that makes the cost-availability ratio minimum. Section 6 conducts comparative verification and sensitivity analysis of the proposed joint preventive maintenance strategy through case studies. Finally, Section 7 summarizes the paper and delineates prospective directions for further research.

2. MODEL DESCRIPTION AND ASSUMPTIONS

J-PM means the PM work carried out jointly by multiple maintenance entities, with this paper primarily considering the manufacturer and the user as the maintenance subjects. Taking the one-dimensional initial warranty of a product as an example, in initial warranty period T_w, PM is conducted with the interval T_0, totaling H_w ($H_w = \lfloor T_w/T_0 \rfloor$,the "$\lfloor \varDelta \rfloor$" signifies decreasing "$\varDelta$ " to the nearest lower integer) PM tasks. PM is executed at times $T_0, 2T_0 \ldots, H_w T_0$, among all PM tasks, a total of "N " PM tasks are conducted by manufacturer. Presupposing that the H_n-th ($1 \leq n \leq N$) PM task is carried out by manufacturer, then the remaining "$H_w - N$ " PM tasks are conducted by user, as illustrated in Figure 1.

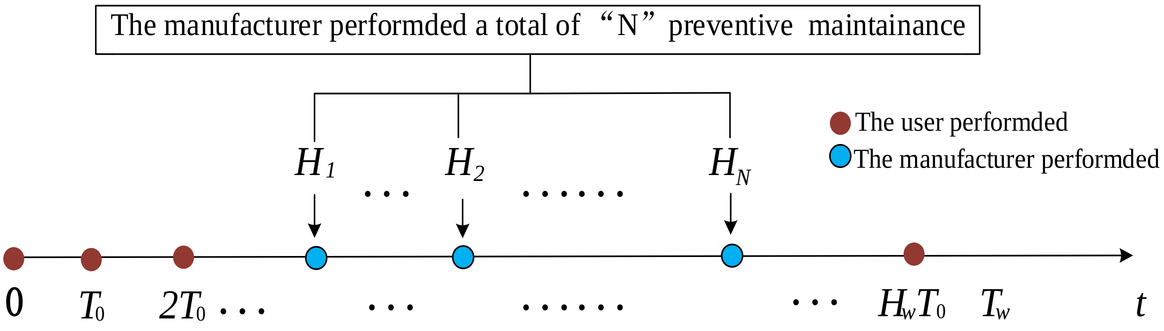


Figure 1. Diagram of J-PM under one-dimensional initial warranty.

This paper advocates for the execution of J-PM during the TEW period, T_0 or $U_0$is used as the interval for J-PM. Due to the fact that the number of performing PM varies with the actual usage rate "r " of users, this paper analyzes based on the usage rate at the time of the highest number of executions. While $r_0 \leq r_w$, the highest number of PM actions is carried out under the specified conditions that $r > r_w$, totaling $H_w = \lfloor (U_w - U_c)/U_0 \rfloor$ PM actions were executed. While $r_0 > r_w$, the highest number of PM actions is carried out under the specified conditions that $r < r_w$, totaling $H_w = \lfloor (T_w - T_c)/T_0 \rfloor$ PM actions were executed. Figure 2 illustrates the scenario($r_0 > r_w > r$) of conducting J-PM for a product within the TEW period$(T_c, T_w) \cap (U_c, U_w)$.

The research in this paper is conducted under the following assumptions:

1.Assuming that the actual usage rate "r " of the user keeps constant during the TEW period, an assumption that can be validated through literature on the analysis of TEW data[24].

2.Manufacturers can estimate the distribution of the actual usage rate of users based on maintenance data and historical warranty data from similar products[25].

3.This paper sets the upper limits of the product's TEW period, denoted as T_w and U_w, within the product's service life cycle. During the initial warranty period $(0, T_c) \cap (0, U_c)$, only corrective maintenance is conducted after product failures.

4.All product failures within the TEW period are repaired by the manufacturer, with the corrective maintenance being minimal and not altering the product's age[26].

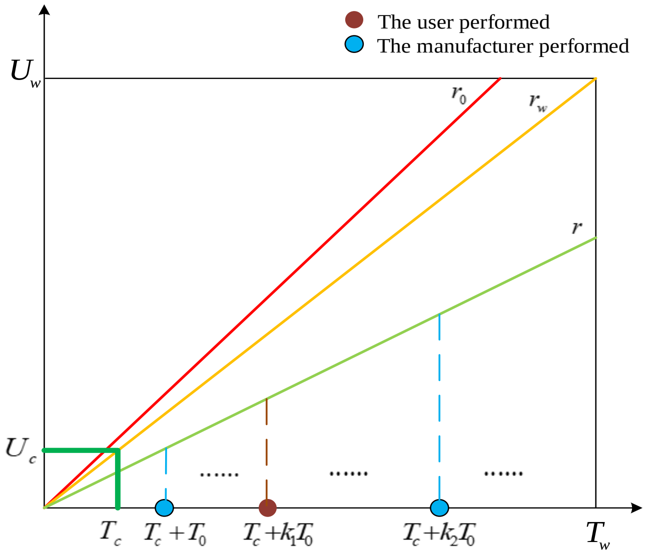


Figure 2. Diagram of J-PM under TEW.

3. TWO-DIMENSIONAL FAILURE RATE FUNCTION

The marginal approach is employed to model product failures within the warranty period, and the usage rate r is included as a covariate to simplify the bi-dimensional product failure modeling problem into a single-dimensional issue. The failure rate function's polynomial form was initially put forth by Hassett[27] and adopted by Huang et al.[28], He et al.[29] and Manna et al.[30], as follows:

$$\psi(t|r) = \delta_0 + \delta_1 t + \delta_2 r + \delta_3 rt \tag{1}$$

The effectiveness of imperfect preventive maintenance can be articulated through the notion of virtual age. As stated by reference[31], after performing PM, the entire virtual age before PM is diminished. Suppose the product undergoes PM at the time t_p, the virtual age before PM is denoted as $v(t_p^-)$, and the virtual age after PM is denoted as $v(t_p^+)$. θ represent the repair factor. After performing PM, the virtual age changes from $v(t_p^-)$ to $v(t_p^+)$ =$\theta v(t_p^-)$, which is a reduction by $(1-\theta)v(t_p^-)$. Therefore, before the next PM, the failure rate of the product is:

$$\psi(t|r) = \psi(t - (1-\theta)v(t_p^-)|r) \tag{2}$$

The repair factor for PM conducted by users is denoted by θ_1, and conducted by manufacturers is denoted by θ_2. When PM is executed by manufacturer, the product's virtual age is reduced more significantly, leading to a more effective repair outcome, so $(1-\theta_2) v(t_p^-) > (1-\theta_1) v(t_p^-)$ and $\theta_2 < \theta_1$. The product's failure rate at the time t $(kT_0 + T_c \le t \le (k+1)T_0 + T_c, k \in [0, H_w]$ and $k \in N)$ after the k-th($k \in [0, H_w], k \in N$) PM is denoted by ψ_k.

