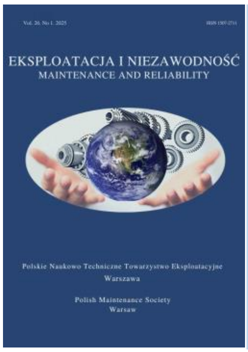

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Further Results on Relative Aging Orders and Comparison of Record Statistics

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Highlights

- Improve the study and understanding of relative aging ordering properties to compare lifetime distributions.
- Highlight the difference between relative aging orders and other well-known stochastic orders in literature.
- Compare items that are close to each other during aging.
- Investigate the preservation properties of two well-known faster aging orders under the upper and lower record values.

Abstract

In reliability engineering, relative aging is an important notion useful for measuring how a system ages relative to another. In recent years, the reliability properties of record statistics used for statistical modeling, such as shock models, have been investigated. This study presents new findings regarding aging faster orders. Several implications of relative aging orders are presented, including further inequalities arising from these stochastic orders. We apply two faster aging orders by comparing distributions using their cumulative hazard rate functions and cumulative reversed hazard rate functions in the upper and lower record values, respectively. In addition, we compare the record statistics in the two-sample problem. The extent to which the aging-induced faster orders are preserved in the record statistics resulting from sequences of independent and identically distributed random lifetimes is investigated. Finally, examples are provided to illustrate these concepts.

Keywords
upper records, lower records, aging faster, cumulative hazard function, cumulative reversed hazard function.

1. Introduction

In reliability engineering, it is useful to know which of the two units ages (or deteriorates) faster than the other (i.e., which unit tends to age quickly in comparison to the other unit). This may depend on the aging rate of the underlying units (see, e.g., Barlow and Proschan [1]). This can be investigated using various concepts related to relative aging. There are a number of other stochastic orderings that are useful for comparing random variables (rvs) from the point of view of their magnitude, such as likelihood ratio ordering or common stochastic ordering. The term "magnitude" was used in Shaked and Shanthikumar [2] to quantify the concept that a rv is more likely to be "larger" than another rv. For example, the well-known usual stochastic order between rvs X and Y means that Y is more likely to get values greater than x, for all $x \in \mathbb{R}$, compared to X. It is clear that this criterion compares the magnitude of the two rvs, where the rv Y tends to take larger values than the rv X from a probabilistic point of view. However, if one device ages faster than the other in terms of relative aging orders, this does not mean that the first device has a stochastically smaller random lifetime than the second device. Note that a random lifetime is a non-negative rv. This separates the role of stochastic orders of magnitude from that of the relative stochastic orders of aging. By setting up a suitable stochastic ordering between the random variables X and Y, which may denote the lifetime of the two devices, respectively, one can study the relative aging of the first device compared to

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the other device. Thus, two concepts of relative aging have been developed in the literature. For this reason, two notions of relative aging order (stochastic ordering) have been defined in the literature. Two classes of stochastic ordering have been proposed to compare the relative aging between lifetime units. The first class, known as transformation ordering, includes convex transformation ordering, star ordering and superadditive ordering. This class of stochastic ordering aims to capture the essence of one distribution being more skewed than the other, indicating that the latter distribution ages faster than the former. Since "skewness" is an aspect of the probability density curve, the above stochastic orders are useful to recognize the object that ages faster. The concept of aging, which describes the change in performance of a unit over time, is crucial for survival analysis and reliability theory. To this end, a variety of measures and measure-based stochastic orderings have been developed to analyze the aging characteristics of lifetime distributions.

1.1. Literature Review

The second strategy described earlier, was proposed by Kalashnikov and Rachev [3] and Sengupta and Deshpande [4], and it expresses relative aging by the increasing property of the hazard ratio. Such ordering of relative aging entails an ordering of variances if the distributions involved have the same mean (e.g., Lai and Xie [5]). Mantel and Stablein [6] applied the problem of crossing hazard functions into a clinical cancer data. Di Crescenzo [7] considered a model where the ratio of two reversed hazard rates is constant. However, Rezaei et al. [8] proposed a relative aging order on the basis of the increasing reversed hazard rate ratio. The second type of stochastic ordering has obvious advantages over the first because it can be used to simulate the occurrence of intersection measures and allows an intuitive interpretation of relative aging. Relative aging comparisons of units based on the cumulative hazard rate function and cumulative reversed hazard rate function have also been applied in the literature (see, for example, Misra and Francis [9]). The study of relative aging orders in the context of reliability analysis of coherent systems have been found to be useful (see, e.g., Hazra and Nanda [10], Hazra and Misra [11] and Misra et al. [12]). Relative aging orders have also been used to compare frailty models (see, Kayid et al. [13]). In statistics, a record value or record statistic is the largest or smallest value obtained from a sequence of random variables. The theory is closely related to that used in order statistics. The term was first introduced by Chandler [14]. For details on distribution theory and various applications of records, the reader is referred to Ahsanullah [15] and Arnold et al. [16]. Record values are also arisen in the context of non-homogenous poisson process as the epoch times of a nonhomogeneous Poisson process follow the same model of record values (see, e.g., Pellerey et al. [17]). In the context of reliability, record values have found many applications. For example, Belzunce et al. [18] used record values as a model for repair times of an item that is continuously minimally repaired. They also studied some stochastic comparisons of load-sharing systems via record values. Furthermore, they studied nonhomogeneous Poisson or pure birth processes from the perspective of some stochastic orders among record values. Wang et al. [19] proposed two mission reliability evaluation algorithms for k-out-of-n: G phased-mission systems with imperfect fault coverage based on record values. Record values have also been applied to estimate the stress-strength reliability (see, e.g., Hassan et al. [20]). Consider a system of components which experiences some kind of shocks such as voltage peaks. Stochastic comparisons of record values have been conducted in the literature by researchers in the past decades. Ahmadi and Arghami [21] obtained some preservation properties of stochastic orders under record values. Belzunce et al. [22] derived several results on (multivariate and univariate) stochastic comparisons of generalized order statistics which include some implications concerning comparison of record values (see, also, Alimohammadi et al. [23], Esna-Ashari et al. [24] and Esna-Ashari et al. [25]). Khaledi and Shojaei [26] established some stochastic ordering results among residual record values in two sample problems. Zhao et al. [27] investigated stochastic comparison of spacings of record values. Khaledi et al. [28] established several stochastic orderings among residual record values as well as inactive record values in two-sample problems. Zhao and Balakrishnan [29] studied the inactive record values and carry out a stochastic comparison of these quantities from two independent samples. Recently, Balakrishnan et al. [30] obtained some sufficient conditions for increasing concave order and location independent more riskier order of lower record values based on stochastic comparisons of minimum order statistics. Kayid [31] derived some results for comparisons of upper records and lower records using relative aging orders in the two-sample problem.

1.2. Mathematical background and plan of the paper

We assume that the sequence of independent and identically distributed (i.i.d.) rvs $\{X_i, i = 1,2, \dots\}$ following a common cumulative distribution function (cdf) F_X, probability density function (pdf) $f_X = F'_X$ and survival function (sf) $\bar{F}_X \equiv 1 - F_X$ with the amount of measurements (of voltage peaks) on shocks. The loads on the system at different points of time induce the shocks. The record statistics (values of the highest stresses reported) of this sequence are of interest to us. The rvs are the lifetime of devices in our context. Hence, we assume that the amounts of values of the sequence of rvs $\{X_i, i = 1,2, \dots\}$ which identifies the record values are non-negative. The record values are, therefore, non-negative. The i-th order statistic of the first n elements of $\{X_i, i = 1,2, \dots\}$, is signified by $X_{i:n}$.

Below, we define the upper record values (denoted by X_{U_n}) and upper record timings $\{T_n, n \geq 1\}$, respectively, as follows:

$$X_{U_n} = X_{T_n:T_n}, \qquad n = 0,1, \dots,$$

where

$$T_0 = 1, \qquad T_n = \min\{j: j > T_{n-1}, X_j > X_{U_{n-1}}\}, \qquad n \geq 1.$$

Let us denote by Λ_F the cumulative hazard function of the cdf F, given by $\Lambda_F(x) = -\ln(\bar{F}(x))$, for $x > 0$. It is commonly known that the pdf of X_{U_n}, represented by $f_{X_{U_n}}(x)$ given by

$$f_{X_{U_n}}(x) = \frac{[\Lambda_F(x)]^{n-1}}{(n-1)!} f_X(x); x \geq 0.$$

We denote by $\bar{F}_{X_{U_n}}(x)$ the sf of X_{U_n}, which is a function of the cumulative hazard function, and it is derived as

$$\bar{F}_{X_{U_n}}(x) = \int_{\Lambda_F(x)}^{+\infty} \frac{s^{n-1}}{(n-1)!} e^{-s} ds = \bar{F}_X(x) \sum_{k=0}^{n-1} \frac{[\Lambda_F(x)]^k}{k!}; \;\; x \geq 0,$$

where the last identity is derived by using the expansion of incomplete gamma function (see e.g. Arnold et al. [16]). In contrast to the upper record values are the lower record values. The nth lower record time $L(n), n = 0,1,2, \dots$ with $L(0) = 1$ is stated as

$$L(0) = 1, \qquad L(n) = \min\{j: j > L(n-1), X_{1:L(n-1)} > X_{1:j}\}, \qquad n = 1,2, \dots.$$

and the n-th lower record is enumerated as $X_{L_n} = X_{1:L(n)}, n = 1,2, \dots$. Let us denote the cumulative reversed hazard function by $\tilde{\Lambda}_F(x) = -\ln(F(x))$, for $x > 0$. The pdf of X_{L_n} can be acquired as

$$f_{X_{L_n}}(x) = \frac{[\tilde{\Lambda}_F(x)]^{n-1}}{(n-1)!} f_X(x); \;\; x \geq 0.$$

Further, the cdf of X_{L_n} is a function of cumulative reversed hazard function and is given by:

$$F_{X_{L_n}}(x) = \int_{\tilde{\Lambda}_F(x)}^{+\infty} \frac{s^{n-1}}{(n-1)!} e^{-s} ds = F_X(x) \sum_{k=0}^{n-1} \frac{[\tilde{\Lambda}_F(x)]^k}{k!}; \;\; x \geq 0$$

The aim of this paper is to provide some further insights into relative aging orders and their implications in the context of reliability and distribution theory. We then present some stochastic orderings of upper and lower data sets by relative aging according to the cumulative hazard rate function and the cumulative inverse hazard rate function.

