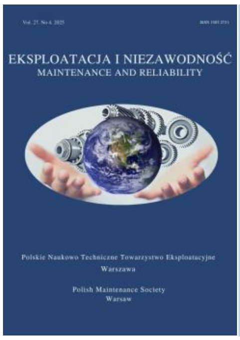


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# Optimal predictive maintenance for a nonstationary gamma process

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**Yue Wang[a], Ganlong Wang[a], Yanxia Wu[a,*], Guoyin Zhang[a], Rui Zheng[b]**

[a] College of Computer Science and Technology, Harbin Engineering University, Harbin 150006, China
[b] School of Management, Hefei University of Technology, Hefei 230009, China

## Highlights

- A heuristic predicitive maintenance policy for a nonstationary gamma process.
- A preventive repair model considering the change of degradation rates.
- Decisions based on information of age, degradation, and the number of performed repairs.
- Formulation of the problem in the semi-Markov decision process framework.



## Abstract

Nonstationary gamma processes have extensive applications in depicting the degradation of many practical systems. This paper proposes a predictive maintenance policy that involves various types of maintenance actions for a nonstationary gamma process. Periodic inspections fully reveal the degradation levels of the system. The information on age, degradation, and the number of conducted preventive maintenance actions is synthesized for decision-making, which distinguishes our model from most existing models considering only degradation states. The objective is to find the maximum number of repairs and the best threshold for preventive maintenance by minimizing the expected average cost in an infinite time horizon. The maintenance problem is addressed as a semi-Markov decision problem. An optimization algorithm is developed to find the optimal values of the decision variables. The effectiveness of the proposed method is verified by a coating system.

**Keywords**
accelerated degradation, gamma process, imperfect maintenance, semi-Markov, dynamic programming

## 1. Introduction

In modern manufacturing industries, complex systems play a vital role in day-to-day operations. However, these systems are prone to degrade over time due to usage and age. When degradation surpasses a critical threshold, the performance of the system becomes unsatisfactory, leading to a system failure. The substantial costs linked with unplanned failures have sparked significant research efforts in maintenance optimization. The primary objective is to identify the optimal timing for preventive repairs or replacements to avert system failures from transpiring. Various preventive maintenance (PM) policies have been proposed in the literature. Gan et al. [1] investigated a PM optimization problem for a production-inventory system exposed to shock environments; Zheng et al. [2] developed a framework to optimize group maintenance for numerical-control machine tools based on real failure data.

Predictive maintenance, also known as condition-based maintenance (CBM), has been widely implemented in practice [3]. Unlike traditional time-based maintenance or age-based

(*) Corresponding author.
E-mail addresses: Y. Wang (ORCID: 0009-0000-3865-8404) 563669810@gq.com, G. Wang (ORCID: 0009-0008-5745-2311) wangganlon@hrbeu.cdu.cn, Y. Wu, (ORCID: 0000-0001-8384-9234) wuyanxia@hrbcu.cdu.cn, G. Zhang (ORCID: 0009-0003-2877-0066) zhangguoyin@hrbeu.cdu.cn, R. Zheng (ORCID: 0000-0002-8913-9265) rui_zheng@hfut.edu.cn,

maintenance, which uses the expertise and knowledge of decision-makers and the distribution of system lifetimes, predictive maintenance makes decisions based on information gathered from condition monitoring. In predictive maintenance decision-making processes, maintenance actions are triggered when there is substantial evidence of significant degradation. Due to the rapid development of condition monitoring, predictive maintenance can recommend a PM action just before unexpected failure [4–6]. For a comprehensive review of predictive maintenance methods, we refer readers to [7–10] and the references therein.

Selecting an appropriate degradation process is essential for predictive maintenance decision-making. Mathematically, the degradation model of a system can be characterized either as discrete states or as continuous states. Typical discrete-state stochastic processes include discrete-time Markov chains [11], semi-Markov models [12], continuous-time Markov chains [13,14], and hidden Markov models [15]. However, these processes often suffer from the drawback of arbitrary classification when fitting real-world degradation processes [16]. A more practical approach utilizes continuous-state models, such as inverse Gaussian processes [17], Wiener processes [18], or gamma processes [19], to describe degradation processes. The gamma process, among these models, is particularly appropriate for gradual degradation, such as steel corrosion and concrete creep [20].