When $k = 0$, the failure rate of the product prior to the first PM is denoted as:

$$\psi_k = \psi(t|r) \tag{3}$$

When $1 \le k < H_1$, if the product has undergone k preventive maintenances performed by the user but has not yet undergone any PM by the manufacturer, failure rate is denoted as:

$$\psi_k = \psi(t - kT_0 + (\theta_1 T_0 + \theta_1^2 T_0 + \ldots + \theta_1^k T_0)|r)) = \psi(t - kT_0 + \sum_{j=1}^{k} \theta_1^j T_0 \,|r) \tag{4}$$

When $k = H_1$, the manufacturer has just completed the first PM, failure rate is denoted as:

$$\psi_k = \psi(t - kT_0 + (\theta_2 \theta_1^{k-H_1} T_0 + \theta_2 \theta_1^{k-H_1+1} T_0 + \ldots + \theta_2 \theta_1^{k-1} T_0)|r)) = \psi(t - kT_0 + \theta_2 \textstyle\sum_{j=k-H_1}^{k-1} \theta_1^j T_0 \,|r) \tag{5}$$

When$H_1 < k < H_2$, the product is between the first and second preventive maintenances conducted by the manufacturer, failure rate is denoted as:

$$\begin{aligned} \psi_k &= \psi(t - kT_0 + ((\theta_1 T_0 + \theta_1^2 T_0 + \ldots + \theta_1^{k-H_1} T_0) + (\theta_2 \theta_1^{k-H_1} T_0 + \theta_2 \theta_1^{k-H_1+1} T_0 + \ldots + \theta_2 \theta_1^{k-1} T_0))|r) \\ &= \lambda(t - kT_0 + \sum_{j=1}^{k-H_1} \theta_1^j T_0 + \theta_2 \sum_{j=k-H_1}^{k-1} \theta_1^j T_0 \,|r) \end{aligned} \tag{6}$$

When$k = H_2$, the manufacturer has just completed the second PM, failure rate is denoted as:

$$\begin{aligned}\psi_k &= \psi(t-kT_0+((\theta_2\theta_1^{k-H_2}T_0+\theta_2\theta_1^{k-H_2+1}T_0+\ldots+\theta_2\theta_1^{k-H_1-1}T_0)\ +(\theta_2^2\theta_1^{k-H_1-1}T_0+\theta_2^2\theta_1^{k-H_1}T_0+\ldots+\theta_2^2\theta_1^{k-2}T_0))|r \\ &= \psi(t-kT_0+\theta_2\sum_{j=k-H_2}^{k-H_1-1}\theta_1^jT_0+\theta_2^2\sum_{j=k-H_1-1}^{k-2}\theta_1^jT_0\,|r)\end{aligned} \tag{7}$$

When$H_2 < k < H_3$, the product is between the second and third preventive maintenances conducted by the manufacturer, failure rate is denoted as:

$$\begin{aligned}&\psi_k = \psi(t-kT_0+((\theta_1T_0+\theta_1^2T_0+\ldots+\theta_1^{k-H_2}T_0)+(\theta_2\theta_1^{k-H_2}T_0+\theta_2\theta_1^{k-H_2+1}T_0+\ldots+\theta_2\theta_1^{k-H_1-1}T_0) \\ &+(\theta_2^2\theta_1^{k-H_1-1}T_0+\theta_2^2\theta_1^{k-H_1}T_0+\ldots+\theta_2^2\theta_1^{k-2}T_0))|r) = \psi(t-kT_0+\sum_{j=1}^{k-H_2}\theta_1^jT_0+\theta_2\sum_{j=k-H_2}^{k-H_1-1}\theta_1^jT_0+\theta_2^2\sum_{j=k-H_1-1}^{k-2}\theta_1^jT_0\,|r)\end{aligned} \tag{8}$$

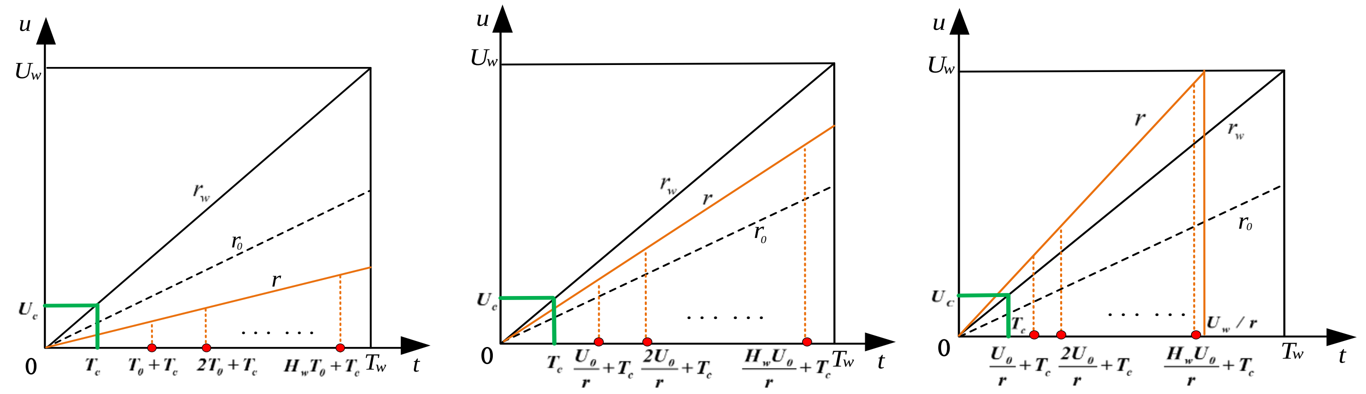
From the above scenarios, we can generalize that:

$$\psi_k = \begin{cases}\psi(t|r) & k=0 \\ \psi(t-kT_0+\sum_{j=1}^{k}\theta_1^jT_0\,|r) & 1\le k<H_1 \\ \psi(t-kT_0+\sum_{i=0}^{n-1}\theta_2^{n-i}(\sum_{j=k-n+i-H_{i+1}+1}^{k-n+i-H_i}\theta_1^jT_0)|r) & k=H_n, 1\le n\le N \\ \psi(t-kT_0+\sum_{i=0}^{n-1}\theta_2^{n-i}(\sum_{j=k-n+i-H_{i+1}+1}^{k-n+i-H_i}\theta_1^jT_0)+\sum_{j=1}^{k-H_n}\theta_1^jT_0\,|r) & H_n<k<H_{n+1}, 1\le n, n+1\le N \\ \psi(t-kT_0+\sum_{i=0}^{N-1}\theta_2^{N-i}(\sum_{j=k-N+i-H_{i+1}+1}^{k-N+i-H_i}\theta_1^jT_0)+\sum_{j=1}^{k-H_N}\theta_1^jT_0\,|r) & H_N<k\le H_w\end{cases} \tag{9}$$

4. MODEL ESTABLISHMENT

The TEW region delineated in this study is represented as a rectangular area, $[T_C, T_W]$denotes the TEW span in terms of usage time and $[U_C, U_W]$denotes the TEW span in terms of usage intensity. The TEW span and the intervals for PM of the product are subject to variation with fluctuations in the user's rate of utilization. Given that the quantitative relationships among r , r_0 and r_w are distinct, the model development necessitates a bifurcated approach, with each scenario encompassing three distinct conditions.

Case 1 $r_0 \le r_w$



(a)$r_0 \le r_w$ and $r \quad < r_0$ (b)$r_0 \le r_w$ and $r_0 \le r \le r_w$ (c)$r_0 \le r_w$ and $r_w < r$

Figure 3. Diagram of TEW span and PM intervals while $r_0 \le r_w$.

While $r_0 \leq r_w$, upon the product's introduction to the market, the frequency and moment of PM to be executed by manufacturer have been predetermined, thus they are independent of the actual usage rate "r ". Assuming that $r_0 \leq r_w$, the usage rates r of equipment vary among different users, the relationships among r ,r_0 and r_w should be discussed in the following three scenarios, as specifically illustrated in Figure 3.