In Section 2 we present some preliminary concepts of stochastic orderings and aging terms. In Section 3, we introduce some implications of relative aging orders in the context of distribution theory. In Section 4, we present a result for the conservation of relative aging order according to the cumulative risk function under upper data sets. Section 5 investigates the preservation of the relative order of lower data sets according to the cumulative inverse hazard rate function. In Section 6, we close the paper with further conclusions and also provide key points for future research in this area.

2. Preliminaries

In this section, we bring some preliminaries that will be used throughout the paper. The definitions of the ageing faster orders utilized in our paper are provided below (see, e.g., Sengupta and Deshpande [4], Rezaei et al. [8] and Misra et al. [12]).

Definition 1 *Let X and Y be two absolutely continuous random variables which represent the lifetimes of devices A and B, respectively. We assume that X and Y have cumulative distribution functions F_X and F_Y, survival functions $\bar{F}_X$ and $\bar{F}_Y$, hazard rate functions $h_X = f_X/\bar{F}_X$ and $h_Y = f_Y/\bar{F}_Y$, reversed hazard rate functions $r_X = f_X/F_X$ and $r_Y = f_Y/F_Y$, respectively. We shall say that the device A ages faster than the device B in:*

- Hazard rate function, written as $X \preceq_c Y$, if

$$\frac{h_X(t)}{h_Y(t)} \text{ is increasing in } t \geq 0.$$

- Reversed hazard rate function, written as $X \preceq_b Y$, if

$$\frac{r_X(t)}{r_Y(t)} \text{ is decreasing in } t > 0.$$

- Cumulative hazard rate function, written as $X \preceq_{c^\star} Y$, if

$$\frac{-\ln(\bar{F}_X(t))}{-\ln(\bar{F}_Y(t))} \text{ is increasing in } t \geq 0.$$

• Reversed hazard rate, denoted by $X \preceq_{b^\star} Y$, if

$$\frac{-\ln(F_X(t))}{-\ln(F_Y(t))} \text{ is decreasing in } t \geq 0.$$

It is known in literature that $X \preceq_c Y$ implies $X \preceq_{c^\star} Y$ and further $X \preceq_b Y$ implies $X \preceq_{b^\star} Y$ (see, e.g., Misra and Francis [9]).

We give the following useful definition of stochastic orders.

Definition 2 *Let us suppose that X and Y represent the lifetime of two devices which follows cdfs F_X and F_Y, respectively. Let $X_t := [X - t|X > t]$, for all $t \geq 0$ for which $F_X(t) < 1$ and $Y_t := [Y - t|Y > t]$ for all $t \geq 0$ for which $F_Y(t) < 1$ denote the residual lifetime of the devices after age t. Suppose that $X_{(t)} := [t - X|X \leq t]$, for all $t > 0$ such that $F_X(t) > 0$ and $Y_t := [t - Y|Y \leq t]$ for all $t > 0$ such that $F_Y(t) > 0$ denote the inactivity time of the devices after time t, respectively. It is said that X is smaller than Y in the*

• Likelihood ratio order, denoted by $X \preceq_{lr} Y$, if $\frac{f_Y(t)}{f_X(t)}$ is increasing in $t \geq 0$.

• Hazard rate order, denoted by $X \preceq_{hr} Y$, if

$$h_X(t) \geq h_Y(t), \qquad for\ all\ t \geq 0.$$

• Reversed hazard rate order, denoted by $X \preceq_{rh} Y$, if

$$r_X(t) \leq r_Y(t), \qquad for\ all\ t > 0.$$

• Usual stochastic order, denoted by $X \preceq_{st} Y$, if

$$\bar{F}_X(t) \leq \bar{F}_Y(t), \qquad for\ all\ t \geq 0.$$

• Probability order, denoted by $X \preceq_{pr} Y$, if

$$P(X > Y) \leq \frac{1}{2}.$$

• Residual probability order, denoted by $X \preceq_{rpr} Y$, if

$$P(X_t > Y_t) \leq \frac{1}{2}, \qquad for\ all\ t \geq 0,$$

or equivalently if, $\frac{\int_t^{+\infty} f_X(x)\bar{F}_Y(x)dx}{\int_t^{+\infty} f_Y(x)\bar{F}_X(x)dx} \geq 1$, for all $t \geq 0$.

• Inactivity probability order, denoted by $X \preceq_{ipr} Y$, if

$$P\left(X_{(t)} > Y_{(t)}\right) \geq \frac{1}{2}, \qquad for\ all\ t > 0,$$

or equivalently if, $\frac{\int_0^t F_X(x)f_Y(x)dx}{\int_0^t f_X(x)F_Y(x)dx} \geq 1$, for all $t > 0$.

It is well-known in literature that $X \preceq_{hr} Y$ implies $X \preceq_{st} Y$ and it further implies $X \preceq_{rpr} Y$. $X \preceq_{rh} Y$ implies $X \preceq_{st} Y$ and further it provides that $X \preceq_{ipr} Y$. We also know that $X \preceq_{rpr} Y$ implies $X \preceq_{pr} Y$ and in parallel $X \preceq_{ipr} Y$ gives $X \preceq_{pr} Y$. In Definition 2, the stochastic orders which consider magnitude of random variables rather than their relative aging behaviours are $\preceq_{lr}, \preceq_{hr}, \preceq_{rh}$ and $\preceq_{st}$ (cf. Shaked and Shanthikumar [2]). The stochastic orders which compare the random lifetimes relative to each other are $\preceq_{pr}, \preceq_{rpr}$ and $\preceq_{ipr}$ (see, e.g., Zardasht and Asadi [32]).

The following definition is regarding the aging property of a life unit.

Definition 3 *Let X be a lifetime random variable with hazard rate h_X and reversed hazard rate r_X. It is said that X has*

• Increasing [decreasing] failure rate (denoted as $X \in IFR$ [$X \in DFR$]), if $h_X(t)$ is increasing [decreasing] in $t \geq 0$.

• Increasing [decreasing] failure rate in average (denoted as $X \in IFRA$ [$X \in DFRA$]), if $\frac{\int_0^t h_X(x)dx}{t}$ is increasing [decreasing] in $t > 0$.

• Decreasing reversed hazard rate (denoted as $X \in DRHR$) if $r_X(t)$ is decreasing in $t > 0$.

It is well-known in literature that $X \in IFR$ [$X \in DFR$] implies $X \in IFRA$ [$X \in DFRA$] (see, e.g, Lai and Xie [33]). The following definition is due to Karlin [34] which be used frequently in the sequel.

Definition 4 *Let $f(x,y)$ be a non-negative function. It is said that f is Totally positive of order 2 (TP_2) in $(x,y) \in \mathcal{X} \times \mathcal{Y}$ where $\mathcal{X}$ and $\mathcal{Y}$ are two arbitrary subsets of $\mathbb{R} = (-\infty, +\infty)$ whenever*

$$\begin{vmatrix} f(x_1,y_1) & f(x_1,y_2) \\ f(x_2,y_1) & f(x_2,y_2) \end{vmatrix} \geq 0,\ for\ all\ x_1 \leq x_2 \in \mathcal{X}, and\ for\ all\ y_1 \leq y_2 \in \mathcal{Y}. \qquad (1)$$

If the direction of the inequality given after determinant in (1) is reversed then it is said that f is Reverse regular of order 2 (RR_2) in $(x,y) \in \mathcal{X} \times \mathcal{Y}$. It is notable that the function $f(x,y)$ is TP_2 (RR_2) in $(x,y) \in \mathcal{X} \times \mathcal{Y}$ if, and only if, $\frac{f(x_2,y)}{f(x_1,y)}$ is increasing (decreasing) in $y \in \mathcal{Y}$, for all $x_1 \leq x_2 \in \mathcal{X}$, where we use the conventions $\frac{0}{0} = 0$ and $\frac{a}{0} = +\infty$, for every $a > 0$.

3. Further insights on aging faster orders

In this section, we concentrate on the aging faster orders to explain their properties and the conclusions that can be obtained by using such orders. By knowing the implications of aging faster orders, further descriptions of these orders can be realized. In this section, using aging faster orders between two random lifetimes under some conditions for the ratio of their hazard rates or the ratio of their reversed hazard rates, which have a finite limit, some bounds on the stress strength reliability are provided. As a result, some connections are found between the probability order, which is a well-known stochastic order in the literature, and aging faster orders. Inequalities concerning the cumulative residual entropy and the cumulative past entropy of two random lifetimes satisfying an aging faster order are derived. In this direction, we show that if the underlying random lifetime has some aging properties, some bounds on its cumulative residual entropy and cumulative past entropy are acquired. The parameter $R:=P(Y>X)$ is very well-known as the stress-strength parameter in reliability. Brown and Rutemiller [35] in evaluation of $P(Y>X)$, when both X and Y are distributed as Weibull, have pointed out that to design as long-lived a product as possible one can consider the quantity $P(Y>X)$ and then choose X or Y when this probability is greater or less than $\frac{1}{2}$, respectively. The quantity $P(Y>X)$ gives the reliability of Y relative to X.

In the following we show that $X \preceq_c Y$ provides an upper bound and a lower bound for $P(Y>X)$ under some condition.

Proposition 5 *Let* $r_0 = lim_{t\to 0}\frac{h_X(t)}{h_Y(t)}$ *and also let* $r_1 = lim_{t\to +\infty}\frac{h_X(t)}{h_Y(t)}$. *Then,*

- If $r_0 > 0$, then $X \preceq_c Y$ implies $P(Y \geq X) \geq \frac{r_0}{r_0+1}$.
- If $r_1 < +\infty$, then $X \preceq_c Y$ implies $P(Y \geq X) \leq \frac{r_1}{r_1+1}$.