A common assumption in predictive maintenance optimization models is that the degradation process is stationary. It indicates that the increments over a given interval depend only on the length of the interval, but are independent of the starting age of the interval. Although this assumption can suit the degradation properties of some systems, there exist some cases where the degradation rate varies as a system gets old [21] or operational environments change [22,23]. For example, structural components such as concrete, steel, or wood degrade faster due to factors like moisture, corrosion, or natural wear and tear. When the degradation rate of a system changes, its conditional reliability over the next interval is not only dependent on the current degradation level and the length of the interval but also influenced by the current age [24]. The predictive maintenance decisions of stationary degradation processes are made mainly based on the degradation levels at decision epochs, while the system age is not considered when making decisions. Such a decision-making framework is not cost-effective for nonstationary degradation processes because of the negligence of system age.

PM for nonstationary degradation processes has received increasing research attention in recent years. Nicolai et al. [25] noticed that the coating systems of steel structures degrade faster as their ages grow and depicted the degradation process as a nonstationary gamma process. An imperfect maintenance policy was proposed and the optimal maintenance decisions were derived by dynamic programming. Zhao et al. [26] considered the effect of shocks on an accelerated damage process and investigated the optimization problem of an opportunistic maintenance policy. Liang et al. [27] investigated the replacement decision of a complex system with failure interaction. Zheng and Zhou [28] proposed a CBM policy for a two-component system with interacted degradation/failure processes. Inspection intervals and replacement policy were jointly optimized. Some other models can be found in [29] for road pavements; in [30] for railway tracks, and in [31] for energy pipelines.

This paper develops a novel predictive maintenance model for a nonstationary gamma process, where the shape parameter is nonlinear of age. The degradation is monitored at equidistant time points. If it is detected to be higher than a predetermined failure threshold, a failure occurs, triggering corrective replacement to return the system as good as new. Otherwise, a decision is made on whether to perform PM actions, which can be either preventive replacement or preventive repair. The former returns the system to a brand-new state. The latter reduces the degradation level to zero, but has no effects on the system age. Moreover, we allow the degradation rate to be changeable after repairs. Such a repair model is quite reasonable. For example, the coating system after a repainting without removing the corrosion completely (repair) has zero degradation from the perspective of condition monitoring, but the degradation rate is higher than that of the brand-new state. Under this policy, the degradation of the system follows a multi-stage nonstationary gamma process, which has been rarely investigated. The information on age, degradation, and the number of conducted repairs is used to make a maintenance decision, which distinguishes our paper from most previous ones where only degradation is considered. The policy utilizes a maximum number of PM actions and a PM threshold to guide the action selection at each inspection epoch. Such a policy can be easily understood by maintenance engineers and thus can be well applied in engineering practice. We aim to find the optimal decision variables so that the expected average cost in an infinite time horizon is minimized.

Our problem constitutes a typical sequential decision problem. One tool to address this problem is the Markov decision process (MDP), which is used for situations where decisions are made at equidistant time points [32]. However, the decision intervals are not equidistant in our problem. Therefore, we choose another tool, the semi-Markov decision process (SMDP), which is particularly applicable for continuous decision making. To formulate the optimization problem as a SMDP, we discretize the nonstationary process into a Markov model. Based on the discretization method, the SMDP quantities are derived and an optimization algorithm is developed.

Our policy has some similarities with the CBM policies proposed in [33,34], which considered a limited number of imperfect maintenance actions. Although the policy is designed in a similar manner, our model has some considerable differences. First, our nonstationary degradation process allows its parameters to be changed after repairs, while the other two references consider stationary degradation processes with changeable parameters or transition probabilities. Second, we formulate our problem as an SMDP and propose an algorithm combining the policy iteration algorithm and numerical analysis methods for solving the problem.

The remainder of this paper is organized as follows. Section 2 provides a description of the system degradation and the maintenance policy. In Section 3, the optimization problem is formulated and optimized as an SMDP. Section 4 demonstrates the effectiveness using the example of a coating system. Finally, conclusions are presented in Section 5.