(1)If$r \ < r_0$, refer to Figure 3(a), $T_c$denotes the beginning of TEW in the usage time dimension,$T_w$denotes the ending of TEW in the usage time dimension, T_0denotes the interval of PM.$H_W = \left\lfloor \frac{(T_w - T_c)}{T_0} \right\rfloor$is the total frequency of PM within TEW, and $T_0 + T_c, 2T_0 + T_c, \ldots\ldots, H_w T_0 + T_c$ is the moment of PM within TEW. when equipment fails in TEW, corrective maintenance is executed, hence the sum cost of corrective maintenance is:

$$\begin{aligned} C_{fz} &= C_f(\int_{T_c}^{T_0+T_c} \psi_0 dt + \int_{T_0+T_c}^{2T_0+T_c} \psi_1 dt + \ldots + \int_{(H_w-1)T_0+T_c}^{H_w T_0+T_c} \psi_{H_w-1} dt + \int_{H_w T_0+T_c}^{T_w} \psi_{H_w} dt) \\ &= C_f(\sum_{j=0}^{H_w-1} \int_{jT_0+T_c}^{(j+1)T_0+T_c} \psi_j dt + \int_{H_w T_0+T_c}^{T_w} \psi_{H_w} dt) \end{aligned} \tag{10}$$

The sum downtime of corrective maintenance is:

$$\begin{aligned} T_{fz} &= T_f(\int_{T_c}^{T_0+T_c} \psi_0 dt + \int_{T_0+T_c}^{2T_0+T_c} \psi_1 dt + \ldots + \int_{(H_w-1)T_0+T_c}^{H_w T_0+T_c} \psi_{H_w-1} dt + \int_{H_w T_0+T_c}^{T_w} \psi_{H_w} dt) \\ &= T_f(\sum_{j=0}^{H_w-1} \int_{jT_0+T_c}^{(j+1)T_0+T_c} \psi_j dt + \int_{H_w T_0+T_c}^{T_w} \psi_{H_w} dt) \end{aligned} \tag{11}$$

It is stipulated that the manufacturer will execute the $H_1, H_2, \ldots, H_N$ -th PM, and the user will execute the remaining $H_w - N$ instances. The sum cost of PM for the product within TEW is:

$$C_{pz} = C_{p1}(H_w - N) + C_{p2}N \tag{12}$$

The sum downtime of PM for the product within TEW is:

$$T_{pz} = T_{p1}(H_w - N) + T_{p2}N \tag{13}$$

At the given condition$r \ \leq r_0$, the total maintenance cost within TEW can be expressed as:

$$C_1(T_0, U_0, N, \boldsymbol{H_n}) = C_{pz} + C_{fz} = C_{p1}(H_w - N) + C_{p2}N + C_f(\sum_{j=0}^{H_w-1} \int_{jT_0+T_c}^{(j+1)T_0+T_c} \psi_j dt + \int_{H_w T_0+T_c}^{T_w} \psi_{H_w} dt) \tag{14}$$

The total downtime within TEW can be expressed as:

$$D_1(T_0, U_0, N, \boldsymbol{H_n}) = T_{pz} + T_{fz} = T_{p1}(H_w - N) + T_{p2}N + T_f(\sum_{j=0}^{H_w-1} \int_{jT_0+T_c}^{(j+1)T_0+T_c} \psi_j dt + \int_{H_w T_0+T_c}^{T_w} \psi_{H_w} dt) \tag{15}$$

(2)If$r_0 \leq r \ \leq r_w$, refer to Figure 3(b), T_w denotes the ending of TEW in the usage time dimension, $\frac{U_0}{r}$denotes the interval of PM.$H_W = \left\lfloor \frac{(T_W - T_c)r}{U_0} \right\rfloor$is the total frequency of PM within TEW, and $\frac{U_0}{r} + T_c, \frac{2U_0}{r} + T_c, \ldots\ldots, \frac{H_w U_0}{r} + T_c$ is the moment of PM within TEW. The sum cost of corrective maintenance is:

$$\begin{aligned} C_{fz} &= C_f(\int_{T_c}^{U_0/r+T_c} \psi_0 dt + \int_{U_0/r+T_c}^{2U_0/r+T_c} \psi_1 dt + \ldots + \int_{(H_w-1)U_0/r+T_c}^{H_w U_0/r+T_c} \psi_{H_w-1} dt + \int_{H_w U_0/r+T_c}^{T_w} \psi_{H_w} dt) \\ &= C_f(\sum_{j=0}^{H_w-1} + \int_{jU_0/r+T_c}^{(j+1)U_0/r+T_c} \psi_j dt + \int_{H_w U_0/r+T_c}^{H_w} \psi_{H_w} dt) \end{aligned} \tag{16}$$

The sum downtime of corrective maintenance is:

$$\begin{aligned} T_{fz} &= T_f(\int_{T_c}^{U_0/r+T_c}\psi_0 dt+\int_{U_0/r+T_c}^{2U_0/r+T_c}\psi_1 dt+\ldots+\int_{(H_w-1)U_0/r+T_c}^{H_wU_0/r+T_c}\psi_{H_w-1}dt+\int_{H_wU_0/r+T_c}^{T_w}\psi_{H_w}dt) \\ &= T_f(\sum_{j=0}^{H_w-1}+\int_{jU_0/r+T_c}^{(j+1)U_0/r+T_c}\psi_j dt+\int_{H_wU_0/r+T_c}^{H_w}\psi_{H_w}dt) \end{aligned} \tag{17}$$

At the given condition $r_0 \leq r \leq r_w$, the total maintenance cost within TEW can be expressed as:

$$C_2(T_0,U_0,N,\boldsymbol{H_n})=C_{pz}+C_{fz}=C_{p1}(H_w-N)+C_{p2}N+C_f(\sum_{i=0}^{H_w-1}+\int_{jU_0/r+T_c}^{(j+1)U_0/r+T_c}\psi_j dt+\int_{H_wU_0/r+T_c}^{T_w}\psi_{H_w}dt) \tag{18}$$

The total downtime within TEW can be expressed as:

$$D_2(T_0,U_0,N,\boldsymbol{H_n})=T_{pz}+T_{fz}=T_{p1}(H_w-N)+T_{p2}N+T_f(\sum_{i=0}^{H_w-1}+\int_{jU_0/r+T_c}^{(j+1)U_0/r+T_c}\psi_j dt+\int_{H_wU_0/r+T_c}^{T_w}\psi_{H_w}dt) \tag{19}$$

(3)If $r > r_w$, refer to Figure 3(c), $\frac{U_W}{r}$denotes the ending of TEW in the usage time dimension, $\frac{U_0}{r}$ denotes the interval of PM. $H_W=\left\lfloor\frac{((U_W/r)-T_c)r}{U_0}\right\rfloor$ is the total frequency of PM within TEW, and $\frac{U_0}{r}+T_c,\frac{2U_0}{r}+T_c,\ldots\ldots,\frac{H_wU_0}{r}+T_c$ is the moment of PM within TEW. The sum cost of corrective maintenance is:

$$\begin{aligned} C_{fz} &= C_f(\int_{T_c}^{U_0/r+T_c}\psi_0 dt+\int_{U_0/r+T_c}^{2U_0/r+T_c}\psi_1 dt+\ldots+\int_{(H_w-1)U_0/r+T_c}^{H_wU_0/r+T_c}\psi_{H_w-1}dt+\int_{N_wU_0/r+T_c}^{U_w/r}\psi_{H_w}dt) \\ &= C_f(\sum_{j=0}^{H_w-1}\int_{jU_0/r+T_c}^{(j+1)U_0/r+T_c}\psi_j dt+\int_{H_wU_0/r+T_c}^{U_w/r}\psi_{H_w}dt) \end{aligned} \tag{20}$$