Further, if $r_0 \geq 1$, then $X \preceq_c Y$ implies $X \preceq_{pr} Y$, and also if $r_1 \leq 1$, then $X \preceq_c Y$ gives $X \succeq_{pr} Y$.

Proof. Since $X \preceq_c Y$, thus $\frac{h_X(t)}{h_Y(t)} \leq r_1 < +\infty$, for all $t \geq 0$. Hence, $f_X(t)\bar{F}_Y(t) \leq r_1 f_Y(t)\bar{F}_X(t)$, for all $t \geq 0$. Since $X \preceq_c Y$ involves only the marginal distributions of X and Y, thus, we can take without loss of generality X and Y as two independent random variables. We have

$$P(Y \geq X) = \int_0^{+\infty} f_X(t)\bar{F}_Y(t)dt \leq r_1 \int_0^{+\infty} f_Y(t)\bar{F}_X(t)dt$$
$$= r_1 P(Y < X) = r_1 - r_1 P(Y \geq X),$$

which further implies that $P(Y \geq X) \leq \frac{r_1}{r_1+1}$. On the other hand, since $X \preceq_c Y$, thus $\frac{h_X(t)}{h_Y(t)} \geq r_0 > 0$, for all $t \geq 0$. Therefore, $f_X(t)\bar{F}_Y(t) \geq r_0 f_Y(t)\bar{F}_X(t)$, for all $t \geq 0$. We have

$$P(Y \geq X) = \int_0^{+\infty} f_X(t)\bar{F}_Y(t)dt \geq r_0 \int_0^{+\infty} f_Y(t)\bar{F}_X(t)dt$$
$$= r_0 P(Y < X) = r_0 - r_0 P(Y \geq X),$$

from which one can get $P(Y \geq X) \geq \frac{r_0}{r_0+1}$. It is now straightforward that if $r_0 \geq 1$, then $\frac{r_0}{r_0+1} \geq \frac{1}{2}$, i.e., $X \preceq_{pr} Y$, and analogously if $r_1 \leq 1$, then $\frac{r_1}{r_1+1} \leq \frac{1}{2}$, i.e., $X \succeq_{pr} Y$. ||

If a system has an age t, it is important to take into account this age, when we compare the remaining lifetimes of the system. In this case X_t and Y_t denote the additional residual lifetime of X and Y given that the systems, have survived up to t. The residual probability (rpr) function is defined as

$$R(t) = P(X_t > Y_t), \qquad t > 0.$$

The study of the properties of rpr function might be important for engineers and system designers to compare the lifetime of the products and, hence to design better products (see for instances, Tan and Lü [36] for some biological background and Lü and Chen [37], Chen et al. [38], and Zhou et al. [39] for some real world applications). The rpr function uniquely determines the distribution function of F (and hence the distribution function of G), under the condition that the ratio of the hazard rates of X and Y is known. In addition, when the ratio of the hazard rates of X and Y is a monotone function of time i.e., when $X \preceq_c Y$ or $X \succeq_c Y$, then the rpr function is also a monotone function of time. Hence, it is revealed that the rpr function as a relative measure of aging is closely related to the concept of aging faster order in the sense of hazard rate function. The rpr function was studied by Zardasht and Asadi [32] to establish several properties of stochastic comparisons based on the rpr function under the reliability operations of mixture and random minima. Further investigation on the rpr concept in the context of stochastic orders and aging notions has been made by Kayid et al. [40].

The following proposition is readily proved. The proof being straightforward is omitted.

Proposition 6 *Let* $r_0 = lim_{t\to 0}\frac{h_X(t)}{h_Y(t)} > 0$ *and also assume that* $r_1 = lim_{t\to +\infty}\frac{h_X(t)}{h_Y(t)} < +\infty$. *Then,*

- If $r_0 \geq 1$, then $X \preceq_c Y$ implies $X \preceq_{rpr} Y$.
- If $r_1 \leq 1$, then $X \preceq_c Y$ implies $X \succeq_{rpr} Y$.

In Proposition 5 and Proposition 6, one may ask whether two conditions $\mathrm{r}_0 > 0$ and $\mathrm{r}_1 < +\infty$ are satisfied. These conditions have been applied in Kayid [31] to develop aging faster orders of upper records and lower records. In the next remark, we illustrate sufficient conditions for $\mathrm{r}_0 > 0$ and $\mathrm{r}_1 < +\infty$. We also illustrate that the orders $\preceq_c$ and $\preceq_{c^*}$ provide some connections between the orders $\preceq_{hr}, \preceq_{\mathrm{st}}$ and $\preceq_{lr}$.

Remark 7

(a) *In general, if* $\frac{h_X(t)}{h_Y(t)}$ *is bounded such that there exist* $k_1 < k_2$, *where* $0 < k_i < +\infty, i = 1,2$ *for which* $k_1 \leq \frac{h_X(t)}{h_Y(t)} \leq k_2$, *for all* $t \geq 0$, *or equivalently if* $k_1 h_Y(t) \leq h_X(t) \leq k_2 h_Y(t)$ *then* $r_0 > 0$ *and* $r_1 < +\infty$. *More specifically, if* $k_1 h_Y(t) \leq h_X(t)$, *for all* $t \geq 0$, *then* $r_0 > 0$ *and if* $h_X(t) \leq k_2 h_Y(t)$, *for all* $t \geq 0$, *then* $r_1 < +\infty$. *If* $k_1 = 1$, *then the former inequality means that* $X \preceq_{hr} Y$ *and if* $k_2 = 1$, *then the latter inequality means that* $X \succeq_{hr} Y$.

(b) *It is notable that* $X \preceq_{hr} Y$ *implies* $X \preceq_{st} Y$ *(see, Theorem 1.B.1 in Shaked and Shanthikumar [2]), however, the converse is not true in general. It is remarkable here that if* $X \preceq_{c^*} Y$, *then* $X \preceq_{st} Y$ *implies* $X \preceq_{hr} Y$. *This is because* $X \preceq_{c^*} Y$ *holds if, and only if,*

$$\frac{h_X(t)}{h_Y(t)} \geq \frac{-\ln(\bar{F}_X(t))}{-\ln(\bar{F}_Y(t))}, for\ all\ t \geq 0. \quad (2)$$

Thus, if $X \preceq_{st} Y$, *then* $\frac{-\ln(\bar{F}_X(t))}{-\ln(\bar{F}_Y(t))} \geq 1$, *for all* $t \geq 0$, *and consequently,* $\frac{h_X(t)}{h_Y(t)} \geq 1$, *for all* $t \geq 0$, *which means* $X \preceq_{hr} Y$.

(c) *Since the order* $\preceq_c$ *implies the order* $\preceq_{c^*}$, *thus if* $X \preceq_c Y$ *and* $X \preceq_{st} Y$, *then* $X \preceq_{hr} Y$. *According to Theorem 1.C.4(a) of Shaked and Shanthikumar [2], if* $X \preceq_{hr} Y$ *and* $X \succeq_c Y$, *then* $X \preceq_{lr} Y$. *This acknowledges that the aging faster orders have an essential role in making connections between well-known magnitude orders.*

The following proposition strengthens the result of Proposition 5(i) as $X \preceq_c Y$ is a stronger condition than $X \preceq_{c^*} Y$.

Proposition 8 *Let* $r_0 = lim_{t\to 0}\frac{h_X(t)}{h_Y(t)} > 0$. *If* $r_0 \geq 1$, *then* $X \preceq_{c^*} Y$ *implies* $X \preceq_{rpr} Y$.

Proof. Suppose that $X \preceq_{c^*} Y$. Now, for all $t \geq 0$, one has the following

$$\frac{f_X(t)\bar{F}_Y(t)}{f_Y(t)\bar{F}_X(t)} = \frac{h_X(t)}{h_Y(t)} \geq \frac{-\ln(\bar{F}_X(t))}{-\ln(\bar{F}_Y(t))} \geq \lim_{t\to 0^+}\frac{-\ln(\bar{F}_X(t))}{-\ln(\bar{F}_Y(t))}$$
$$= \lim_{t\to 0^+}\frac{h_X(t)}{h_Y(t)} = r_0,$$

where the first inequality is due to (2), the second inequality holds because $\frac{-\ln(\bar{F}_X(t))}{-\ln(\bar{F}_Y(t))}$ is increasing in $t \geq 0$ and the last inequality follows by using the L'Hôpital's rule. Now, since for all $t \geq 0$, $f_X(t)\bar{F}_Y(t) \geq r_0 . f_Y(t)\bar{F}_X(t)$, thus $\frac{\int_0^x f_X(t)\bar{F}_Y(t)dt}{\int_0^x f_Y(t)\bar{F}_X(t)dt} \geq r_0$. Hence, if $r_0 \geq 1$, then from definition $X \preceq_{rpr} Y$. ||

It is remarkable here that in the context of Proposition 5, we could find the lower bound $\frac{r_0}{r_0+1}$ under a weaker condition. Specifically, if $X \preceq_{c^*} Y$ and $r_0 > 0$, then $P(Y > X) \geq \frac{r_0}{r_0+1}$. It is notable here that in Proposition 5, Proposition 6 and Proposition 8, if $r_0 = 0$ and $r_1 = +\infty$ then all the derived inequalities and achieved bounds become obvious.

The following proposition presents a lower bound and an upper bound for the sf of X based on the sf of Y.

Proposition 9 The following assertions are satisfied:

- Let $r_0 = \lim_{t\to 0}\frac{h_X(t)}{h_Y(t)} > 0$. Then, $X \preceq_{c^*} Y$ implies $\bar{F}_X(t) \leq \bar{F}_Y^{r_0}(t)$, for all $t \geq 0$.
- Let $r_1 = \lim_{t\to +\infty}\frac{h_X(t)}{h_Y(t)} < +\infty$. Then, $X \preceq_{c^*} Y$ implies $\bar{F}_X(t) \geq \bar{F}_Y^{r_1}(t)$, for all $t \geq 0$.

