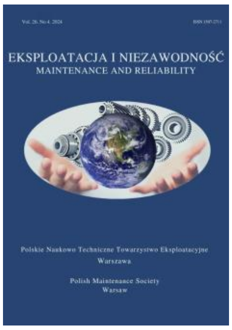

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Exponential Dispersion accelerated degradation modelling and reliability assessment considering initial value and processes heterogeneity

Indexed by: Web of Science Group

Runcao Tian[a], Fan Zhang[b,c], Hongguang Du[a], Peng Wang[b,*]

[a] School of Safety Science and Engineering, Civil Aviation University of China, China
[b] Institute of Science and Technology Innovation, Civil Aviation University of China, China
[c] Key Laboratory of Civil Aviation Aircraft Airworthiness Certification Technology, Civil Aviation University of China, China

Highlights

- A TED degradation process considering the random initial values and stochastic effects of degradation rate is proposed.
- The accelerated degradation model that considers the heterogeneity of trajectories subject to stress levels is developed.
- Combining the MLE and the improved stochastic EM algorithm to complete the parameters estimation.

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Abstract

In the production and operation, inherent variability and uncertainty necessitate addressing unit-to-unit heterogeneity in initial performance values and degradation processes. This article presents a bi-stochastic exponential dispersion process (BS-ED) designed to account for heterogeneity in both initial performance values and degradation processes. First, based on the ED process, the time and acceleration covariates are introduced to form a nonlinear accelerated ED process, and a random effect coefficient associated with the accelerated stress is incorporated to consider the heterogeneity of the process. Meanwhile, through the modelling of degradation time-shift, a degradation model considering the stochastic initial value of the product performance is developed. To effectively conduct the statistic inference of the BS-ED process, an improved stochastic EM algorithm is proposed, and the information matrix and Ito calculus are combined to estimate the confidence intervals. Finally, the stability of the method is verified by simulation and analyzed by two real cases.

Keywords

stochastic processes, exponential dispersion degradation process with bi-stochastic, accelerated testing, degradation modeling, parameter estimation

1. Introduction

According to the provisions of reliability testing standards such as RTCA DO-160G 1, IEC 60300 2 and JEDEC JESD22 series 3, lots of products are subjected to standard tests before they leave the factory to ensure their inherent reliability. Such tests cause a certain degree of degradation of the product's initial performance. Moreover, before products are put into use, there exists a long or short process of storage and transport, which ultimately lead to the wear and tear. Therefore, products have a certain amount of initial degradation before put into use. At the same time, different individuals also have heterogeneity between degradation paths due to differences in raw material processes and manufacturing. As the degradation process is random and dynamic during the operation stage, the initial degradation data and degradation paths also have random and dynamic characteristics. It is, therefore, necessary to develop methods to accurately describe the degradation law of products with random initial performance and heterogeneity of degradation paths.

(*) Corresponding author.
E-mail addresses: R. Tian (ORCID: 0009-0008-5212-2911) tian_run_cao@yeah.net, F. Zhang (ORCID: 0009-0009-8532-0233) zhang_fan_2023@yeah.net, H. Du (ORCID: 0009-0003-9040-0621) du_hong_guang@yeah.net, P. Wang (ORCID: 0000-0002-8738-0602) 2021092093@cauc.edu.cn