## 2. Problem description

Consider a repairable system with its degradation dependent not only on age but also on the repair number. Let $X = \{X_t, t \geq 0\}$ be the degradation process of the system. Assuming that repairs are performed at time points $s_1, s_2, \cdots$, the degradation increments over the time interval $(s, s+t)$, where $s_k < s < s+t < s_{k+1}$, are independent and follow a gamma distribution. The probability density of $X_{s+t} - X_s = x$ is

$$\gamma(x|\omega(k,s,t),\theta) = \frac{\theta^{\omega(k,s,t)} x^{\omega(k,s,t)-1} e^{-\theta x}}{\Gamma\big(\omega(k,s,t)\big)} \tag{1}$$

where $\omega(k,s,t)$ is the shape parameter dependent on the age at the beginning of the interval $s$, the length of the interval $t$, and the repair number $k$; $\theta > 0$ is the scale parameter; $\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} \mathrm{d}x$ is the gamma function [35]. The degradation process is nonstationary because $\omega(k,s,t)$ is not only dependent on $t$.

Suppose that preventive repair reduces the degradation level to 0, but does not influence the age of the system. Moreover, we allow the degradation rate to be influenced by repairs. In particular, the shape parameter after the $k$th ($k = 0,1,\cdots$) repair is

$$\omega(k,s,t) = a_k[(s+t)^{b_k} - s^{b_k}] \tag{2}$$

where $a_k > 0$ and $b_k \geq 1$ are parameters that depends on the repair number $k$ ($k = 0,1,\cdots$). From the perspective of the system's lifetime, the degradation process follows a multi-phase gamma process and the phases are separated by preventive repairs.

Let $y$ denote a degradation increment. The probability that the degradation increments over a time interval $(s, s+t)$, where $s_k < s < s+t < s_{k+1}$, are greater than $y$ is given by

$$\mathbb{P}(X_t \geq y) = \int_y^\infty \gamma(x|\omega(k,s,t),\theta)\mathrm{d}x = \frac{\Gamma(\omega(k,s,t), y\theta)}{\Gamma\big(\omega(k,s,t)\big)} \tag{3}$$

where $\Gamma(a,x) = \int_x^\infty z^{a-1} e^{-z} \mathrm{d}z$ is the incomplete gamma function for $x \geq 0$ and $a \geq 0$.

The degradation level can be fully revealed by inspections carried out periodically at $\Delta, 2\Delta, \cdots$. The inspection cost is $C_I$. The time of each inspection is negligible, which is a quite reasonable assumption because condition monitoring can usually be carried out without stopping the system.

When the degradation level is available, the decision maker can determine whether a maintenance action is required. This paper considers a predictive maintenance policy that involves multiple maintenance actions such as preventive replacement, corrective replacement, and preventive repair. Either preventive or corrective replacement will return the age and the degradation level to 0. Normally, the preventive replacement cost, $C_P$, is less than the corrective replacement $C_C$. The preventive repair cost is $C_R$. Different from the widely adopted assumption of negligible durations of PM actions, we assume that preventive repair takes an expected time $T_R$, preventive replacement takes an expected time $T_P$, and corrective replacement takes an expected time $T_C$. Such a setting is realistic. First, although most existing policies consider negligible maintenance durations for modelling simplicity, real maintenance actions take time. Second, the durations have effects on the average cost rate and the optimal policy.

In case the degradation of the system exceeds a critical level $D$, it cannot meet the normal demand and thus a failure occurs. The failure is only known at inspection epochs. If the system keeps operating in a failure state, a cost $C_D$ is incurred per unit time.

We define a policy by $\xi = (K, L)$, where $K$ is an integer representing the maximum times of preventive repairs over a lifecycle, and $L > 0$ indicates the PM threshold, including preventive repair and preventive replacement. In particular, at the $n$th decision epoch, the maintenance actions are determined by the degradation level $X_{n\Delta}$ and the current times of preventive repairs $K_{n\Delta}$ as follows:

- If $X_{n\Delta} < L$, no maintenance action is taken.
- If $L \le X_{n\Delta} < D$ and $K_{n\Delta} < K$, preventive repair is performed.
- If $L \le X_{n\Delta} < D$ and $K_{n\Delta} = K$, preventive replacement is conducted.
- If $X_{n\Delta} \ge D$, corrective replacement is taken.

In our problem, $K$ and $L$ are two decision variables that can be optimized by minimizing the expected average cost in the long run.