Proof. By definition, $X \preceq_{c^*} Y$ provides that $\frac{-\ln(\bar{F}_X(t))}{-\ln(\bar{F}_Y(t))}$ is increasing in $t \geq 0$, thus

$$\frac{-\ln(\bar{F}_X(t))}{-\ln(\bar{F}_Y(t))} \geq \lim_{t\to 0}\frac{-\ln(\bar{F}_X(t))}{-\ln(\bar{F}_Y(t))} = r_0,$$

from which one gets $-\ln(\bar{F}_X(t)) \geq r_0 .(-\ln(\bar{F}_Y(t)))$, for all $t \geq 0$, or equivalently, $\bar{F}_X(t) \leq \bar{F}_Y^{r_0}(t)$, for all $t \geq 0$. In a similar manner, since $X \preceq_{c^*} Y$, thus

$$\frac{-\ln(\bar{F}_X(t))}{-\ln(\bar{F}_Y(t))} \le \lim_{t\to+\infty} \frac{-\ln(\bar{F}_X(t))}{-\ln(\bar{F}_Y(t))} = r_1,$$

leading to $-\ln(\bar{F}_X(t)) \le r_1.(-\ln(\bar{F}_Y(t)))$, for all $t \ge 0$, or equivalently, $\bar{F}_X(t) \ge \bar{F}_Y^{r_1}(t)$, for all $t \ge 0$. ||

In the context of Proposition 9, let Y^* follow the sf $\bar{F}_Y^{r_0}(t)$ and also let $\tilde{Y}$ follow the sf $\bar{F}_Y^{r_1}(t)$. Then, this proposition gives some limiting conditions on the ratio of the hazard rate functions of X and Y under which $X \preceq_{c^*} Y$ yields $X \preceq_{st} Y^*$ and $\tilde{Y} \preceq_{st} X$. Note that the rvs Y^* and $\tilde{Y}$ follow the proportional hazard rates (PHR) model (see, e.g., Cox [41] and Kumar and Klefsjö [42]). Indeed, we established that if $X \preceq_{c^*} Y$ and $r_0 > 0$, then the random lifetime X is dominated, in the sense of the usual stochastic order, by a member of distributions which have a hazard rate being proportional to the hazard rate of Y. Further, we demonstrated that if $X \preceq_{c^*} Y$ and $r_1 < +\infty$, then the random lifetime X dominates, in terms of the usual stochastic order, a member of distributions which have a hazard rate being proportional to the hazard rate of Y. It is notable here that if $X \preceq_{c^*} Y$, then $r_0 \le r_1$. Let us denote by $Exp(\lambda)$, a non-negative random variable with exponential distribution having mean $1/\lambda$. Note then that X is IFR if, and only if, $X \preceq_c Exp(\lambda)$. Furthermore, X is $IFRA$ if, and only if, $X \preceq_{c^*} Exp(\lambda)$. The exponential distribution is a standard life distribution which confirms that there is no wear and tear, and as a result, a component with exponential lifetime never ages. Comparison between the exponential distribution and other life distributions may, therefore, develop a method to assess the degree of aging of component.

The following corollary presents a lower bound and an upper bound for the probability $P(X < Exp(\lambda))$. The proof of part (i) of this corollary obtains from Proposition 8 and the proof of part (ii) of this corollary is obtained by applying Proposition 5 into the special case where Y is an exponential random variable with mean $1/\lambda$.

Corollary 10 *Let X and $Exp(\lambda)$ be two independent random variables. Then:*

- If $h_X(0) > 0$, and X is $IFRA$, then $P(X < Exp(\lambda)) \ge \frac{h_X(0)}{h_X(0)+\lambda}$.
- If $h_X(\infty) < +\infty$ and X is IFR, then $P(X < Exp(\lambda)) \le \frac{h_X(\infty)}{h_X(\infty)+\lambda}$.

It is a well-known principle that between any two rational numbers, there is an irrational number. Thus, one can always find an irrational number such as r_0 or r_1 between two rational numbers. Therefore, it is commonly acceptable that for each r_0, there exist n_i and m_i, for $i = 1,2$, such that $\frac{m_1}{n_1} \le r_j \le \frac{m_2}{n_2}$, where $j = 0,1$. In view of this point, the following corollary presents a conclusion of Proposition 9.

Corollary 11 *Let $X_1, X_2, \dots, X_n$ be a random sample from F_X and also let $Y_1, Y_2, \dots, Y_m$ be another random sample from F_Y. Let $X_{1:n} = min\{X_1, X_2, \dots, X_n\}$ and $Y_{1:m} = min\{Y_1, Y_2, \dots, Y_m\}$.*

- Let $X \preceq_{c^*} Y$ such that $r_0 = \lim_{t\to 0} \frac{h_X(t)}{h_Y(t)} \ge 1$. Then, there exist $m \ge n \in \mathbb{N}$, such that $X_{1:n} \preceq_{st} Y_{1:m}$.
- Let $X \preceq_{c^*} Y$ such that $1 \le r_1 = \lim_{t\to+\infty} \frac{h_X(t)}{h_Y(t)} < +\infty$. Then, there exist $m' \ge n' \in \mathbb{N}$, such that $X_{1:n'} \succeq_{st} Y_{1:m'}$.

In the context of information theory, Rao et al. [43] proposed the cumulative residual entropy (CRE) as

$$\mathcal{E}(X) = -\int_0^{+\infty} \bar{F}(x)\ln(\bar{F}(x))dx.$$

Properties of the CRE and its dynamic version and some other generalization of this measure together with their properties are discussed in detail in Asadi and Zohrevand [44], Navarro et al. [45], Psarrakos and Navarro [46], Psarrakos and Toomaj [47], Tahmasebi and Mohammadi [48], Mohamed et al. [49], among others.

Next, we establish a result which provides a lower bound and an upper bound for the CRE of X.

Proposition 12 *The following assertions are satisfied:*

- Let $r_0 = \lim_{t\to 0} \frac{h_X(t)}{h_Y(t)}$. Let $r_0 > 0$ and $X \succeq_{st} Y$. Then, $X \preceq_{c^*} Y$ implies that $\mathcal{E}(X) \ge r_0 \mathcal{E}(Y)$.
- Let $r_1 = \lim_{t\to+\infty} \frac{h_X(t)}{h_Y(t)}$. Let $r_1 < +\infty$ and $X \preceq_{st} Y$. Then, $X \preceq_{c^*} Y$ implies that $\mathcal{E}(X) \le r_1 \mathcal{E}(Y)$.

Proof. Part (i): Since $X \succeq_{st} Y$, thus $\frac{\bar{F}_X(t)}{\bar{F}_Y(t)} \ge 1$, for all $t \ge 0$. Thus, for all $t \ge 0$, one has the following

$$\frac{-\bar{F}_X(t)\ln(\bar{F}_X(t))}{-\bar{F}_Y(t)\ln(\bar{F}_Y(t))} \ge \frac{-\ln(\bar{F}_X(t))}{-\ln(\bar{F}_Y(t))} \ge \lim_{t\to 0} \frac{-\ln(\bar{F}_X(t))}{-\ln(\bar{F}_Y(t))} = r_0,$$

where the second inequality is due to the fact that $\frac{-\ln(\bar{F}_X(t))}{-\ln(\bar{F}_Y(t))}$

is increasing in $t \geq 0$, since $X \preceq_{c^*} Y$. Therefore, $-\bar{F}_X(t)\ln(\bar{F}_X(t)) \geq -r_0\bar{F}_Y(t)\ln(\bar{F}_Y(t))$, for all $t \geq 0$. Hence,

$$\mathcal{E}(X) = -\int_0^{+\infty} \bar{F}_X(t)\ln(\bar{F}_X(t))dt$$

$$\geq -r_0\int_0^{+\infty} \bar{F}_Y(t)\ln(\bar{F}_Y(t))dt = r_0\mathcal{E}(Y).$$

Thus the proof of Part (i) is completed. The proof of Part (ii) can be similarly obtained. ||

The following corollary presents an application of Proposition 12.

Corollary 13 *Let* $c_0 = lim_{t\to 0}h_X(t)$ *and let* $c_1 = lim_{t\to+\infty}h_X(t)$. *Then,*

• If X is $IFRA$, if there exists a $\lambda > 0$ such that $c_1 \leq \lambda$ and also if $c_0 > 0$, then, $\mathcal{E}(X) \geq \frac{c_0}{\lambda^2}$.

• If X is $DFRA$, and there exists a $\lambda > 0$ such that $c_0 \leq \lambda$ where $c_0 > 0$, then, $\mathcal{E}(X) \geq \frac{c_0}{\lambda^2}$.

• If X is $IFRA$, and if there exists a $\lambda > 0$ such that $c_0 \geq \lambda$ and $c_1 < +\infty$, then, $\mathcal{E}(X) \leq \frac{c_1}{\lambda^2}$.

• If X is $DFRA$ and there exists a $\lambda > 0$ such that $c_1 \geq \lambda$ where $c_1 < +\infty$, then, $\mathcal{E}(X) \leq \frac{c_1}{\lambda^2}$.

Proof. We only prove Part (i). The other parts can be proved analogously. Suppose that Y has an exponential distribution with mean $1/\lambda$. We shall write $Exp(\lambda)$ in place of Y. Then, since X is IFRA, thus $X \preceq_{c^*} Exp(\lambda)$. On the other hand, since X is IFRA, thus $-\frac{1}{t}\ln(\bar{F}(t))$ is increasing in $t > 0$. Thus, since from assumption $r_1 \leq \lambda$, thus

$$-\frac{1}{t}\ln(\bar{F}(t)) \leq \lim_{t\to+\infty} -\frac{1}{t}\ln(\bar{F}(t)) = \lim_{t\to+\infty} h_X(t) = c_1$$

$$\leq \lambda, for\ all\ t > 0,$$

which holds if, and only if, $\bar{F}_X(t) \geq \exp(-\lambda t)$, for all $t \geq 0$. This is equivalent to saying that $X \geq_{st} Exp(\lambda)$. Now, $r_0 = \frac{c_0}{\lambda}$ which is positive by assumption. Now, by using Proposition 12(i) we conclude that $\mathcal{E}(X) \geq \frac{c_0}{\lambda}\mathcal{E}(Y)$. By some routine calculation, we get $\mathcal{E}(Y) = \mathcal{E}(Exp(\lambda)) = \frac{1}{\lambda}$. Hence, $\mathcal{E}(X) \geq \frac{c_0}{\lambda^2}$ and the required result of Part (i) is obtained.