The degradation behaviors of products need to be modeled first. Among the stochastic degradation modeling, the exponential dispersion (ED) process, a generalized stochastic process that more accurately describes products with complex degradation mechanisms 4, has been proposed to facilitate the development of degradation modeling. Many scholars have studied the degradation modeling by ED processes. In the field of mathematics, Peter [4] and Zhou et al. 5 mathematically derived the solution and approximation model of ED process in order to facilitate the model solution in combination with the reliability engineering problem. Hong 6 firstly introduced exponential dispersion-like process in the degradation modeling, which unifies and extends the traditional degradation stochastic process. Chen et al. 7, 8, 9 proposed a nonlinear degradation path modeling based on ED process and introduced the measurement error of the test with life prediction. Wang et al. 10 paid attention to the quantitative relationship between the parameters of the degradation model under different stress levels, investigated the mechanism equivalence of the ED process, and proposed a procedure for mechanism equivalence test of ADT through the joint application of normality tests and parameter hypothesis tests. Yan et al. 11, 12 established an accelerated ED process degradation test model based on the optimal design of the test and realized the application on the fatigue life assessment of flax fiber reinforced composites. In addition, the traditional test methods, especially the life test, require a certain amount of samples, and it is more difficult to determine the failure of the product to obtain the life information quickly 13. Therefore, it is necessary to conduct the accelerated degradation test to shorten the time period of obtaining information by subjecting the product to more severe and comprehensive working conditions. In recent years, many scholars have introduced accelerated stress covariates into degradation modeling. For wiener processes, Wang 14 and Ye et al. 15 have established a variety of degradation test models considering accelerated stress covariates in conjunction with the wiener process. In the study of Gamma process, Limon 16 and Wang et al.17 considered the relationship between the parameters of the degradation model and the stress covariates to establish the Gamma degradation-acceleration stress covariate model. In the inverse Gaussian (IG) process, Ma 18 and He et al. 19 established the accelerated degradation model based on the IG process, combined with the accelerated stress relationship, and used the Maximum likelihood estimation (MLE) for parameter estimation. Notwithstanding the foregoing, few scholars have considered the problem of different initial performance and degradation paths of products in the accelerated test framework, especially lacking the practical integration with ED processes.

Currently, some scholars have considered the stochastic nature of the initial performance, which initially given a framework for modeling the degradation of products in the presence of heterogeneous initial performance. For instance, Shen et al. 20 assumed that the product underwent pre-degradation. Considering that its initial performance satisfied the normal distribution, they introduced normal random variables to explain the heterogeneity of the initial degradation by using the Wiener process. Zheng et al. 21 considered the establishment of the Wiener process based on the stochastic process model with initial degradation and degradation rate correlation. Luis et al. 22 considered degradation modelling with random initial degradation levels and random thresholds based on Gamma process. Xiao et al. 23 studied the wiener process considering stochastic initial degradation from the perspective of optimal design of experiments and implemented the optimal design of experiments. Further, Yan et al. 24 gave a specific method for analyzing left-truncated data based on an experimental optimization design for the Wiener process, which is similar to initial value random data. However, most of these models are used for products satisfying specific distributions, which have certain limitations and are not fully applicable to products with complex degradation mechanisms. In the actual degradation process, the degradation paths may be nonlinear and the degradation increments are not necessarily monotonic and the distributions are not purely symmetric. Compared to existing models, the ED process can cover the cases of monotonous and non-monotonous degradation, and can also describe the products with biased distribution of degradation amount. It is more flexible in degradation modelling and more suitable for describing complex degradation processes.

Some scholars have introduced random effects in degradation modeling so as to take the heterogeneity of the degradation process into account, by which the modeling is closer to the actual situation of batch products. First, many scholars have investigated the random effects associated with Wiener processes, mainly in terms of the variability of means and variances: Hou 25, Tang 26, and Sun et al. 27 established a variety of improved random-effects Wiener processes that consider multiple performance parameter random effects and their dependencies, which well quantify the existence of heterogeneity. Xiao et al. 28 proposed a random effect with heteroskedastic measurement error, which can be used from the perspective of the random effect of the error to further measure the inter-individual differences. Wang et al. 29 proposed a Wiener process in which the degradation rate satisfies a generalized inverse Gaussian process to explain the units' heterogeneity in the degradation process with a more general model, which greatly facilitates the modelling of Wiener process based on random effects. In addition, many scholars studied the heterogeneity of the degradation paths present in the Gamma process and the IG process, a common approach is to make its degeneracy parameter satisfy the stochastic process: Ye et al. 30 firstly proposed the use of the IG process for the degradation modelling, considering the random effect that the degeneracy parameters therein obey the Gamma distribution; and Hao 31 and Sun, et al. 32 proposed an extended inverse Gaussian process with biased-normal stochastic effect and consideration of the measurement error to deal with products with asymmetric degradation behaviors; Liu et al. 33 used Bayesian model averaging to highlight inter-unit heterogeneity based on both the IG process and the Gamma process, which used straightly a Bayesian model to measure the degradation heterogeneity between units; Hao 34 and Wang et al. 35 proposed a degradation model considering the time transformations and shape scale stochasticity based on the Gamma process, which makes the modeling of random effects of stochastic processes more flexibly. However, few scholars have considered the influence of stress on the random effect parameters under acceleration, which may cause the degradation paths of similar products to have different degrees of heterogeneity in accelerated tests or when working under severe operating conditions.