The sum downtime of corrective maintenance is:

$$\begin{aligned} T_{fz} &= T_f(\int_{T_c}^{U_0/r+T_c}\psi_0 dt+\int_{U_0/r+T_c}^{2U_0/r+T_c}\psi_1 dt+\ldots+\int_{(H_w-1)U_0/r+T_c}^{H_wU_0/r+T_c}\psi_{H_w-1}dt+\int_{H_wU_0/r+T_c}^{U_w/r+T_c}\psi_{H_w}dt) \\ &= T_f(\sum_{j=0}^{H_w-1}\int_{jU_0/r+T_c}^{(j+1)U_0/r+T_c}\psi_j dt+\int_{H_wU_0/r+T_c}^{U_w/r}\psi_{H_w}dt) \end{aligned} \tag{21}$$

At the given condition$r \geq r_w$, the total maintenance cost within TEW can be expressed as:

$$C_3(T_0,U_0,N,\boldsymbol{H_n})=C_{pz}+C_{fz}=C_{p1}(H_w-N)+C_{p2}N+C_f(\sum_{j=0}^{H_w-1}\int_{jU_0/r+T_c}^{(j+1)U_0/r+T_c}\psi_j(t|r)dt+\int_{H_wU_0/r+T_c}^{U_w/r}\psi_{H_w}dt) \tag{22}$$

The total downtime within TEW can be expressed as:

$$D_3(T_0,U_0,N,\boldsymbol{H_n})=T_{pz}+T_{fz}=T_{p1}(H_w-N)+T_{p2}N+T_f(\sum_{j=0}^{H_w-1}\int_{jU_0/r+T_c}^{(j+1)U_0/r+T_c}\psi_j(t|r)dt+\int_{H_wU_0/r+T_c}^{U_w/r}\psi_{H_w}dt) \tag{23}$$

Based on the above, when the product usage rate distribution function of the user is $F(r)$ and $r_0 \leq r_w$, the expected warranty cost of the product within TEW is:

$$C(T_0,U_0,N,\boldsymbol{H_n}) = \int_{r_l}^{r_0} C_1(T_0,U_0,N,\boldsymbol{H_n})\,dF(r) + \int_{r_0}^{r_w} C_2(T_0,U_0,N,\boldsymbol{H_n})\,dF(r) + \int_{r_w}^{r_u} C_3(T_0,U_0,N,\boldsymbol{H_n})\,dF(r) \tag{24}$$

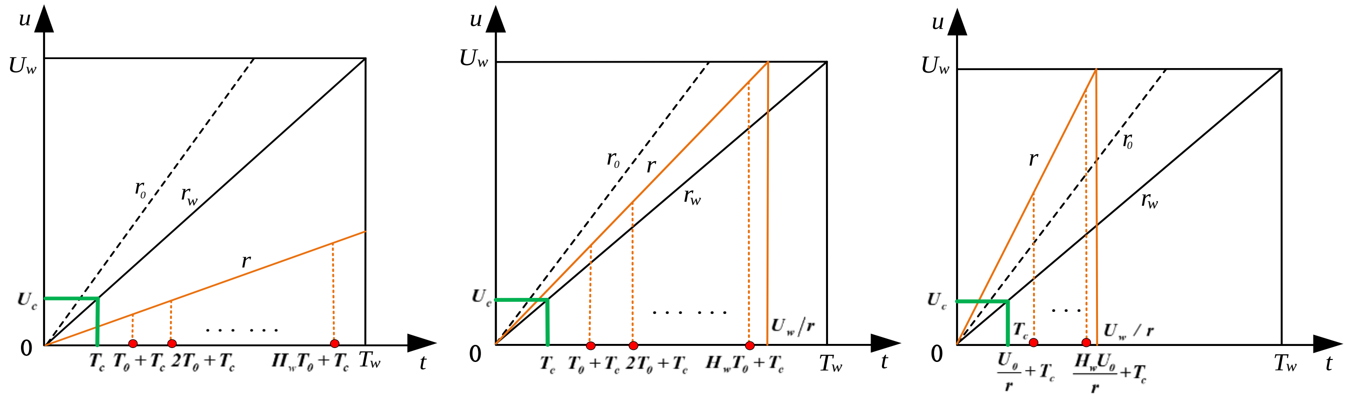
The expected warranty availability of the product within TEW is:

$$\begin{aligned} A(T_0,U_0,N,\boldsymbol{H_n}) &= \int_{r_l}^{r_0}[T_w - D_1(T_0,U_0,N,\boldsymbol{H_n})]/T_w dF(r) + \int_{r_0}^{r_w}[T_w - D_2(T_0,U_0,N,\boldsymbol{H_n})]/T_w dF(r) \\ &+ \int_{r_w}^{r_u}[U_w/r - D_3(T_0,U_0,N,\boldsymbol{H_n})]/(U_w/r)dF(r) \end{aligned} \tag{25}$$

r_l represents the lower limit of usage rate, which is the lowest usage rate. r_u represents the upper limit of usage rate, which is the highest usage rate.

Case 2 $r_0 > r_w$

While $r_0 > r_w$, the frequency and moment of PM to be executed by manufacturer are independent of the actual usage rate "r ". Assuming that $r_0 > r_w$, the usage rates r of equipment vary among different users, there are three scenarios, as specifically illustrated in Figure 4.



(a)$r_0 > r_w$ and $r < r_w$ (b)$r_0 > r_w$ and $r_w \le r \le r_0$ (c)$r_0 > r_w$ and $r_0 < r$

Figure 4. Diagram of TEW span and PM intervals while $r_0 > r_w$.

(1)If $r < r_w$, refer to Figure 4(a), T_w denotes the ending of TEW in the usage time dimension, T_0 denotes the interval of PM. $H_W = \left\lfloor \frac{(T_w - T_c)}{T_0} \right\rfloor$ is the total frequency of PM within TEW, and $T_0+T_c, 2T_0+T_c, \ldots\ldots, H_wT_0+T_c$ is the moment of PM within TEW.

At the given condition $r < r_w$, the total maintenance cost within TEW is:

$$C_4(T_0,U_0,N,\boldsymbol{H_n}) = C_{fz} + C_{pz} = (H_w - N)C_{p1} + NC_{p2} + C_f\left(\sum_{j=0}^{H_w-1}\int_{jT_0+T_c}^{(j+1)T_0+T_c}\psi_j dt + \int_{H_wT_0+T_c}^{T_w}\psi_{H_w}dt\right) \tag{26}$$

The total downtime within TEW can be expressed as:

$$D_4(T_0,U_0,N,\boldsymbol{H_n}) = T_{fz} + T_{pz} = (H_w - N)T_{p1} + NT_{p2} + T_f\left(\sum_{j=0}^{H_w-1} + \int_{jT_0+T_c}^{(j+1)T_0+T_c}\psi_j dt + \int_{H_wT_0+T_c}^{T_w}\psi_{H_w}dt\right) \tag{27}$$

(2)If $r_w \le r \le r_0$, refer to Figure 4(b), $\frac{U_W}{r}$ denotes the ending of TEW in the usage time dimension, T_0 denotes the interval of PM. $H_W = \left\lfloor \frac{\left(\left(\frac{U_W}{r}\right) - T_c\right)}{T_0} \right\rfloor$ is the total frequency of PM within TEW, and $T_0+T_c, 2T_0+T_c, \ldots\ldots, H_wT_0+T_c$ is the moment of PM within TEW.