Now, we show that $X \preceq_b Y$ gives an upper bound and a lower bound for $P(Y > X)$ under some limiting conditions. The proof is similar to the one of Proposition 5 and hence we omit it.

Proposition 14 *Set* $\tilde{r}_0 = lim_{t\to 0^+}\frac{r_X(t)}{r_Y(t)}$ *and set* $\tilde{r}_1 = lim_{t\to+\infty}\frac{r_X(t)}{r_Y(t)}$. *Then,*

• If $\tilde{r}_0 < +\infty$, then $X \preceq_b Y$ yields $P(Y \geq X) \geq \frac{1}{\tilde{r}_0+1}$.

• If $\tilde{r}_1 > 0$, then $X \preceq_b Y$ implies $P(Y \geq X) \leq \frac{1}{\tilde{r}_1+1}$.

In addition, if $\tilde{r}_0 \leq 1$, then $X \preceq_b Y$ implies $X \preceq_{pr} Y$, and if $\tilde{r}_1 \geq 1$, then $X \preceq_b Y$ gives $X \succeq_{pr} Y$.

In the context of Proposition 14, we give some illustrations of the conditions $\tilde{r}_0 < +\infty$ and $\tilde{r}_1 > 0$ in the following remark. The orders $\preceq_b$ and $\preceq_{b^*}$ make some links between the orders $\preceq_{rh}, \preceq_{st}$ and $\preceq_{lr}$ as we illustrate in this remark.

Remark 15

(a) *If* $\frac{r_X(t)}{r_Y(t)}$ *is bounded such that there exist* $k'_1 < k'_2$, *where* $0 < k'_i < +\infty, i = 1,2$ *for which* $k'_1 \leq \frac{r_X(t)}{r_Y(t)} \leq k'_2$, *for all* $t > 0$, *or equivalently if* $k'_1 r_Y(t) \leq r_X(t) \leq k'_2 r_Y(t)$ *then* $\tilde{r}_0 < +\infty$ *and* $\tilde{r}_1 > 0$. *To be more specific, if* $k'_1 r_Y(t) \leq r_X(t)$, *for all* $t > 0$, *then* $\tilde{r}_1 > 0$ *and if* $r_X(t) \leq k'_2 r_Y(t)$, *for all* $t > 0$, *then* $\tilde{r}_0 < +\infty$. *If* $k'_1 = 1$ *and* $k'_2 = 1$ *then the foregoing inequalities reduce to* $X \succeq_{rh} Y$ *and* $X \preceq_{rh} Y$.

(b) *It is well-known that* $X \preceq_{rh} Y$ *implies* $X \preceq_{st} Y$ *(see, Theorem 1.B.42 of Shaked and Shanthikumar [2]). The reversed implication is not correct anyway. We remark here that if* $X \preceq_{b^*} Y$, *then* $X \succeq_{st} Y$ *implies* $X \succeq_{rh} Y$. *Note that* $X \preceq_{b^*} Y$ *if, and only if,*

$$\frac{r_X(t)}{r_Y(t)} \geq \frac{-\ln(F_X(t))}{-\ln(F_Y(t))}, for\ all\ t > 0.$$

Hence, provided that $X \succeq_{st} Y$ *holds true, then* $\frac{-ln(F_X(t))}{-ln(F_Y(t))} \geq 1$, *for all* $t > 0$, *and as a result,*

$$\frac{r_X(t)}{r_Y(t)} \geq 1, \text{ for all } t > 0, \text{ i.e., } X \succeq_{rh} Y.$$

(c) It is to be mentioned that the order $\preceq_b$ *implies the order* $\preceq_{b^*}$, *thus if* $X \preceq_b Y$ *and* $X \succeq_{st} Y$, *then* $X \succeq_{rh} Y$. *According to Theorem 1.C.4(b) of Shaked and*

Shanthikumar [2], if $X \preceq_{rh} Y$ and $X \preceq_b Y$, then $X \preceq_{lr} Y$.

If it has become apparent that a system has failed before the time t, it is important to take into account the accurate time of its failure. To achieve this goal, the inactivity time of the system at the time t at which the failure of the system has been recorded after delay, is an important random variable. In such a situation $X_{(t)}$ and $Y_{(t)}$ are assumed to be the inactivity times of the system with lifetime X and the system with life length Y provided that the systems, have failed prior to the time t. The inactivity probability (*ip*) function is given by (cf. Abouelmagd et al. [50])

$$R^*(t) = P(X_{(t)} > Y_{(t)}), \qquad t > 0.$$

Let X and Y denote the lifetime of devices A and B, respectively. The *ip* function then measures the probability for device A to be failed before the device B, provided that both devices fail before time point t. The following proposition can be proved in a similar manner with Proposition 6 and Proposition 8.

Proposition 16 *The following assertions hold true:*

- Let $\tilde{r}_0 = \lim_{t\to 0^+} \frac{r_X(t)}{r_Y(t)} \leq 1$ and $X \preceq_b Y$. Then, $X \preceq_{ipr} Y$.
- Let $\tilde{r}_1 = \lim_{t\to +\infty} \frac{r_X(t)}{r_Y(t)} \geq 1$ and $X \preceq_{b^\star} Y$. Then, $X \succeq_{ipr} Y$.

The following result is similar to the result of Proposition 9 presents a lower bound and an upper bound for the sf of X based on the sf of Y.

Proposition 17 *The following implications hold:*

- Let $\tilde{r}_0 = \lim_{t\to 0^+} \frac{r_X(t)}{r_Y(t)} < +\infty$. Then, $X \preceq_{b^*} Y$ provides that $F_X(t) \geq F_Y^{\tilde{r}_0}(t)$, for all $t \geq 0$.
- Let $\tilde{r}_1 = \lim_{t\to +\infty} \frac{r_X(t)}{r_Y(t)} > 0$. Then, $X \preceq_{b^*} Y$ provides that $F_X(t) \leq F_Y^{\tilde{r}_1}(t)$, for all $t \geq 0$.

In Proposition 17, suppose that Y^{**} follows the cdf $F_Y^{\tilde{r}_0}(t)$ and suppose that $\check{Y}$ follows the cdf $F_Y^{\tilde{r}_1}(t)$. Then, this proposition presents some limiting conditions on the ratio of the reversed hazard rate functions of X and Y under which $X \preceq_{b^*} Y$ implies $X \preceq_{st} Y^{**}$ and $\check{Y} \preceq_{st} X$. The rvs Y^{**} and $\check{Y}$ have distributions with proportional reversed hazard rates (PRHR) model (see, for example, Di Crescenzo [7] and Gupta and Gupta [51]). In Proposition 17, it was shown that if $X \preceq_{b^*} Y$ and $\tilde{r}_0 < +\infty$, then X is dominated, in the sense of the usual stochastic order, by a member of distributions which have a reversed hazard rate being proportional to the reversed hazard rate of Y. We also proved that if $X \preceq_{b^*} Y$ and $\tilde{r}_1 > 0$ then X dominates, in the usual stochastic order, a member of distributions which have a reversed hazard rate being proportional to the reversed hazard rate of Y. Note that these conclusions together with conclusions given after Proposition 9 could represent useful applications of these results in the context of the relevant PHR and PRHR models.

In information theory, Di Crescenzo and Longobardi [52] proposed the cumulative past entropy (CPE) as

$$\mathcal{E}^\star(X) = -\int_0^{+\infty} F(x)\ln(F(x))dx.$$

In the following, we find a lower bound and an upper bound for the CPE of X. Its proof is similar to the one of Proposition 12 and hence we omit it.

Proposition 18 *We have the following assertions:*

- Let $\tilde{r}_0 = \lim_{t\to 0^+} \frac{r_X(t)}{r_Y(t)}$. Let $\tilde{r}_0 < +\infty$ and $X \succeq_{st} Y$. Then, $X \preceq_{b^*} Y$ yields $\mathcal{E}^\star(X) \leq \tilde{r}_0\mathcal{E}^\star(Y)$.
- Let $\tilde{r}_1 = \lim_{t\to +\infty} \frac{r_X(t)}{r_Y(t)}$. Let $\tilde{r}_1 > 0$ and $X \preceq_{st} Y$. Then, $X \preceq_{b^*} Y$ yields $\mathcal{E}^\star(X) \geq \tilde{r}_1\mathcal{E}^\star(Y)$.

The next corollary presents a conclusion of Proposition 17.

Corollary 19 *Consider $X_1, X_2, \dots, X_n$ as a random sample from F_X and $Y_1, Y_2, \dots, Y_m$ as a random sample from F_Y. Denote $X_{n:n} = max\{X_1, X_2, \dots, X_n\}$ and $Y_{m:m} = max\{Y_1, Y_2, \dots, Y_m\}$.*

- Let $X \preceq_{b^\star} Y$ such that $\tilde{r}_0 = \lim_{t\to 0} \frac{r_X(t)}{r_Y(t)} \leq 1$. Then, there exist $m \geq n \in \mathbb{N}$, for which $X_{n:n} \preceq_{st} Y_{m:m}$.
- Let $X \preceq_{b^\star} Y$ such that $\tilde{r}_1 = \lim_{t\to +\infty} \frac{r_X(t)}{r_Y(t)} \geq 1$. Then, there exist $m' \geq n' \in \mathbb{N}$, for which $X_{n':n'} \succeq_{st} Y_{m':m'}$.