## 3. Mathematical formulation

### 3.1. Discretization method

Generating a finite discrete state space is necessary for SMDP formulation. We set a large enough integer $N$ and define $N\Delta$ as the maximum useful lifetime. It means that, at $N\Delta$ if the system is not in the failure state, preventive replacement is mandatory. As mentioned in [36], the results will not be affected as long as $N$ is large enough. In practice, $N$ can be determined according to the experience of maintenance engineers or the manual of the system.

Then, the degradation process $X$ is partitioned into a Markov model with state space $\Omega = \{0,1,\cdots,M\}$. Let $\varepsilon = D/M$ be the discretization interval. We consider the degradation to be in state 0 if the degradation level is 0, in state $i \in \{1,\dots,M-1\}$ if the degradation level lies in the interval$((i-1)\varepsilon, i\varepsilon]$, and in state $M$ if the degradation level meets $((M-1)\varepsilon, +\infty)$. For any state $i \in \{1,\dots,M\}$, its degradation level is approximated by the mid-point $(i-0.5)\varepsilon$. Given that the current repair number is $k$, the transition probability from $X_s = i$ to $X_{s+t} = i'$ is given as follows:

(1) When $0 = i < i' < M$, the increments are within the interval $((i'-1)\varepsilon, j\varepsilon)$. Thus,

$$\begin{aligned} P_{0,i'}(s,s+t) &= \int_{(i'-1)\varepsilon}^{i'\varepsilon} \gamma(x|\omega(k,s,t),\theta)\mathrm{d}x \\ &= \frac{\Gamma(\omega(k,s,t),(i'-1)\varepsilon\theta) - \Gamma(\omega(k,s,t),i'\varepsilon\theta)}{\Gamma(\omega(k,s,t))} \end{aligned} \tag{4}$$

(2) When $i = 0$ and $i' = M$, the increments are greater than $(M-1)\varepsilon$. Thus,

$$\begin{aligned} P_{0,M}(s,s+t) &= \int_{(M-1)\varepsilon}^{+\infty} \gamma(x|\omega(k,s,t),\theta)\mathrm{d}x \\ &= \frac{\Gamma(\omega(k,s,t),(M-1)\varepsilon\theta)}{\Gamma(\omega(k,s,t))} \end{aligned} \tag{5}$$

(3) When $0 < i < i' < M$, the increments are between $(i' - i - 0.5)\varepsilon$ and $(i' - i + 0.5)\varepsilon$. Thus,

$$\begin{aligned} P_{i,i'}(t,t+\Delta t) &= \int_{(i'-i-0.5)\varepsilon}^{(i'-i+0.5)\varepsilon} \gamma(x|\omega(k,s,t),\theta)\mathrm{d}x \\ &= \frac{\Gamma(\omega(k,s,t),(i'-i-0.5)\varepsilon\theta) - \Gamma(\omega(k,s,t),(i'-i+0.5)\varepsilon\theta)}{\Gamma(\omega(k,s,t))} \end{aligned} \tag{6}$$

(5) When $0 \le i = i' \le M$, the transition probability can be calculated by

$$P_{i,i}(s,s+t) = 1 - \sum_{i' \neq i} P_{i,i'}(s,s+t) \tag{7}$$

Such discretization methods have been widely used in [19,37,38]. When the discretization level is sufficiently small, the obtained results of these methods are quite close.

Let $S = \{(n,i,k) | n = 0,\cdots,N; i \in \Omega; k = 0,\cdots,K\}$ be the system state space. The state $(n,i,k)$ indicates the system has an age $n\Delta$, a degradation state $i$, and has experienced $k$ times of preventive repairs. As described in Section 2, the set of actions is given by $A = \{0,1,2,3\}$. Action 0 denotes no maintenance, action 1 is preventive repair, action 2 means preventive replacement, and action 3 indicates corrective replacement. Let $a_{(n,i,k)} \in A$ be the action of state $(n,i,k) \in S$. For a policy defined by $K$ and $L$, i.e., $\xi = (K, L)$, $a_{(n,i,k)}$ can be given as follows

$$a_{(n,i,k)}(\xi)=\begin{cases}0 & i<L, n<N\\ 1 & L\le i<M, k<K, n<N\\ 2 & i<M, n=N \text{ or } L\le i<M, k=K, n<N\\ 3 & i=M\end{cases} \quad (8)$$