In essence, while current research addresses the stochastic initial properties and diverse degradation paths, respectively, accelerated ED modeling lacks comprehensive perspectives. Additional studies are vital to enhance model accuracy and applicability for complex product degradation behaviors. This work introduces an ED process with bi-stochastics (BS-EDP) model to account for initial performance variability and degradation heterogeneity. Key contributions include: (1) Models for degradation time-shift and parameter stochasticity based on the ED process. (2) A BS-ED accelerated degradation model that considers stress-induced heterogeneity. (3) An improved stochastic EM algorithm for estimating unobservable parameters. (4) Interval estimation and confidence intervals via information matrix and Ito integral. The model addresses the life extrapolation and reliability of complex, high-reliability products by considering initial state, degradation heterogeneity, and stress effects.

The rest of this paper is organized as follows. Section 2 describes the proposed BS-ED model. Statistical inference is discussed in Section 3, consisting of the point estimation methods based on overall MLE and improved stochastic EM algorithms, as well as the parameter interval estimation methods based on the observation information matrices. Section 4 leverages the simulated degradation data to verify the effectiveness and convergence of the method. Section 5 takes the real GaAs laser and LED chips as objects to demonstrate the applicability of the proposed model. Section 6 is a summary of our work. Meanwhile, the key symbols related to the model are defined in in Table 1.

Table 1. The key symbol definition table.

Symbol	Explain	Symbol	Explain
$Y(t)$	Independent smooth increments	$E(\cdot)$	Average value
$ED(\eta\Delta t,\lambda)$	The ED distribution	$Var(\cdot)$	Standard deviation
t	Time	$\Lambda(t)$	Time function
η	Degeneracy parameter	q	Time covariate
λ	Scale parameter	α, β	Parameters of acceleration model
$c(\cdot)$	The regularization function	S	The standard stress

Symbol	Explain	Symbol	Explain
$\kappa(\cdot)$	The cumulative function	ε	Quality characteristics
Δy	The degree of degradation	τ	Initial value parameter
ρ	Shape parameter	μ	Mean value of heterogeneity
σ	Standard deviation of heterogeneity	Z	Indicator function
D	Random process deviation function	ω	Failure threshold
FHT	The first hitting time	$f_{ED}(t)$	The PDF of ED with random initial values after SAM
α_x, β_x	Heterogeneity parameters for accelerated models	$F_T(t)$	The CDF of life under the ED with random initial values by B-S
$f_T(t)$	The PDF of life under the ED with random initial values by B-S	$R_{ED}(t)$	The reliability function of life under the ED with random initial values
$F(t)$	The CDF of life of the BS-EDP	$f_{(a)}(\eta)$	The PDF of η (under accelerated)
$R(t)$	The reliability function of life of the BS-EDP	$f_{\bar{T}}(t)$	The PDF of life under the linear ED with random initial values by B-S
t_p	The p-th quantile	$f(t)$	The PDF of life of the BS-EDP
z_p	the p-th quantile of the standard normal distribution	n	Number of observations per component
L	The log-likelihood function	k	Number of parameters of the model
MTTF	The mean time to failure	m	Number of components

2. ED process considering heterogeneity of initial value and degradation processes

2.1. ED accelerates the degradation process

2.1.1. Nonlinear ED degradation processes

The ED process, as a generalized stochastic process, contains common stochastic processes, such as Wiener process, Gamma process, IG process, composite Poisson process, and etc. In some products which have complex failure mechanism, such as integrated chips, their degradation profiles are suitable to be described as ED processes, and the existing research results show that the ED model can improve the estimation accuracy of reliability analysis of products with complex degradation mechanisms [6].