4. Aging faster orders of upper records

There are two results regarding the preservation of the order "$\preceq_c$" under upper records. In this section, we first restate those results from Kayid [31].

Proposition 20 *(Kayid [31]).*

Let $X \succeq_{st} Y$ and also let $m \geq n \in \mathbb{N}$. Then, $X \preceq_c Y$ implies $X_{U_m} \preceq_c Y_{U_n}$.

The result of Proposition 20 is applicable when the underlying distributions from which the upper records are adopted, are ordered according to the usual stochastic order in the opposite direction of the relative aging ordering among the underlying distributions. It is shown by Proposition 20 that since "$m \geq n$" thus the initial upper records are less frailer than the final upper records in terms of the relative hazard rate order.

Let $\psi_n(u) = \frac{u\xi'_n(u)}{\xi_n(u)}$ where

$$\xi_n(u) = (n-1)! \sum_{i=0}^{n-1} \frac{(-\ln(u))^{i-n+1}}{i!}, \text{for } u \in (0,1) \qquad (3)$$

The following result, however, relaxes the two conditions of Proposition 20. Despite, the conditions "$r_0 > 0$" and "$r_1 < +\infty$" are necessary through this development.

Proposition 21 *(Kayid [31]).*

If $\sup_{0<u<1} \frac{\psi_n(u)}{\psi_m(u^{r_1})} \leq r_0$ then, $X \preceq_c Y$ implies $X_{U_m} \preceq_c Y_{U_n}$.

From Sengupta and Deshpande [4], it is known that $X \preceq_c Y$ is equivalent to $-\ln(\bar{F}_Y(X))$ is IFR and, further, $X \preceq_{c^\star} Y$ is equivalent to $-\ln(\bar{F}_Y(X))$ is $IFRA$. Since the IFR property implies the $IFRA$ property, and the converse is not true in general thus $X \preceq_{c^\star} Y$ does not imply $X \preceq_c Y$. However, $X \preceq_c Y$ implies $X \preceq_{c^\star} Y$ as mentioned earlier. The following example illustrates a situation where $X \preceq_{c^\star} Y$ but $X \npreceq_c Y$

Example 22 *Suppose that X and Y are two non-negative rvs with respective sfs*

$$\bar{F}_X(x) = \begin{cases} \exp(-x^2), & \text{if } 0 \leq x \leq 1 \\ \exp(-x^3), & \text{if } x \geq 1 \end{cases},$$

$$\bar{F}_Y(x) = \begin{cases} \exp(-x(x+1)), & \text{if } 0 \leq x \leq 1 \\ \exp(-x^2(1+x)), & \text{if } x \geq 1. \end{cases}$$

Now, one can see readily that $\frac{-\ln(\bar{F}_X(x))}{-\ln(\bar{F}_Y(x))} = \frac{x}{x+1}$ which is increasing in $x \geq 0$. This means $X \preceq_{c^*} Y$. However, one can see that

$$h_X(t) = \begin{cases} 2t, & \text{if } 0 \leq t \leq 1 \\ 3t^2, & \text{if } t > 1 \end{cases},$$

$$h_Y(t) = \begin{cases} 1+2t, & \text{if } 0 \leq t \leq 1 \\ 2t+3t^2, & \text{if } t > 1. \end{cases}$$

It is observed that $\frac{h_X(1)}{h_Y(1)} = \frac{2}{3}$ while $\lim_{t \to 1^+} \frac{h_X(t)}{h_Y(t)} = \frac{3}{5}$. Therefore, $\frac{h_X(t)}{h_Y(t)}$ is not increasing in $t \geq 0$, and this clarifies that $X \npreceq_c Y$.

We now present a result similar to Proposition 20 in which the order "$\preceq_c$" is replaced with the weaker order "$\preceq_{c^*}$".

Theorem 23 *Let $X \succeq_{st} Y$ and also let $m \geq n \in \mathbb{N}$. Then, $X \preceq_{c^\star} Y$ implies $X_{U_m} \preceq_{c^\star} Y_{U_n}$.*

Proof. We only need to prove that $X_{U_n} \preceq_{c^*} Y_{U_n}$ under assumptions of the theorem. From Kayid [31], the hazard rate functions of X_{U_n} and Y_{U_n} are respectively given by

$$h_{X_{U_n}}(t) = \frac{h_X(t)}{\xi_n(\bar{F}_X(t))} \text{ and } h_{Y_{U_n}}(t) = \frac{h_Y(t)}{\xi_n(\bar{F}_Y(t))},$$

where ξ_n is given in (3). Furthermore, the sfs of X_{U_n} and Y_{U_n} are respectively obtained as

$$\bar{F}_{X_{U_n}}(t) = d_n(\bar{F}_X(t)) \text{ and } \bar{F}_{Y_{U_n}}(t) = d_n(\bar{F}_Y(t)),$$

where $d_n(u) = u \sum_{k=0}^{n-1} \frac{(-\ln(u))^k}{k!}$. It is to be mentioned here that X_{U_n} and Y_{U_n} have distorted survival functions from the ones of X and Y, respectively, through the distortion d_n for all $n \in \mathbb{N}$, see Section 2.4 in Navarro [53] for more details about distortion functions. From Equation (2), $X \preceq_{c^*} Y$ if, and only if,

$$\frac{h_X(t)}{h_Y(t)} \geq \frac{\Lambda_{F_X}(t)}{\Lambda_{F_Y}(t)}, \text{for all } t \geq 0. \qquad (4)$$

Similarly, $X_{U_n} \preceq_{c^*} Y_{U_n}$ if, and only if,

$$\frac{h_{X_{U_n}}(t)}{h_{Y_{U_n}}(t)} \geq \frac{\Lambda_{F_{X_{U_n}}}(t)}{\Lambda_{F_{Y_{U_n}}}(t)}, \text{for all } t \geq 0,$$

or equivalently, if

$$\frac{h_X(t).\xi_n(\bar{F}_Y(t))}{h_Y(t).\xi_n(\bar{F}_X(t))} \geq \frac{-\ln\left(d_n(\bar{F}_X(t))\right)}{-\ln\left(d_n(\bar{F}_Y(t))\right)}, \text{for all } t \geq 0.$$

From the inequality (4), it suffices to prove that

$$\frac{-\ln(\bar{F}_X(t))}{-\ln(d_n(\bar{F}_X(t)))\xi_n(\bar{F}_X(t))} \geq \frac{-\ln(\bar{F}_Y(t))}{-\ln\left(d_n(\bar{F}_Y(t))\right)\xi_n(\bar{F}_Y(t))}, \text{for all } t \geq 0. \qquad (5)$$

Since from assumption $X \succeq_{st} Y$, thus $\bar{F}_X(t) \geq \bar{F}_Y(t)$, for all $t \geq 0$. Hence, (5) holds true when

$$\frac{-\ln(u)}{-\ln(d_n(u))\xi_n(u)} \text{ is increasing in } u \in (0,1).$$

We can equivalently prove that

$$\frac{y}{\xi_n(e^{-y}).(-\ln(d_n(e^{-y})))} \text{ is decreasing in } y > 0.$$

This is also equivalent to saying that

$$\frac{1}{y}\xi_n(e^{-y}).\left(-\ln\left(d_n(e^{-y})\right)\right) \text{ is increasing in } y > 0. \qquad (6)$$

Note that since $\xi_n(u) = \frac{d_n(u)}{ud_{n'}(u)}$, thus for every $y > 0$,

$$-\ln(d_n(e^{-y})) = \int_0^y \frac{e^{-x}d_{n'}(e^{-x})}{d_n(e^{-x})}dx = \int_0^y \frac{dx}{\xi_n(e^{-x})}.$$

Now, one can see that (6) is satisfied if, and only if,

$$\frac{1}{y}\int_0^y \frac{\xi_n(e^{-y})}{\xi_n(e^{-x})}dx \text{ is increasing in } y > 0. \qquad (7)$$

By the change of variable $x = zy$ in the integral (7), we obtain

$$\frac{1}{y}\int_0^y \frac{\xi_n(e^{-y})}{\xi_n(e^{-x})}dx = \int_0^1 \frac{\xi_n(e^{-y})}{\xi_n(e^{-zy})}dz = \int_0^1 \frac{\sum_{i=0}^{n-1}\frac{y^{i-n+1}}{i!}}{\sum_{i=0}^{n-1}\frac{(zy)^{i-n+1}}{i!}}dz$$

$$= \int_0^1 \frac{\sum_{i=0}^{n-1}\frac{y^i}{i!}}{\sum_{i=0}^{n-1} z^{i-n+1}\frac{y^i}{i!}}dz.$$

Therefore, it is sufficient to prove that, for every $z \in (0,1)$,

$$\phi(y,z) = \frac{\sum_{i=0}^{n-1}\frac{y^i}{i!}}{\sum_{i=0}^{n-1} z^{i-n+1}\frac{y^i}{i!}} \text{ is increasing in } y > 0.$$

Let us define the function K of $y > 0$ and $j = 1,2$, for fixed $z \in (0,1)$ as follows:

$$K(j,y) = \sum_{i=0}^{n-1} \psi_1(y,i)\psi_2(i,j),$$

where $\psi_1(y,i) = \frac{y^i}{i!}$ for $i = 0,1,\dots,n-1$ and $y > 0$ $\psi_2(i,1) = z^{i-n+1}$ and $\psi_2(i,2) = 1$ for $i = 0,1,\dots,n-1$. Note that $\phi(y,z) = \frac{K(2,y)}{K(1,y)}$ is increasing in $y > 0$, if, and only if, $K(j,y)$ is TP_2 in $(j,y) \in \{1,2\} \times (0,+\infty)$. It is readily seen that $\psi_1(y,i)$ is TP_2 in $(y,i) \in (0,+\infty) \times \{0,1,\dots,n-1\}$. Further, since $z \in (0,1)$, thus $\psi_2(i,j)$ is also TP_2 in $(i,j) \in \{0,1,\dots,n-1\} \times \{1,2\}$. Thus, by using the general composition theorem of Karlin [34] we deduce that $K(j,y)$ is TP_2 in $(j,y) \in \{1,2\} \times (0,+\infty)$. Thus, the required result is validated. ||

Note that in the proof Theorem 23, we used Proposition 20 from which , for $m \geq n$, one gets $X_{U_m} \preceq_c X_{U_n}$ and thus $X_{U_m} \preceq_{c^*} X_{U_n}$. The following example presents an application of Theorem 23.