The average cost of policy $\xi$ can be derived from the renewal theory. In this policy, a renewal cycle completes at a preventive or corrective replacement. The average cost is given by the expected cost over a renewable cycle divided by its expected length. In particular, the average cost of policy $\xi$ is

$$g(\xi)=\frac{\sum_{(n,i,k)\in S}\Pi_{(n,i,k)}(\xi)\cdot c_{(n,i,k)}(\xi)}{\sum_{(n,i,k)\in S}\Pi_{(n,i,k)}(\xi)\cdot \tau_{(n,i,k)}(\xi)} \quad (9)$$

where $\Pi_{(n,i,k)}(\xi)$ denotes the balanced probability of state $(n,i,k)$ under policy $\xi$, $c_{(n,i,k)}(\xi)$ represents the expected cost of state $(n,i,k)$ until the next decision epoch under policy $\xi$, and $\tau_{(n,i,k)}(\xi)$ is the expected sojourn time of state $(n,i,k)$ until the next decision epoch under policy $\xi$. It is worth mentioning that $\Pi_{(n,i,k)}(\xi)$ are always determined by $p_{(n,i,k),(n',i',k')}(\xi)$, which is the probability of transiting from state $(n,i,k)\in S$ to $(n',i',k')\in S$ under policy $\xi$. These quantities are also referred to as SMDP quantities [39].

Among all possible policies, the optimal policy $\xi^*$ generates the lowest average cost. It means for any possible policy $\xi$,

$$g(\xi^*)\le g(\xi) \quad (10)$$

### 3.2. SMDP quantities

In what follows, we will derive the SMDP quantities of state $(n,i,k)\in S$ under policy $\xi$.

(1) If $a_{(n,i,k)}(\xi)=0$, i.e., the decision is not to take any maintenance, the system will move to state $(n+1,j,k)\in S$, with transition probability

$$p_{(n,i,k),(n+1,j,k)}(0)=P_{i,j}(n\Delta,(n+1)\Delta) \quad (11)$$

In this transition situation, no matter what the next system state is, the sojourn time is equal to the inspection interval Δ. That is,

$$\tau_{(n,i,k)}(0)=\Delta \quad (12)$$

At the next decision epoch, an inspection of cost $C_I$ is carried out. Moreover, the system may fail by the next inspection epoch, incurring a cost $C_D$ per unit time. In sum, the expected cost is

$$c_{(n,i,k)}(0)=C_I+C_D\int_0^\Delta P_{i,M}(n\Delta,t)\,\mathrm{d}t \approx C_I+C_D\delta\sum_{i=1}^{I}P_{i,M}(n\Delta,i\delta) \quad (13)$$

where $\delta=\Delta/I$ with $I$ a sufficiently large integer.

(2) If $a_{(n,i,k)}(\xi)=1$, i.e., the decision is to repair the system preventively, the system state will jump to $(n,0,k+1)\in S$. We note that state $(n,0,k+1)$ is just an intermediate state because no maintenance should be taken just after a repair. Taking the intermediate state as a start, the next state can be $(n+1,j,k+1)\in S$ with probability

$$p_{(n,i,k),(n+1,j,k+1)}(1)=P_{0,j}(n\Delta,(n+1)\Delta) \quad (14)$$

In this transition situation, the sojourn time is the sum of the expected repair time $T_R$ and the inspection interval Δ. That is,

$$\tau_{(n,i,k)}(1)=T_R+\Delta \quad (15)$$

Calculating the expected cost should consider three parts: The inspection cost $C_I$; The repair cost $C_R$; and the possible failure cost by the next inspection epoch. As a result,

$$c_{(n,i,k)}(1)=C_R+C_I+C_D\int_0^\Delta P_{0,M}(n\Delta,t)\,\mathrm{d}t \quad (16)$$