The traditional ED process is determined by the drift rate and the dispersion rate and has three basic features:

1) It initiates from a value of 0, denoted as $Y(0) = 0$;

2) Consistent with the three types of traditional stochastic processes, $Y(t)$ has statistically independent smooth increments, i.e., for any $t_1 < t_2 \leqslant t_3 < t_4$, the stochastic process increments $Y(t_4) - Y(t_3)$ and $Y(t_2) - Y(t_1)$ are independent of each other;

3) For any $\Delta t = t_{i+1} - t_i$, there exists $Y(t_{i+1}) - Y(t_i) \sim ED(\eta\Delta t, \lambda)$.

Here, $ED(\eta\Delta t, \lambda)$ represents the ED distribution, η signifies the degeneracy parameter (drift rate), and λ corresponds to the scale parameter (diffusivity). The probability density is described as follows:

$$f(\Delta y \mid \eta, \lambda) = c(\Delta y \mid \Delta t, \lambda) \cdot exp\{\lambda[\Delta y \cdot \omega(\eta) - \Delta t \cdot \kappa(\omega(\eta))]\} \quad (1)$$

where $c(\cdot)$ and $\kappa(\cdot)$ represent stochastic functional relationships, $c(\cdot)$ is the regularization function that ensures that the cumulative distribution function of the ED distribution is standardized, $\kappa(\cdot)$ is the cumulative function and is quadratically differentiable. while Δy denotes the degree of degradation. To make the description of the degradation distribution for computational convenience, we introduce a stochastic process shape parameter, referred to as $\eta^\rho = \kappa''(\omega(\eta)), \eta = \kappa'(\omega(\eta))$, where η denotes the degradation rate, which is related to the working environment, ρ is the shape parameter of the degradation model, which is related only to the product itself and the degradation mechanism, and λ is the scale parameter, which determines the scale of the distribution of the degradation amount. From the above definition of the ED process, the mean and variance of y are given by

$$\begin{cases} E(\Delta y) = \eta t \\ \frac{Var(\Delta y) = \eta^\rho t}{\lambda} \end{cases} \quad (2)$$

Also combining the definition of the generalized degenerate trajectory, establish the nonlinear functions, i.e., $\Lambda(t)$, and $\Lambda(0) = 0$. In mathematical statistics and engineering practice, exponential and power functions are often used to describe and fit nonlinear processes, i.e., $\Lambda(t) = t^q$ or $\Lambda(t) = exp(qt) - 1$, and the time interval is $\Delta\Lambda(t)$, then the final transformation is a generalized ED distribution, i.e., $Y(\Delta\Lambda(t)) \sim ED(\eta\Delta\Lambda(t), \rho, \lambda, q)$.

2.1.2. Accelerated stress model

In order to obtain the life index and reliability level of high-reliability products, it is necessary to introduce accelerated tests to obtain degradation data quickly. In accelerated testing, the test stress covariate needs to be considered, at which point the drift rate η will be stress-dependent and satisfy the Arrhenius model [6], the Arrhenius model, originated in the field of chemistry, is nowadays widely used in accelerated test modelling, and is generally used to describe the relationship between the characteristic quantity of a product's life and the applied temperature stress, i.e.,

$$\eta(T) = \alpha\, exp\left[\frac{E_a}{k_B T}\right] \tag{3}$$

where α is a constant, called the product coefficient, related to the product's own failure mechanism and test method; E_a is the activation energy of the chemical reaction; k_B is Boltzmann's constant, i.e., 8.617 eV/°C; and T is the temperature stress, generally the Kelvin temperature in K.

The Arrhenius model shows that the amount of degradation is proportional to the rate of reaction, and for the purpose of parameter estimation and processing, the model can be expressed as follows:

$$\eta = \alpha\, exp(\beta S) \tag{4}$$

where α and β are the parameters to be found in the acceleration model and S is the standard stress.