Example 24 *We suppose that X and Y follow the distributions specified in Example 22. It can be checked conveniently that $X \geq_{st} Y$. As clarified in Example 22, it is also known that $X \preceq_{c^*} Y$. Therefore, according to Theorem 23, $X_{U_m} \preceq_{c^*} Y_{U_n}$, for every $m \geq n \in \mathbb{N}$. Let us choose $m = n = 2$. Since $d_2(u) = u(1 - ln(u))$ and because $\bar{F}_{X_{U_2}}(t) = d_2(\bar{F}_X(t))$ and $\bar{F}_{Y_{U_2}}(t) = d_2(\bar{F}_Y(t))$, thus one gets:* $\frac{-\ln(\bar{F}_{X_{U_2}}(t))}{-\ln(\bar{F}_{Y_{U_2}}(t))} = \frac{-\ln\{\bar{F}_X(t).(1-\ln(\bar{F}_X(t)))\}}{-\ln\{\bar{F}_Y(t).(1-\ln(\bar{F}_Y(t)))\}}$. Figure 1 shows that this ratio is an increasing function of t. Therefore, the result of Theorem 23 is acknowledged and $X_{U_2} \preceq_{c^*} Y_{U_2}$. In Figure 2 and Figure 3, we also plotted the ratio of $\frac{-\ln(\bar{F}_{X_{U_3}}(t))}{-\ln(\bar{F}_{Y_{U_2}}(t))}$ and $\frac{-\ln(\bar{F}_{X_{U_4}}(t))}{-\ln(\bar{F}_{Y_{U_2}}(t))}$ which exhibit that they are increasing in t, and, thus, $X_{U_3} \preceq_{c^*} Y_{U_2}$ and $X_{U_4} \preceq_{c^*} Y_{U_2}$, respectively. Note that the result of Proposition 20 is not applicable here because $X \not\preceq_c Y$.

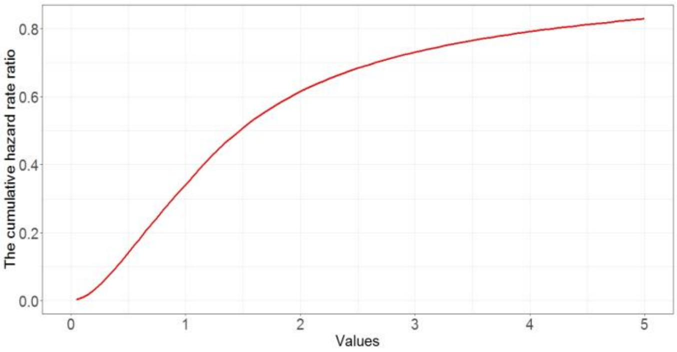

Figure 1. The plot of the function $-\ln\left(\bar{F}_{X_{U_m}}(t)\right)/-\ln\left(\bar{F}_{Y_{U_n}}(t)\right)$ in Example 24 for $m = n = 2$ for values of $0 < t < 5$.

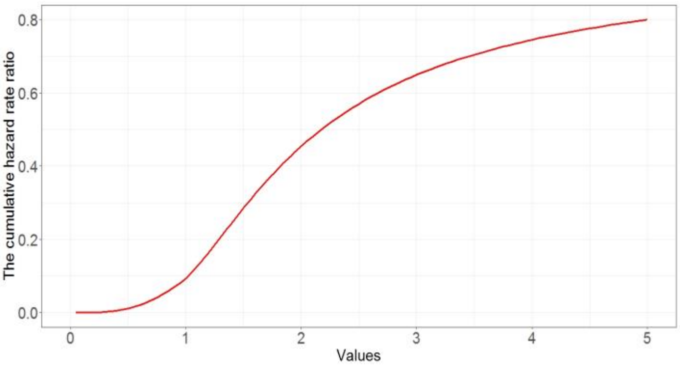

Figure 2. The plot of the function $-\ln\left(\bar{F}_{X_{U_m}}(t)\right)/-\ln\left(\bar{F}_{Y_{U_n}}(t)\right)$ in Example 24 for $m = 3, n = 2$ for values of $0 < t < 5$.

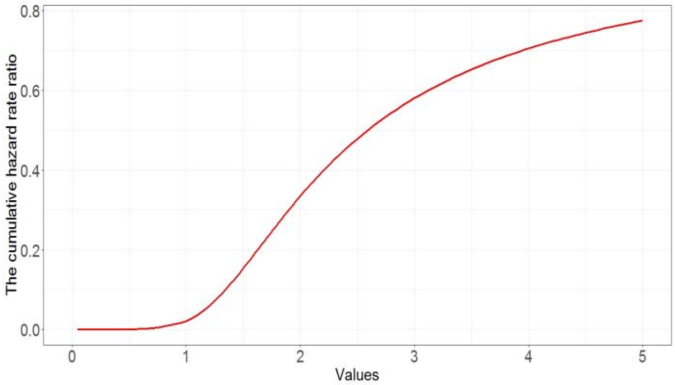

Figure 3. The plot of the function $-\ln\left(\bar{F}_{X_{U_m}}(t)\right)/-\ln\left(\bar{F}_{Y_{U_n}}(t)\right)$ in Example 24 for $m = 4, n = 2$ for values of $0 < t < 5$.

5. Aging faster orders of lower records

Kayid [31] established two results concerning the preservation of the order "$\preceq_b$" under lower records. We present those results below:

Proposition 25 *(Kayid [31]).*

Let $X \preceq_{st} Y$ and also let $m \geq n \in \mathbb{N}$. Then, $X \preceq_b Y$ implies $X_{L_m} \preceq_b Y_{L_n}$.

In Proposition 25 the underlying distributions from which the lower records are taken, are assumed to be ordered according to the usual stochastic order in the same direction of the relative ordering between the underlying distributions. By using Proposition 25 it is realized that when "$m \geq n$" then $X_{L_m} \preceq_b X_{L_n}$ which indicates the reversed hazard rate of initial lower records grows faster than the reversed hazard rate of the terminal lower records. The next result which relaxes the conditions presented in Proposition 25 is useful when the underlying distributions are closely related to each other so that $\tilde{r}_0 = \lim_{t\to 0} \frac{r_X(t)}{r_Y(t)} < +\infty$ and also $\tilde{r}_1 = \lim_{t\to +\infty} \frac{r_X(t)}{r_Y(t)} > 0$.

Proposition 26 *(Kayid [31]).*

If $\sup_{0<u<1} \frac{\psi_n\left(u^{\frac{1}{\tilde{r}_0}}\right)}{\psi_m(u)} \leq \tilde{r}_1$ then, $X \preceq_b Y$ implies $X_{L_m} \preceq_b Y_{L_n}$.

As mentioned before, $X \preceq_b Y$ implies $X \preceq_{b^*} Y$. Despite, the converse is not true in general as $X \preceq_{b^\star} Y$ does not imply $X \preceq_b Y$. The example below presents a situation in which $X \preceq_{b^\star} Y$ but $X \npreceq_b Y$

Example 27 *Let us assume that X and Y are two non-negative rvs with respective cdfs*

$$F_X(x) = \begin{cases} \exp\left(-\frac{1}{x^3}\right), & \text{if } 0 < x \leq 1 \\ \exp\left(-\frac{1}{x^2}\right), & \text{if } x > 1 \end{cases},$$

$$F_Y(x) = \begin{cases} \exp\left(-\frac{x+1}{x^3}\right), \text{if } 0 < x \leq 1 \\ \exp\left(-\frac{x+1}{x^2}\right), \text{if } x > 1. \end{cases}$$

Routinely, one realizes that $\frac{-\ln(F_X(x))}{-\ln(F_Y(x))} = \frac{1}{x+1}$ which is decreasing in $x > 0$. This provides that $X \preceq_{b^*} Y$. In spite of that, we observe that

$$r_X(t) = \begin{cases} \frac{3}{t^4}, & \text{if } 0 < t \leq 1 \\ \frac{2}{t^3}, & \text{if } t > 1 \end{cases},$$

$$r_Y(t) = \begin{cases} \frac{3}{t^4} + \frac{2}{t^3}, \text{if } 0 < t \leq 1 \\ \frac{2}{t^3} + \frac{1}{t^2}, \text{if } t > 1. \end{cases}$$

We then see that $\frac{r_X(1^-)}{r_Y(1^-)} = \frac{3}{5}$ while $\lim_{t\to 1^+} \frac{r_X(t)}{r_Y(t)} = \frac{2}{3}$. Obviously, the ratio $\frac{r_X(t)}{r_Y(t)}$ is not decreasing in $t > 0$ and this confirms that $X \npreceq_b Y$.

We now establish a result similar to Proposition 25 in the case when the order "$\preceq_b$" is replaced with the weaker order "$\preceq_{b^*}$".