(3) If $a_{(n,i,k)}(\xi)=2$, i.e., replacing the system preventively is the decision, the system will immediately switch to the intermediate state (0,0,0), and then at the next decision epoch the system state will be $(1,j,0)\in S$ with probability

$$p_{(n,i,k),(1,j,0)}(2)=P_{0,j}(0,\Delta) \quad (17)$$

The calculation of the expected sojourn time is similar to that under action 1. That is,

$$\tau_{(n,i,k)}(2)=T_P+\Delta \quad (18)$$

Similar to the expected cost under action 1, the expected cost under action 2 is given by

$$c_{(n,i,k)}(2)=C_P+C_I+C_D\int_0^\Delta P_{0,M}(0,t)\,\mathrm{d}t \quad (19)$$

(4) If $a_{(n,i,k)}(\xi)=3$, i.e., corrective replacement is taken for the current state, the calculation of SMDP quantities is the same as that of $a_{(n,i,k)}(\xi)=2$. The transition probabilities are given by

$$p_{(n,i,k),(1,j,0)}(3)=P_{0,j}(0,\Delta) \quad (20)$$

The expected sojourn time is

$$\tau_{(n,i,k)}(3)=T_C+\Delta \quad (21)$$

The expected cost is

$$c_{(n,i,k)}(3)=C_C+C_I+C_D\int_0^\Delta P_{0,M}(0,t)\,\mathrm{d}t \quad (22)$$

So far, we have derived the SMDP quantities of the optimization problem. Solving the following set of equations generates balanced probabilities:

$$\begin{cases}\Pi_{(n,i,k)} = \sum_{(n',i',k')\in S} p_{(n',i',k'),(n,i,k)}\Pi_{(n',i',k')} \\ \sum_{(n,i,k)\in S} \Pi_{(n,i,k)} = 1\end{cases} \tag{23}$$

Using Eq. (9), the expected average cost for any policy can be calculated. Then the optimal policy can be obtained by enumerating the expected average costs for all possible policies. However, such an optimization procedure is time-consuming. In what follows, we will introduce an efficient optimization algorithm.

### 3.3. Optimization algorithm

Let $V(n,i,k)$ be the value function of state $(n,i,k)\in S$. Then under a policy $\xi$, the value functions of all states and the expected cost $g(\xi)$ can be obtained by solving a set of equations as follows:

$$\begin{cases}V(n,i,k) = c_{(n,i,k)} - g(\xi)\cdot\tau_{(n,i,k)} + \sum_{(n',i',k')\in S} p_{(n',i',k'),(n,i,k)}\cdot V(n',i',k') \\ V(n_0,i_0,k_0) = 0\end{cases} \tag{24}$$

where $(v_0,i_0,k_0)\in S$ is an arbitrarily selected state. Such an equation set can be solved conveniently by matrix. This method is usually integrated into an algorithm to address optimization problems with constant control limits; see for example [40,41].

An optimization algorithm developed by integrating Eq. (24) and golden section method for our problem is presented as follows:

**Step 1:** Initiate maximum repair number $K=0$;

**Step 2:** Find the optimal threshold $L^*(K)$ for a fixed $K$ by Eq. (24) and the golden section method [42]. Denote $\xi^*(K) = (K, L^*(K))$ be the optimal policy under fixed $K$.

**Step 3:** If $g(\xi^*(K)) > g(\xi^*(K-1))$, the algorithm stops with $L^* = L^*(K-1)$ and $K^* = K-1$; Otherwise, go to Step 2 with $K = K+1$.

## 4. Numerical applications

This section uses the proposed method for the maintenance decision-making of the coating system in steel structures. The coating of the system loses with time. The degradation level can be measured by the corrosion areas in the graph taken at regular intervals ( $\Delta = 1$ ). The degradation is modelled as a nonstationary gamma process. Typical maintenance actions include repainting (repair) and replacement. The difference between the two actions is whether to remove corrosion completely before repainting the entire surface. The degradation level after repainting is reduced to 0, but the degradation rate is still high because of the corrosion under the repainted coating. The parameters of the gamma process are $a_k = 0.25$ and $b_k = 2$ for $k = 0,1,\cdots$, and $\theta = 1$ [43]. The failure level is $D = 25$. Once the level is exceeded, corrective replacement is carried out. The other parameters are as follows: $T_R = 0.2$, $T_P = T_C = 0.5$; $C_I = 1$, $C_R = 2$, $C_P = 8$, $C_C = 10$. The parameters of the optimization algorithm are set as follows: $M = 10$ (i.e., $\varepsilon = D/M = 2.5$), $N = 50$.