At the given condition $r_w \le r \le r_0$, the total maintenance cost within TEW can be expressed as:

$$C_5(T_0,U_0,N,\boldsymbol{H_n}) = C_{pz} + C_{fz} = C_{p1}(H_w - N) + C_{p2}N + C_f\left(\sum_{j=0}^{H_w-1}\int_{jT_0+T_c}^{(j+1)T_0+T_c}\psi_j dt + \int_{H_wT_0+T_c}^{U_w/r}\psi_{H_w}dt\right) \tag{28}$$

The total downtime within TEW can be expressed as:

$$D_5(T_0,U_0,N,\boldsymbol{H_n}) = T_{pz} + T_{fz} = T_{p1}(H_w - N) + T_{p2}N + T_f\left(\sum_{j=0}^{H_w-1} \int_{jT_0+T_c}^{(j+1)T_0+T_c} \psi_j dt + \int_{H_wT_0+T_c}^{U_w/r} \psi_{H_w} dt\right) \tag{29}$$

(3)If $r_0 < r$, refer to Figure 4(c), $\frac{U_W}{r}$ denotes the ending of TEW in the usage time dimension, $\frac{U_0}{r}$ denotes the interval of PM. $H_W = \left\lfloor \frac{\left(\left(\frac{U_W}{r}\right)-T_c\right)}{\left(\frac{U_0}{r}\right)} \right\rfloor$ is the total frequency of PM within TEW, and $\frac{U_0}{r} + T_c, 2\frac{U_0}{r} + T_c, \ldots\ldots, \frac{H_wU_0}{r} + T_c$ is the moment of PM within TEW.

At the given condition $r_0 < r$, the total maintenance cost within TEW can be expressed as:

$$C_6(T_0,U_0,N,\boldsymbol{H_n}) = C_{pz} + C_{fz} = C_{p1}(H_w - N) + C_{p2}N + C_f\left(\sum_{j=0}^{H_w-1} \int_{jU_0/r+T_c}^{(j+1)U_0/r+T_c} \psi_j dt + \int_{H_wU_0/r+T_c}^{U_w/r} \psi_{H_w} dt\right) \tag{30}$$

The total downtime within TEW can be expressed as:

$$D_6(T_0,U_0,N,\boldsymbol{H_n}) = T_{pz} + T_{fz} = T_{p1}(H_w - N) + T_{p2}N + T_f\left(\sum_{j=0}^{H_w-1} \int_{jU_0/r+T_c}^{(j+1)U_0/r+T_c} \psi_j dt + \int_{H_wU_0/r+T_c}^{U_w/r} \psi_{H_w} dt\right) \tag{31}$$

Based on the above, when the product usage rate distribution function of the user is $F(r)$ and $r_0 > r_w$, the product's warranty cost of the TEW is:

$$C(T_0,U_0,N,\boldsymbol{H_n}) = \int_{r_l}^{r_w} C_4(T_0,U_0,N,\boldsymbol{H_n})\, dF(r) + \int_{r_w}^{r_0} C_5(T_0,U_0,N,\boldsymbol{H_n})\, dF(r) + \int_{r_0}^{r_u} C_6(T_0,U_0,N,\boldsymbol{H_n})\, dF(r) \tag{32}$$

The product's warranty availability of the TEW is:

$$\begin{aligned} A(T_0,U_0,N,\boldsymbol{H_n}) = {} & \int_{r_l}^{r_w} [T_w - D_4(T_0,U_0,N,\boldsymbol{H_n})]/T_w dF(r) + \int_{r_w}^{r_0} [\tfrac{U_w}{r} - D_5(T_0,U_0,N,\boldsymbol{H_n})]/(\tfrac{U_w}{r}) dF(r) \\ & + \int_{r_0}^{r_u} [\frac{U_w}{r} - D_6(T_0,U_0,N,\boldsymbol{H_n})]/(\frac{U_w}{r}) dF(r) \end{aligned} \tag{33}$$

Finally, the Cost-Availability ratio function of the product's TEW stage can be expressed as:

$$V(T_0,U_0,N,\boldsymbol{H_n}) = \frac{C(T_0,U_0,N,\boldsymbol{H_n})}{A(T_0,U_0,N,\boldsymbol{H_n})} \tag{34}$$

5. ALGORITHM DESIGN AND OPTIMIZATION

This paper provides a detailed introduction to an algorithm designed to obtain the optimal J-PM scheme for a product during its TEW period. For each combination $(T_0,U_0,N,\boldsymbol{H_n})$, we first analyze the warranty cost $C(T_0,U_0,N,\boldsymbol{H_n})$ and warranty availability $A(T_0,U_0,N,\boldsymbol{H_n})$ of the product , then derive the cost-availability ratio $V(T_0,U_0,N,\boldsymbol{H_n})$ model, and finally solve for the combination $(T_0^{\#},U_0^{\#},N^{\#},\boldsymbol{H_n^{\#}})$ that makes the cost-availability ratio minimum. Since the cost-availability ratio model consists of various factors $(T_0,U_0,N,\boldsymbol{H_n})$, this is a complex combinatorial optimization problem that requires the application of an appropriate optimization algorithm.

The peculiarity of this optimization problem lies in the interdependencies among the decision variables, where the outer-layer (T_0,U_0) decision variables' values affect the inner-layer $(N,\boldsymbol{H_n})$ decision variables' values. Consequently, it is necessary to first determine the (T_0,U_0) and then the $(N,\boldsymbol{H_n})$, which characterizes this as a bi-level optimization problem. Given the multitude of decision variables, the solution space is vast. Intelligent optimization algorithms have proven to be highly effective for such problems as they do not require the derivative information that is derived from the objective function. Considering that there are fewer outer-layer decision variables, we opt for a pattern-seeking method for their resolution; whereas, the inner-layer decision variables are more, a genetic algorithm is employed for their solution.

Unlike the simplest gradient method, which searches along the direction of steepest descent, the pattern-seeking method alternates between pattern seek and axis seek in feasible descending directions, offering faster solution speeds and higher reliability in computation. The particular mechanism of the method is outlined below: Initially, based on a given starting point Y_0, the value of objective function is calculated. In the first iteration, a scalar of "x" is referred to as the size of grid; pattern vectors are defined as (x, 0), (0, x), (-x, 0) and (0, -x), which are termed direction vectors. Subsequently, the method adds these direction vectors to the starting point Y_0, and calculates the subsequent points Y_0+(x, 0), Y_0+(0, x), Y_0+(-x, 0) and Y_0+(0, -x), continuing until a point is found where the value of objective function is less than that at Y_0. If a point of this nature exists, a solution is successfully found by the method, and this point is consequently set as Y_1. The method then continues with the second round of iteration, doubling the existing grid size. During the second round of iteration, the grid calculates the following points: Y_1+(2x, 0), Y_1+(0, 2x), Y_1+(-2x, 0), Y_1+(0, -2x). The method searches for a point where the value of objective function is less than that at Y_1, and this point is set as Y_2. The algorithm continues by doubling the grid size for the third iteration, and so on.

This research employs a fusion of the genetic algorithm and pattern-seeking method to resolve the model introduced, and the flowchart of the algorithm is shown in Figure 5.