Theorem 28 *Let $X \preceq_{st} Y$ and also let $m \geq n \in \mathbb{N}$. Then, $X \preceq_{b^\star} Y$ implies $X_{L_m} \preceq_{b^\star} Y_{L_n}$.*

Proof. It is enough to show that $X_{L_n} \preceq_{b^\star} Y_{L_n}$. Kayid [31] derived the reversed hazard rate functions of X_{L_n} and Y_{L_n}, respectively, as

$$r_{X_{L_n}}(t) = \frac{r_X(t)}{\xi_n(F_X(t))} \; and \; r_{Y_{L_n}}(t) = \frac{r_Y(t)}{\xi_n(F_Y(t))},$$

where ξ_n is given in Equation (3). In addition, the cdfs of X_{L_n} and Y_{L_n} are respectively acquired as

$$F_{X_{L_n}}(t) = d_n(F_X(t)) \; and \; F_{Y_{L_n}}(t) = d_n(F_Y(t)),$$

where d_n is defined as in the proof of Theorem 23. It is noteable that X_{L_n} and Y_{L_n} have distorted distribution functions from the ones of X and Y, respectively, where d_n for all $n \in \mathbb{N}$ is the distortion function (see Section 2.4 in Navarro [53]). In the spirit of the Equation 8, $X \preceq_{b^*} Y$ if, and only if,

$$\frac{r_X(t)}{r_Y(t)} \geq \frac{\tilde{\Lambda}_{F_X}(t)}{\tilde{\Lambda}_{F_Y}(t)}, for \; all \; t \geq 0. \tag{8}$$

In the same manner, $X_{L_n} \preceq_{b^*} Y_{L_n}$ if, and only if,

$$\frac{r_{X_{L_n}}(t)}{r_{Y_{L_n}}(t)} \geq \frac{\tilde{\Lambda}_{F_{X_{L_n}}}(t)}{\tilde{\Lambda}_{F_{Y_{L_n}}}(t)}, for \; all \; t \geq 0,$$

or equivalently, if

$$\frac{r_X(t).\xi_n(F_Y(t))}{r_Y(t).\xi_n(F_X(t))} \geq \frac{-\ln\left(d_n(F_X(t))\right)}{-\ln\left(d_n(F_Y(t))\right)}, for \; all \; t \geq 0.$$

From (8) it is enough to demonstrate that

$$\frac{-\ln(F_X(t))}{-\ln(d_n(F_X(t)))\xi_n(F_X(t))} \geq \frac{-\ln(F_Y(t))}{-\ln\left(d_n(F_Y(t))\right)\xi_n(F_Y(t))}, for \; all \; t \geq 0. \tag{9}$$

From assumption, $X \preceq_{st} Y$, which means $F_X(t) \geq F_Y(t)$, for all $t \geq 0$. Hence, (9) is satisfied if

$$\frac{-\ln(u)}{-\ln(d_n(u))\xi_n(u)} \text{ is increasing in } u \in (0,1).$$

This is what we already proved in the proof of Theorem 23. Hence, the proof is completed.||

In the proof of Theorem 28, we used Proposition 25 by which for $m \geq n$, it follows that $X_{L_m} \preceq_b X_{L_n}$ and, consequently,

$X_{L_m} \preceq_{b^*} X_{L_n}$.

The following example illustrates an application of Theorem 28.

Example 29 *Consider* X *and* Y *as two random lifetime with distributions* F_X *and* F_Y *as given in Example 27. We see that* $X \preceq_{st} Y$. *From Example 27,* $X \preceq_{b^*} Y$. *On that account, Theorem 28 concludes that* $X_{L_m} \preceq_{b^*} Y_{L_n}$, *for all* $m \geq n \in \mathbb{N}$. *We take* $m = 3$ *and* $n = 2$. *Since* $d_2(u) = u(1 - ln(u))$ *and* $d_3(u) = u\left(1 - ln(u) + \frac{(ln(u))^2}{2}\right)$, *thus:* $F_{X_{L_3}}(t) = d_3(F_X(t))$ *and* $F_{Y_{L_2}}(t) = d_2(F_Y(t))$. *One has*

$$\frac{-\ln(F_{X_{L_3}}(t))}{-\ln(F_{Y_{L_2}}(t))} = \frac{-\ln\left\{F_X(t).\left(1-\ln(F_X(t))-\frac{(-\ln(F_X(t)))^2}{2}\right)\right\}}{-\ln\{F_Y(t).(1-\ln(F_Y(t)))\}}.$$

The Figure 4 exhibits that the above ratio is decreasing in t. Hence, the result of Theorem 28 is validated and $X_{L_3} \preceq_{b^*} Y_{L_2}$. In Figure 5 and Figure 6, we further plotted the ratio of $\frac{-\ln(F_{X_{L_4}}(t))}{-\ln(F_{Y_{L_2}}(t))}$ and $\frac{-\ln(F_{X_{L_5}}(t))}{-\ln(F_{Y_{L_2}}(t))}$ which show that they are decreasing in t, and, thus, $X_{L_4} \preceq_{b^*} Y_{L_2}$ and $X_{L_5} \preceq_{b^*} Y_{L_2}$, respectively. We remark here that the result of Proposition 25 cannot be applied in this example as $X \npreceq_b Y$.

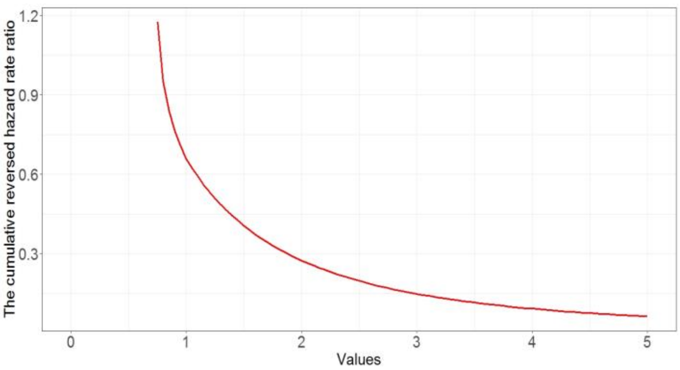

Figure 4. The plot of the function $-\ln\left(F_{X_{L_m}}(t)\right)/-\ln\left(F_{Y_{L_n}}(t)\right)$ in Example 29 for $m = 3, n = 2$ for values of $0 < t < 5$.

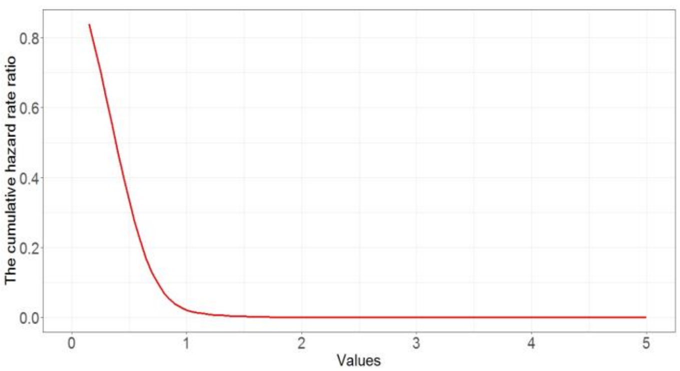

Figure 5. The plot of the function $-\ln\left(F_{X_{L_m}}(t)\right)/-\ln\left(F_{Y_{L_n}}(t)\right)$ in Example 29 for $m = 4, n = 2$ for values of $0 < t < 5$.

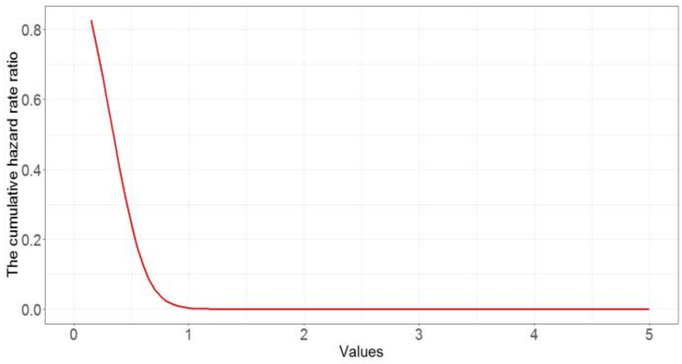

Figure 6. The plot of the function $-\ln\left(F_{X_{L_m}}(t)\right)/-\ln\left(F_{Y_{L_n}}(t)\right)$ in Example 29 for $m = 5, n = 2$ for values of $0 < t < 5$.

Before concluding the paper, we make a few points about the possibility of extending our results to the case of k-record values. It has been shown that the proofs of the main results of this paper do not remain valid when changing the distributions of the record values as necessary. Therefore, the problem of preserving faster aging orders for k-record values is still an open problem.

6. Conclusion

In this study, we achieved two goals. The first is to improve the study and understanding of the relative aging ordering properties to compare lifetime distributions. In fact, in the framework of relative aging orders, it does not matter how well an item begins to work but how fast it will age with time. This highlights the difference between relative aging orders and other well-known stochastic orders in the literature, which only consider the magnitude of the random variables. Several results show that faster aging orders are closely related to the commonly used notion of the probability order. It was also shown that relative aging orders can be used to obtain further inequalities in cumulative entropies. Further bounds for the survival function and the cumulative distribution function of the random variables involved are derived. In deriving the bounds or inequalities, it was assumed that the ratio of (reversed) hazard rates at time t approaches finite values as t approaches 0 and ∞.

The second goal of this study was to investigate the preservation properties of two aging faster orders, namely the relative cumulative hazard rate order and the relative cumulative reversed hazard rate order under the upper and lower record values, respectively. The problem of preserving stochastic orderings under record statistics is a useful subject in reliability. Consider a situation where original data from two heterogeneous populations are available. The data may represent the lifetimes of the two types of electrical devices. The position of each population with respect to the other population may be known because one type of device ages faster than the other. However, if the record data are available in the context of upper and/or lower records, the question can be asked whether the same relative aging sequence determines the position of one population relative to the other population in terms of record values. Another advantage of the preservation properties of the relative aging order in record statistics is that so much data may be needed to test the relative aging order between records in situations where the same relative aging order has already been established in the original distributions. However, datasets, especially the last (extreme) datasets, are very rarely repeated. Therefore, in practice, it is not easy to perform a statistical test to determine the intended relative aging trend between datasets arising from recorded values. It was demonstrated that the usual stochastic ordering of the underlying distributions is a sufficient condition for the aforementioned preservation properties.

In future, we will investigate the preservation properties of relative aging orderings under the structure of $k$record values, generalized order statistics, and sequential order statistics. The preservation properties of the relative aging orders during the formation of complex systems were also investigated.

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