At the same time, stress needs to be standardized to achieve the unity of different units of measure [19], let S_i be the standardized stress level, and the standardization equation is:

$$S_i = S(X_i) = \frac{(X_i - X_{i0})}{(X_{iH} - X_{i0})}, i \in [1, N] \tag{5}$$

where X_{i0} is the daily operating stress level of the i-th stress X_i, and X_{iH} is the accelerated limit stress level, which is the failure or mechanism change level, of the i-th stress X_i. In the acceleration model, the standardization methods are different for different stresses [19], and the commonly used stress types are shown in Table 2.

Table 2. Normalization methods for different types of commonly used accelerated stresses.

Stress S	Normalization	Stress S	Normalization
temperature	$X_i = \frac{1}{S_i'}$	vibratory	$X_i = S_i'$
humidity	$X_i = log(S_i')$	electrical stress	$X_i = log(S_i')$

In this way, the modelling of accelerated degradation processes can be achieved by incorporating accelerated models into the underlying ED degradation process. i.e., $Y(\Delta\Lambda(t)) \sim ED(\alpha\, exp(\beta S) \cdot \Delta\Lambda(t), \rho, \lambda, q)$.

2.2. Modelling of ED considering bi-stochastic in initial values of performance and degradation process

In this section, we propose the BS-EDP to consider the performance initial value and the degradation process bi-stochastic, which can be mathematically written as:

$$Y(t) \sim ED(N(\mu, \sigma^2) \cdot \Lambda(t), \rho, \lambda, q, \varepsilon, \tau) \tag{6}$$

The main features of the BS-EDP are as follows:

(1) Introducing the time-shift parameter τ and the initial value parameter of quality characteristics ε, which equate the natural difference between the performance characteristics of the product after production and the characteristics at the beginning of the test, respectively, which have a real physical meaning. Due to the improvement of the manufacturing process level, the value of ε is generally very small for the same batch of products, and the better the process level, the closer it is to zero.

(2) Considering the presence of random effects in the degradation rate parameter in the ED distribution, η satisfies a Gaussian distribution, i.e., $\eta \sim N(\mu, \sigma^2)$, which allows the heterogeneity among individual products to be transformed into random effects.

(3) Considering the random effects of accelerated stress, we consider that the random effect parameters of η are all stress-sensitive and follow the Arrhenius formula with $\mu = \alpha_\mu\, exp(\beta_\mu S)$, $\sigma = \alpha_\sigma\, exp(\beta_\sigma S)$.

(4) Incorporate the time covariate q to describe generalized degradation trajectories, including both linear and non-linear degradation processes.

(5) For the same product, ρ is determined by the product itself and does not change without a change in the failure mechanism.

The specific model construction process is as follows:

For processes with stochastic initial values of performance, initial degradation values that depend on the time shift τ can be correctly incorporated into the ED process. Considering the stochastic nature of the initial degradation, the expression form of the degradation model with time shift is as follows:

$$Y\big(\Lambda(t)\big) = \varepsilon + Y(\Lambda(t+\tau)) \tag{7}$$

At the beginning of the test, i.e., at $t = 0$:

$$Y(0) = \sigma + Y\big(\Lambda(\tau)\big) \tag{8}$$

and one has,

$$Y(0) - \varepsilon = Y\big(\Lambda(\tau)\big) \sim ED(\eta \cdot \Lambda(\tau), \lambda, \rho) \tag{9}$$

Thereby, $Y(0)$ obeys an ED process with mean $\sigma + \eta \cdot \Lambda(\tau)$ and variance $\frac{\eta^{\rho} \cdot \Lambda(\tau)}{\lambda}$.

Therefore, the mean value of the generalized degenerate process of ED considering random initial values is $\sigma + \eta \cdot \Lambda(t+\tau)$, the variance is $\frac{\eta^{\rho} \cdot \Lambda(t+\tau)}{\lambda}$.