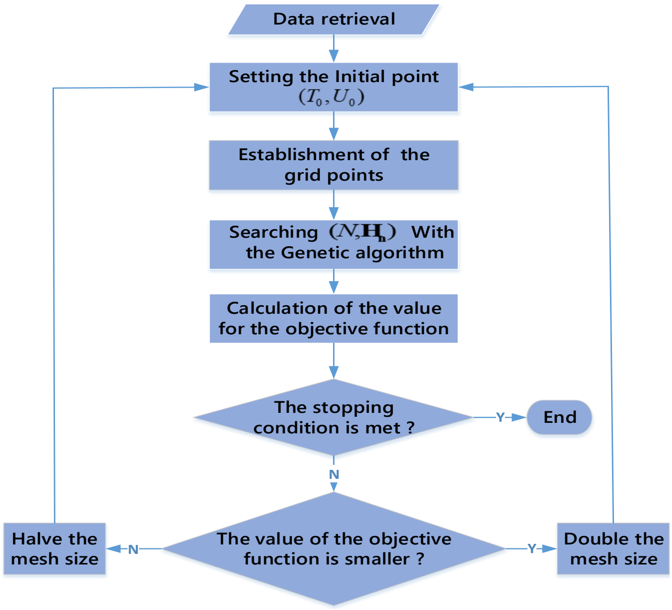


Figure 5. Algorithmic resolution process flowchart in this research.

Based on this, we have developed a MATLAB program to find the combination $(T_0^{\#}, U_0^{\#}, N^{\#}, \boldsymbol{H}_n^{\#})$ that make the cost-availability ratio minimum.

6. A NUMERICAL EXAMPLE

In this part, the objective of decision-making is to make the cost-availability ratio minimum during the product's TEW period. We provide the numerical examples to prove the advantages of J-PM over independent preventive maintenance executed by manufacturers and users. Here, J-PM denotes the strategy where users and manufacturers jointly execute preventive maintenance, U-PM denotes the strategy where users independently execute preventive maintenance, and M-PM denotes the strategy where manufacturers independently execute preventive maintenance. When examining the impact of model parameters on the decision-making objective, sensitivity analysis is conducted on key parameters using the method of controlling variables, and corresponding warranty recommendations are provided.

Considering a scenario where a user has purchased large engineering vehicle from a manufacturer, and the initial warranty period for the equipment has expired, with both user and manufacturer conducting J-PM during the TEW. The range of TEW for the product is characterized by the usage time limit $[T_c = 0.3 years, T_w = 3.5 years]$ and usage intensity limit $[U_c = 0.3 \times 10^4 km, U_w = 3.5 \times 10^4 km]$. The repair factor for user is denoted as $\theta_1 = 0.75$, and for manufacturer is denoted as $\theta_2 = 0.3$. The time required for the product to undergo corrective maintenance is $T_f = 5$ days, The cost required for the product to undergo corrective maintenance is $C_f = 3000$ CNY. The time taken by user to execute a PM is $T_{p1} = 1$ days, and by manufacturer is $T_{p2} = 2$ days. The cost of a PM executed by user is $C_{p1} = 800$CNY, and by manufacturer is $C_{p2} = 900$ CNY. The failure rate function's parameters within the TEW period of the product are $\delta_0 = 0.7, \delta_1 = 0.5, \delta_2 = 0.6, \delta_3 = 0.8$. The usage rate of the users follows a Weibull distribution, characterized by a shape parameter of $p = 1.9$and a scale parameter of $q = 3.8$.

A. Comparison of optimal solutions for different strategies

The minimum cost-availability ratio for product repairs within the TEW period varies with different (T_0, U_0) combinations. In this paper, under the optimal warranty strategy for the product, the cost, availability and cost-availability ratio corresponding to some (T_0, U_0) combinations are depicted in Figures 6-8, respectively.

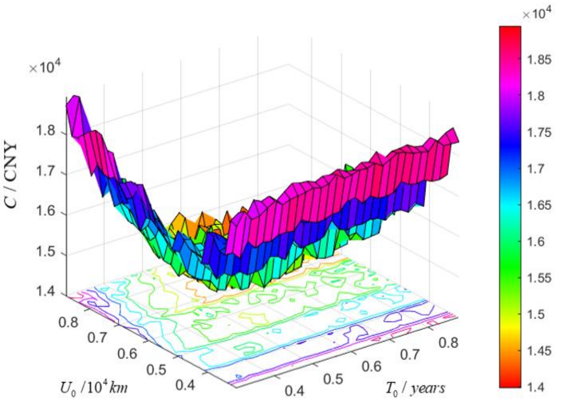


Figure 6. The variation of TEW cost in relation to the changes in $T_0$and U_0.

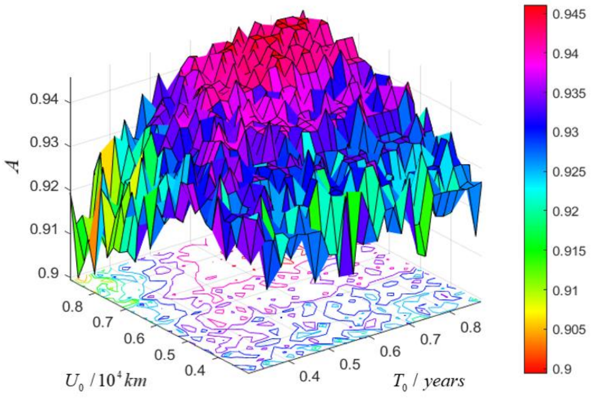


Figure 7. The variation of TEW availability in relation to the changes in $T_0$and U_0.

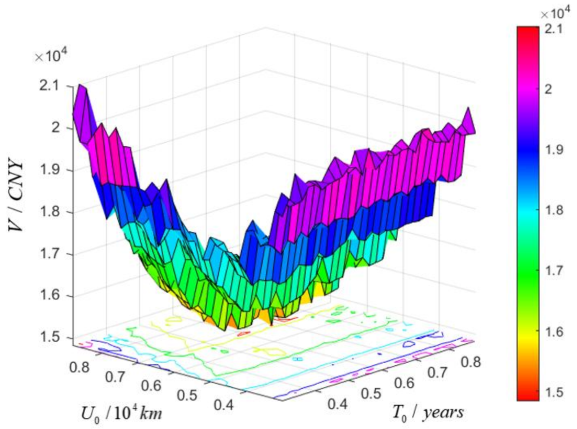


Figure 8. The variation of TEW cost-availability ratio in relation to the changes in $T_0$and U_0.

In Figure 8, when $T_0 \in (0.3,0.9) years$ and $U_0 \in (0.3,0.9) \times 10^4 km$, there exists a unique global optimal combination $(T_0^{\#}, U_0^{\#}, N^{\#}, \boldsymbol{H}_{\boldsymbol{n}}^{\#})$ that makes the product warranty cost-availability ratio minimum, because the cost-availability ratio function exhibits concavity in terms of T_0 and U_0.The pattern-seeking method and genetic algorithm, after iterating 37 times, meets the preset precision requirements and terminates, yielding the optimal J-PM plan is as follows:$T_0^{\#} = 0.66$ $U_0^{\#} = 0.76$ $N^{\#} = 3$ $\boldsymbol{H}_{\boldsymbol{n}}^{\#} = (1,3,4)$ $C(T_0^{\#}, U_0^{\#}, N^{\#}, \boldsymbol{H}_{\boldsymbol{n}}^{\#}) = 14827.34$ $A(T_0^{\#}, U_0^{\#}, N^{\#}, \boldsymbol{H}_{\boldsymbol{n}}^{\#}) = 0.938$ $V(T_0^{\#}, U_0^{\#}, N^{\#}, \boldsymbol{H}_{\boldsymbol{n}}^{\#}) = 15812.33$. The algorithm's iterative procedure is shown in Figure 9.