### 4.1. Optimization results

The preventive policy is optimized by the proposed algorithm. Fig. 1 illustrates the expected average costs for various combinations of the maximum number of preventive repairs $K$ and the PM threshold $L$. As observed, for each value of $K$, the average cost initially decreases and then increases as $L$ increases from 3 to 10. When the threshold $L$ is very low, PM is frequently conducted, incurring excessive cost. If the threshold $L$ is very large, PM is only carried out when degradation approaches the failure threshold. Such a maintenance plan increases the failure probability, thereby also resulting in higher maintenance cost. Consequently, for each value of $K$, the optimal threshold $L^*(K)$ is neither too low nor to high. For any fixed threshold $L$ given in the figure, the average cost decreases first and then increases when $K$ varies from 0 to 4. It means that the effect of preventive repair is positive when $K$ is low, but it becomes negative for excessive repair cost when $K$ is large. The contour plot containing the isolines of average costs is also given in Fig. 1, where the minimal average cost is reached when $K=2$ and $L=8$. Therefore, the optimal policy is $(K^*, L^*) = (2{,}8)$, and the corresponding average cost is $g(K^*, L^*) = 1.70$. It is also interesting to notice that, as $L$ (or $K$) increases, the average cost tends to be less sensitive to the value of $K$ (or $L$). Thus, if the optimal policy cannot be implemented due to some practical reasons, the decision-maker can consider a policy with one variable a little larger than the optimal value. For example, the policy (3,8) can also be regarded as an alternative if the policy (2,8) cannot be implemented.

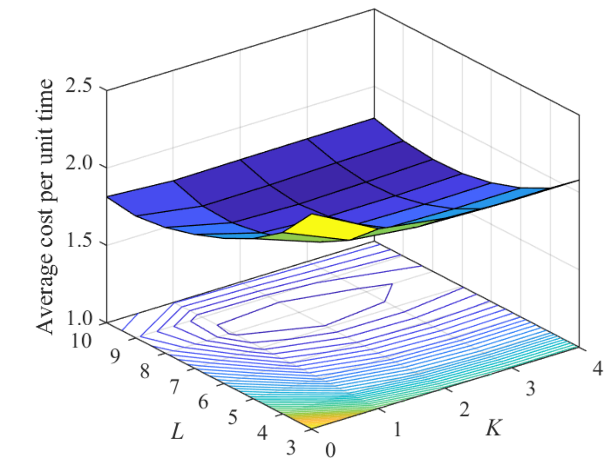


Fig. 1. Average costs for different combinations of $L$ and $K$.

Fig. 2 shows how to apply the obtained policy to a simulated degradation path from the coating system. The optimization results show that PM is conducted when the deterioration state is 8, 9, or 10. Thus, the PM threshold is the lower bound of state 8, which can be calculated by $(8-1)\varepsilon = 17.5$. The PM threshold is found to be exceeded at the 9th and the 13th inspection epochs. At these two inspection epochs, the failure threshold is not crossed, and thus preventive repair is conducted to reduce the degradation level to 0. When the threshold is hit at the third time, the maximum number of preventive repairs is reached. Thus, the lifecycle ends with a preventive replacement.

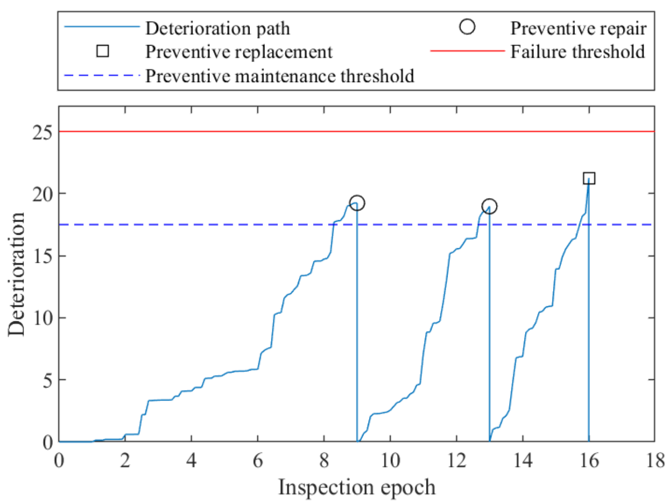


Fig. 2. The application of the obtained policy.