According to the Saddle-point approximation method (SAM) [4], the probability density function (PDF) can be approximated as:

$$f_{ED}(t) \cong \sqrt{\frac{\lambda}{2\pi \cdot \Lambda(t+\tau)^{1-\rho}\left[Y\big(\Lambda(t+\tau)\big)-\varepsilon\right]^{\rho}}} \cdot exp[\lambda \cdot \Lambda(t+\tau) \cdot D] \tag{10}$$

where D is the random process deviation function of the ED distribution, denoted as:

$$D = \begin{cases} -\frac{1}{2}\left[\frac{Y[\Lambda(t+\tau)]-\varepsilon}{\Lambda(t+\tau)} - \eta\right]^2, \rho = 0 \\ \frac{Y[\Lambda(t+\tau)]-\varepsilon}{\Lambda(t+\tau)} - \frac{Y[\Lambda(t+\tau)]-\varepsilon}{\Lambda(t+\tau)} \cdot ln\left[\frac{Y[\Lambda(t+\tau)]-\varepsilon}{\eta\Lambda(t+\tau)}\right] - \eta, \rho = 1 \\ 1 - ln\left[\frac{\eta\Lambda(t+\tau)}{Y[\Lambda(t+\tau)]-\varepsilon}\right] - \frac{Y[\Lambda(t+\tau)]-\varepsilon}{\eta\Lambda(t+\tau)}, \rho = 2 \\ \frac{\eta^{1-\rho}[Y[\Lambda(t+\tau)]-\varepsilon]}{(1-\rho)\Lambda(t+\tau)} - \frac{\left[max\left(\frac{Y[\Lambda(t+\tau)]-\varepsilon}{\Lambda(t+\tau)}, 0\right)\right]^{2-\rho}}{(1-\rho)(2-\rho)} - \frac{\eta^{2-\rho}}{2-\rho}, \rho \neq 1,2 \end{cases} \tag{11}$$

It follows that when ρ takes different values, the corresponding D forms may be different: when $\rho = 0$, a Wiener process; when $\rho = 2$, a Gamma process; when $\rho = 3$, an IG process; and when $\rho \in (1,2)$, a composite Poisson process.

Distribution forms that have a closed form and consider the first hitting time (*FHT*) are important in reliability applications. Assuming that the failure threshold of the product is a fixed value, denoted by ω, the *FHT* is defined as [40]:

$$FHT = inf\{t: Y(t) \geq \omega | Y(0) < \omega\} \tag{12}$$

The FHT of the product obeys the ED distribution, combined with Eq. (9), the cumulative distribution function (CDF) and the PDF of the product life under the degradation process of ED with a random initial value can be obtained by the derivation of the Birnbaum-Saunders (B-S) formula [8], respectively:

$$F_T(t) \cong \Phi\left(\sqrt{\frac{\lambda}{\eta^{\rho}}}\left(\eta\sqrt{\Lambda(t+\tau)} - \frac{\omega-\varepsilon}{\sqrt{\Lambda(t+\tau)}}\right)\right) \tag{13}$$

$$f_T(t) = \frac{\partial F_T(t)}{\partial t} \cong \frac{\Lambda'(t+\tau)}{2}\left(\eta + \frac{\omega-\varepsilon}{\Lambda(t+\tau)}\right)\sqrt{\frac{\lambda}{\Lambda(t+\tau)\eta^{\rho}}} \cdot \varphi\left(\sqrt{\frac{\lambda}{\eta^{\rho}}}\left(\eta\sqrt{\Lambda(t+\tau)} - \frac{\omega-\varepsilon}{\sqrt{\Lambda(t+\tau)}}\right)\right) \tag{14}$$