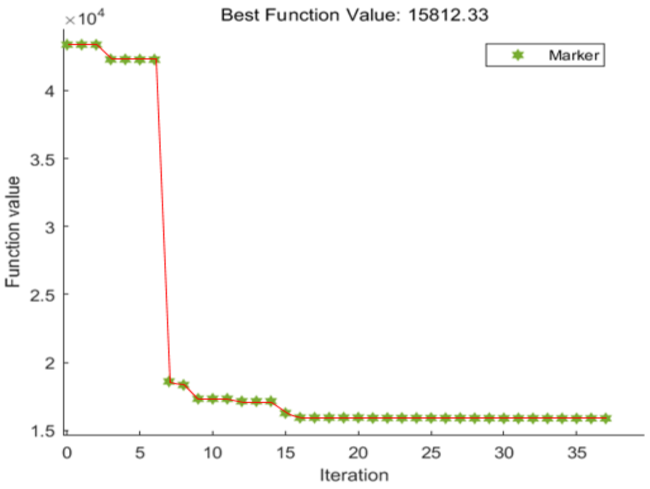


Figure 9. The iterative procedure of the algorithm.

The optimal solutions for different PM strategies during the TEW are presented in Table 1.

Table 1. Optimal solutions of different PM strategies for product during the TEW.

	$T_0^{\#}$	$U_0^{\#}$	N	$\boldsymbol{H}_{\boldsymbol{n}}^{\#}$	C	A	V
J-PM	0.66	0.76	3	(1,3,4)	**14827.34**	**0.938**	**15812.33**
M-PM	0.58	0.69	-	-	14993.05	0.929	16138.91
U-PM	0.31	0.36	-	-	20089.95	0.889	22598.37

Table 1 reveals that the J-PM strategy implemented during the TEW outperforms the M-PM and the U-PM strategies in terms of warranty cost, warranty availability, and warranty cost-availability ratio. Compared to the M-PM strategy, implementing the J-PM strategy can reduce the optimal warranty cost-availability ratio by 2%; Compared to the U-PM strategy, implementing the J-PM strategy can reduce the optimal warranty cost-availability ratio by 30%.

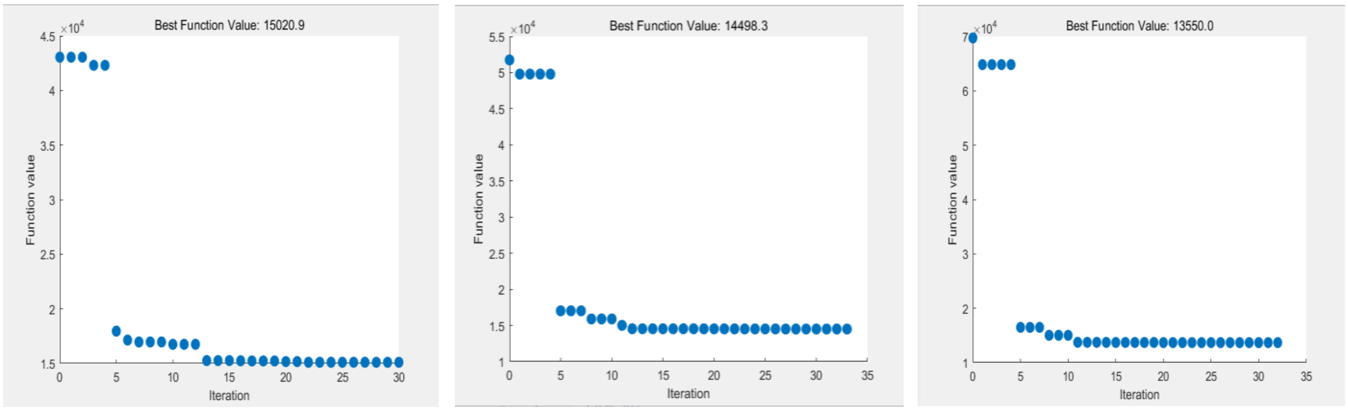
When the M-PM strategy is employed, the optimal cost-availability ratio of the TEW is close to that of the J-PM strategy.

However, when implementing the U-PM strategy, the maintenance effect is far inferior to that of the J-PM strategies. These data eloquently demonstrate the advantages of J-PM strategy over the M-PM and U-PM strategy.

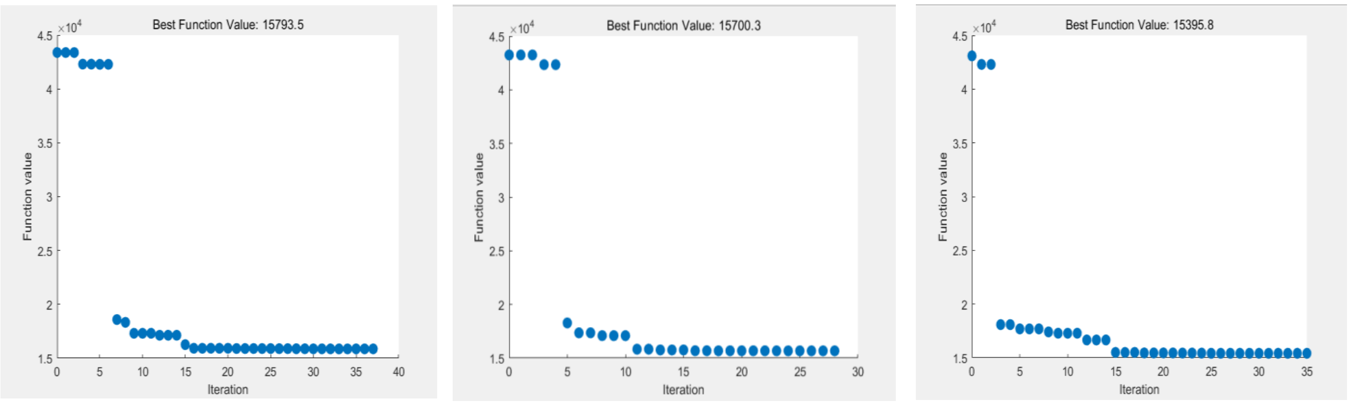
B. The sensitivity analysis of usage rate parameters "*p*, *q*"

This segment will further investigate the influence of pivotal parameters on the implementation of the optimal J-PM strategy. During the sensitivity analysis, each trial involves altering the value of a single parameter while the rest remain fixed.

Different users exhibit varying rates of product utilization. To analyze the influence of utilization rate distribution parameters on the implementation of J-PM strategies, this section employs the method of controlled variables to study different combinations of the shape parameter " *p* " and the scale parameter " *q* " . Figure 10 illustrates the iterative process of the algorithm under various usage rate distribution parameters.



(a) p=2.9, q=3.1 (b) p=3.9, q=3.1 (c) p=4.9, q=3.1



(d) p=1.9, q=4.1 (e) p=1.9, q=5.1 (f) p=1.9, q=6.1

Figure 10. The iterative process of the algorithm under different usage rate parameters.

Table 2 presents the optimal solutions for implementing J-PM under different usage rate distribution parameters.