### 4.2. Sensitivity analysis

This section investigates the effects of key parameters on the optimality of our problem. First, we consider the scale parameter of the gamma process $\theta$. Table 1 shows the optimal policies under different values of $\theta$ and their average costs. According to the property of the gamma process, the expectation of $X_t$ is $\omega(k,t)/\theta$. Thus, increasing the value of $\theta$ indicates a slower degradation rate. As can be seen from Table 1, when $\theta$ varies from 4 to 8, the average cost decreases. Table 1 also shows that $L^*$ (or $K^*$) increases in $\theta$ if $K^*$ (or $L^*$) is fixed.

Table 1. Optimization results for scale parameter $\theta$.

| $\theta$ | $K^*$ | $L^*$ | Average cost |
|---|---|---|---|
| 4 | 2 | 8 | 1.33 |
| 5 | 2 | 8 | 1.30 |
| 6 | 1 | 9 | 1.27 |
| 7 | 2 | 9 | 1.25 |
| 8 | 3 | 9 | 1.21 |

It is also interesting to investigate the effect of the inspection interval $\Delta$. Table 2 shows the optimal policies for different values of $\Delta$. When $\Delta$ increases from 1.5 to 3.5, the average cost decreases first and then increases, reaching the minimum when $\Delta = 3.0$. Therefore, it provides an effective method to determine the optimal inspection interval.

Table 2. Optimization results for inspection interval $\Delta$.

| $\Delta$ | $K^*$ | $L^*$ | Average cost |
|---|---|---|---|
| 1.5 | 2 | 7 | 1.41 |
| 2.0 | 1 | 7 | 1.27 |
| 2.5 | 2 | 7 | 1.18 |
| 3.0 | 2 | 7 | 0.99 |
| 3.5 | 2 | 6 | 1.09 |

The cost of preventive repair has a great effect on the optimality of our problem. Table 3 lists the optimal policies for different values of $C_R$. When $C_R$ varies from 1 to 5, the cost effectiveness of preventive repair reduces, leading to obvious changes to the optimal policies. First, the maximum number of preventive repairs reduces from 5 to 0, which means no preventive repair will be carried out. Second, the PM threshold is increasing due to the low PM effect. Last, the expected average cost increases, which highlights the significance of preventive repair.

Table 3. Optimization results for preventive repair cost $C_R$.

| $C_R$ | $K^*$ | $L^*$ | Average cost |
|---|---|---|---|
| 1 | 5 | 7 | 1.54 |
| 2 | 2 | 8 | 1.70 |
| 3 | 1 | 9 | 1.78 |
| 4 | 0 | 9 | 1.79 |
| 5 | 0 | 9 | 1.79 |

The optimal policies for different values of preventive replacement cost $C_P$ are given in Table 4. If $C_P$ is very low, it becomes very cost effective to preventively replace the system when the degradation level is not high. The increase of $C_P$ increases the expected length of lifecycle, highlighting the significance of preventive repair.

Table 4. Optimization results for preventive replacement cost $C_P$.

| $C_P$ | $K^*$ | $L^*$ | Average cost |
|---|---|---|---|
| 2 | 0 | 7 | 1.18 |
| 4 | 0 | 8 | 1.39 |
| 6 | 1 | 8 | 1.58 |
| 8 | 2 | 8 | 1.70 |
| 10 | 5 | 9 | 1.75 |

## 5. Conclusions

This paper develops a predictive maintenance model for a nonstationary gamma process under periodic inspection. An action is selected among no maintenance, preventive repair, preventive replacement, and corrective replacement at each inspection (decision) epoch. To specify which action to select, we set two decision variables in the policy: the maximum number of preventive repairs and the PM threshold. The optimal decision variables are determined with the objective of minimizing the expected average cost in an infinite time horizon. We treat our optimization problem as an SMDP and develop an efficient optimization algorithm. The coating system is applied to validate the proposed approach and the effects of some key parameters are examined.

Some related topics can be investigated in the future. We note that our policy which follows the most widely used control limit form is suboptimal from an economic perspective. Thus, the first interesting extension is to investigate the optimal structural properties, which may be a control-limit policy. Second, in sensitivity analysis we provide a method to optimize the inspection interval. It is interesting to determine what action to select and when to perform the next inspection at each decision epoch. Lastly, we can consider another situation where corrective repair is performed when a failure occurs. Although this extension will not bring too much technical challenge, it will produce a practical policy that can be applied in asset management.

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