In particular, when the degenerate trajectory is linear, the PDF is:

$$f_{\bar{T}}(t) \cong \frac{\eta \cdot (t+\tau)+\omega-\varepsilon}{2(t+\tau)}\sqrt{\frac{\lambda\eta^{-\rho}}{2\pi(t+\tau)}} \cdot exp\left(-\frac{\lambda}{2\eta^{\rho}}\left(\eta\sqrt{t+\tau} - \frac{\omega-\varepsilon}{\sqrt{t+\tau}}\right)^2\right) \tag{15}$$

where Φ is the CDF of the standard normal distribution and φ is the PDF of the standard normal distribution. According to this approximation, some characteristic functions have closed form expressions. Finally, the reliability index can be obtained as:

$$R_{ED}(t) = 1 - F_T(t) \cong 1 - \Phi\left(\sqrt{\frac{\lambda}{\eta^{\rho}}}\left(\eta\sqrt{\Lambda(t+\tau)} - \frac{\omega-\varepsilon}{\sqrt{\Lambda(t+\tau)}}\right)\right) \tag{16}$$

In addition, due to the presence of random effects in the degeneracy parameter in the ED distribution, we make η satisfy the Gaussian distribution, i.e., $\eta \sim N(\mu, \sigma^2)$, which allows us to transform the heterogeneity into random effects. The PDF of η is:

$$f(\eta) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{(\eta-\mu)^2}{2\sigma^2}\right) \tag{17}$$

Further, considering the random effect of accelerated stress, in order to more accurately measure the random effect between the accelerated stress and the degradation rate parameter, we consider the random effect parameter of η to be stress-sensitive, and the PDF of η under the accelerated test is:

$$f_a(\eta) = \frac{1}{\alpha_{\sigma}\, exp(\beta_{\sigma}S)\sqrt{2\pi}} exp\left\{-\frac{\left(\eta - \alpha_{\mu}\, exp(\beta_{\mu}S)\right)^2}{2\left(\alpha_{\sigma}\, exp(\beta_{\sigma}S)\right)^2}\right\} \tag{18}$$

Considering the above random effects, the CDF and PDF of BS-EDP can be derived from Eq. (12):

$$F(t) = \int_0^{+\infty} F_T(t \mid \eta) f(\eta) d\eta = \int_0^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{(\eta-\mu)^2}{2\sigma^2}\right) \times \Phi\left(\sqrt{\frac{\lambda}{\eta^{\rho}}}\left(\eta\sqrt{\Lambda(t+\tau)} - \frac{\omega-\varepsilon}{\sqrt{\Lambda(t+\tau)}}\right)\right) d\eta \tag{19}$$

$$\begin{aligned} f(t) &= \int_0^{+\infty} f_T(t \mid \eta) f(\eta) d\eta \\ &= \int_0^{+\infty} \frac{\Lambda'(t+\tau)}{2\sigma} \sqrt{\frac{\lambda\eta^{-\rho}}{2\pi\cdot\Lambda(t+\tau)}} \left(\eta + \frac{\omega-\varepsilon}{\Lambda(t+\tau)}\right) \times \\ &\exp\left(-\frac{(\eta-\mu)^2}{2\sigma^2}\right) \times \varphi\left(\sqrt{\frac{\lambda}{\eta^{\rho}}}\left(\eta\sqrt{\Lambda(t+\tau)} - \frac{\omega-\varepsilon}{\sqrt{\Lambda(t+\tau)}}\right)\right) d\eta \end{aligned} \tag{20}$$

In addition, it can be learnt that the reliability of the product is

$$R(t) = 1 - F(t) = 1 - \int_0^{+\infty} F_T(t \mid \eta) f(\eta) d\eta \tag{21}$$

To more fully characterize the life, the approximate mean time to failure (MTTF) and p-th quantile (t_p) of $F(t)$ are derived as [7]:

$$\begin{aligned} \text{MTTF} &= E(E(t \mid \eta)) \cong E\left(\Lambda^{-1}\left(\frac{\omega-\varepsilon}{\eta} + \frac{\eta^{\rho-2}}{2\lambda} - \tau\right)\right) \\ &= \frac{1}{\sigma\sqrt{2\pi}} \int_0^{+\infty} \Lambda^{-1}\left(\frac{\omega-\varepsilon}{\eta} + \frac{\eta^{\rho-2}}{2\lambda} - \tau\right) \exp\left(-\frac{(\eta-\mu)^2}{2\sigma^2}\right) d\eta \end{aligned} \tag{22}$$

$$t_p \cong \frac{1}{\sigma\sqrt{2\pi}} \int_0^{+\infty} \Lambda^{-1}(A) \cdot exp\left(-\frac{(\eta-\mu)^2}{2\sigma^2}\right) d\eta \tag{23}$$