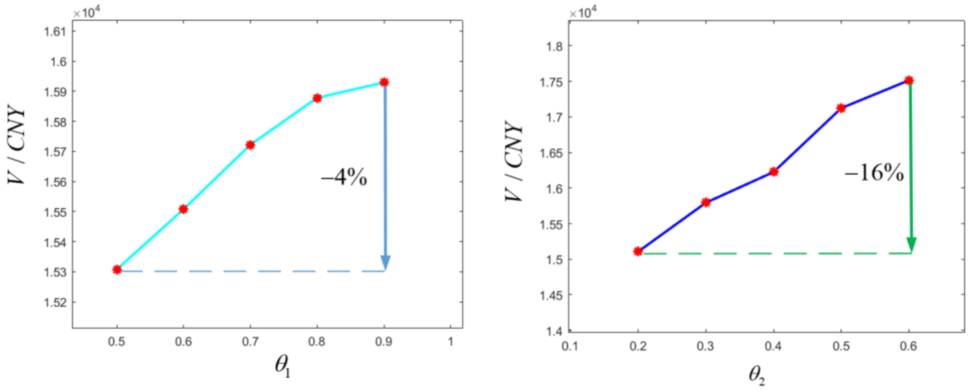
Table 2. Optimal warranty solutions under varying usage rate distribution parameters.

	$T_0^\#$	$U_0^\#$	N	$\boldsymbol{H_n^\#}$	C	A	V
p=2.9, q=3.1	0.62	0.78	2	(3, 4)	14284.90	0.951	**15020.93**
p=3.9, q=3.1	0.62	0.77	1	3	13613.89	0.939	**14498.29**
p=4.9, q=3.1	0.63	0.79	3	(1,2,4)	12845.44	0.948	**13550.04**
p=1.9, q=4.1	0.66	0.76	1	2	14924.87	0.945	**15793.51**
p=1.9, q=5.1	0.65	0.76	2	(3,4)	14711.20	0.937	**15700.32**
p=1.9, q=6.1	0.67	0.75	2	(2,4)	14533.62	0.944	**15395.78**

From Table 2, it is evident that different combinations of "p, q" yield distinct optimal warranty plans. Therefore, when devising optimal warranty plans during the TEW period for a product, it is crucial to understand the utilization rate distribution of the users. As either "p" or "q" increases, the optimal cost-availability ratio gradually decrease, leading to better maintenance outcomes. It has also been observed that compared to "q", changes in "p" exert a greater influence on cost-availability ratio of the optimal warranty plan.

C. The sensitivity analysis of repair factors "$\boldsymbol{\theta_1, \theta_2}$"

In order to analyze the influence of different repair factors from users and manufacturers on the implementation of J-PM strategies, this section also employs the control variable method to conduct separate θ_1 and θ_2 studies. Initially, the manufacturer's repair factor $\theta_2 = 0.3$ is held constant, and then the variation of the user's repair factor θ_1 under the optimal strategy is analyzed for its effects on product warranty cost-availability ratio. The results are depicted in Figure 11(a). Subsequently, keeping the $\theta_1 = 0.75$ unchanged, the variation in the θ_2 under the optimal strategy is analyzed for its impact on product cost-availability ratio. The results are illustrated in Figure 11(b).



(b)The user's repair factor θ_1 (b)The manufacturer's repair factor θ_2

Figure 11. The influence of repair factor on cost-availability ratio.

In Figure 11(a), when the θ_1 changes from 0.9 to 0.5, it leads to a 4% decrease in the cost-availability ratio for the optimal warranty plan. In Figure 11(b), when the $\theta_2$changes from 0.6 to 0.2, it causes a 16% drop in the cost-availability ratio for the optimal warranty plan of the product. It can be inferred that enhancing the repair and maintenance skills of both manufacturers and users can reduce the warranty cost-availability ratio. However, compared to variable θ_1 , the optimal warranty cost-availability ratio is more sensitive to changes in variable θ_2.

7. CONCLUSION

This paper introduces a novel J-PM strategy, which is jointly executed by users and manufacturers during the TEW period of a product to more effectively meet the needs of both parties. We have designed a J-PM plan under a TEW framework, aiming to minimize the cost-availability ratio by optimizing the intervals of PM, the frequency of manufacturer interventions, and the optimal timing. The innovation of this study lies in considering the effect of inadequate maintenance on the two-dimensional failure rate function, which takes into account both time and intensity dimensions simultaneously. Furthermore, we have integrated warranty costs and system availability in decisional process, constructing a function of the cost-availability ratio to guide decision-making. Considering the intricate nature of the model, we have introduced a hybrid optimization approach combining pattern-seeking method and genetic algorithm to ensure the effective identification of the optimal J-PM plan. Through extensive numerical studies, we have validated the effectiveness of the J-PM strategy and conducted a sensitivity analysis.

This paper presents a case study where a comparative analysis was conducted between the J-PM strategy and the divided PM strategy, supplemented by sensitivity analysis, leading to the following conclusions: (1) The J-PM strategy demonstrates superior performance over the divided PM strategy with regard to extended warranty cost, extended warranty availability, and extended warranty cost-availability ratio, exhibiting a win-win characteristic that benefits both manufacturers and users. (2) The distribution of usage rates significantly affects the choice of the ideal J-PM strategy; hence, it is crucial to precisely know the parameters *"p, q"* when preparing to determine actions. (3)Improving the manufacturer's maintenance skills and achieve a lower value of factor $\theta_2$can lead to a more significant reduction in the cost-effectiveness ratio, achieving a more desirable warranty outcome.

Future research directions for this study include: (1) The introduction of performance incentive mechanisms in future studies. (2) The study presupposes an unvarying usage rate for same users; future research could relax this assumption to enhance the practical applicability of the results. (3) Exploring more efficient solution algorithms is also a potential direction for future research.

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Notation

T_w	the end of two-dimensional extended warranty within usage time range
U_w	the end of two-dimensional extended warranty within usage intensity range
T_c	the begin of two-dimensional extended warranty within usage time range
U_c	the begin of two-dimensional extended warranty within usage intensity range
T_0	interval of usage time dimension
U_0	interval of usage intensity dimension
H_w	preventive maintenance frequency
N	manufacturers perform preventive maintenance frequency
$\boldsymbol{H_n}$	the moment for manufacturers to perform preventive maintenance
r	actual usage rate of users
r_0,$r_0 = \frac{(U_0+U_c)}{(T_0+T_c)}$	shape parameters of preventive maintenance area
r_w,$r_w = U_w/T_w$	shape parameters of bidimensional warranty area
$\psi(t\|r)$	Product failure rate function
θ_1	repair factors for preventive maintenance performed by users
θ_2	repair factors for preventive maintenance performed by manufacturers

ψ_k	product failure rate function after the k-th preventive maintenance
C_f	corrective repair cost after a single failure
T_f	corrective repair time after a single failure
C_{p1}	single preventive maintenance cost of user's execution
C_{p2}	single preventive maintenance cost of manufacturer's execution
T_{p1}	single preventive maintenance time of user's execution
T_{p2}	single preventive maintenance time of manufacturer's execution
C_{fz}	total cost of corrective repair during the extended warranty period
T_{fz}	total downtime of corrective repair during the extended warranty period
C_{pz}	total cost of preventive maintenance during the extended warranty period
T_{pz}	total downtime of preventive maintenance during the extended warranty period
$F(r)$	user usage distribution function
$C(T_0, U_0, N, \boldsymbol{H_n})$	product maintenance cost during extended warranty period
$A(T_0, U_0, N, \boldsymbol{H_n})$	product availability during extended warranty period
$V(T_0, U_0, N, \boldsymbol{H_n})$	cost-availability ratio over the course of extended warranty