A is expressed as:

$$\begin{aligned} A = \frac{1}{2}\eta^{\rho-2}\lambda^{-1}\Big({z_p}^2 + 2(\omega-\varepsilon)\lambda\eta^{1-\rho} + \\ z_p\sqrt{z_p^2 + 4(\omega-\varepsilon)\lambda\eta^{1-\rho}}\Big) - \tau \end{aligned} \tag{24}$$

where $\Lambda^{-1}(\cdot)$ is the inverse function of $\Lambda(\cdot)$, and z_p is the p-th quantile of the standard normal distribution.

Ultimately, the parameter vector of the BS-EDP can be expressed as $\Theta = (\mu, \sigma, \lambda, \rho, \tau, \varepsilon, q)$, and in the case of accelerated degradation, the parameter vector can be expressed as $\Theta = (\alpha_\mu, \beta_\mu, \alpha_\sigma, \beta_\sigma, \lambda, \rho, \tau, \varepsilon, q)$.

3. Statistical inference

3.1. Overall MLE model

For test data, parameter estimation methods need to be used to solve for the unknown parameters of the BS-EDP and realize the extrapolation of reliability indicators. MLE is a parameter estimation method that provides a way to evaluate the model parameters using given observations, and is widely used in the field of reliability engineering [13]. In this work, the log-likelihood function of all sample observations in the test can be expressed as:

$$L(\boldsymbol{\theta}) = \sum_{i=1}^{n}\sum_{j=1}^{m} \ln f_{\Delta y}(\Delta y_{ij} \mid \boldsymbol{\theta}) \tag{25}$$

where Δy_{ij} denotes the degradation increment of the component and $\boldsymbol{\theta}$ is the parameter to be solved.

Then, the overall logarithmic MLE based on the ED distribution can be obtained by considering the initial performance parameters randomized as:

$$\begin{aligned} L(\eta,\lambda,\rho,\tau,\varepsilon) &= \sum_{i=1}^{n}\sum_{j=1}^{m} \ln\left[\sqrt{\frac{\lambda\cdot\Delta\Lambda_{ij}^{\rho-1}}{2\pi\left(\Delta Y_{ij}-\varepsilon Z\right)^{\rho}}} \cdot \exp\left(\lambda\cdot\Delta\Lambda_{ij}\cdot D\right)\right] \\ &= \frac{mn}{2}\ln\left(\frac{\lambda}{2\pi}\right) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[(1-\rho)\ln\Delta\Lambda_{ij} + \rho\ln\left(\Delta Y_{ij}-\varepsilon Z\right) - 2\lambda\cdot\Delta\Lambda_{ij}\cdot D\right] \end{aligned} \tag{26}$$

where D is denoted as follows:

$$D = \begin{cases} -\frac{1}{2}\left[\frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}} - \eta\right]^2, \rho = 0 \\ \frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}} - \frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}} \cdot ln\left[\frac{\Delta Y_{ij}-\varepsilon Z}{\eta\cdot\Delta\Lambda_{ij}}\right] - \eta, \rho = 1 \\ 1 - ln\left[\frac{\eta\cdot\Delta\Lambda_{ij}}{\Delta Y_{ij}-\varepsilon Z}\right] - \frac{\Delta Y_{ij}-\varepsilon Z}{\eta\cdot\Delta\Lambda_{ij}}, \rho = 2 \\ \frac{\eta^{1-\rho}(\Delta Y_{ij}-\varepsilon Z)}{(1-\rho)\Delta\Lambda_{ij}} - \frac{\left[max\left(\frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}}, 0\right)\right]^{2-\rho}}{(1-\rho)(2-\rho)} - \frac{\eta^{2-\rho}}{2-\rho}, \rho \neq 1,2 \end{cases} \tag{27}$$

and

$$\Delta\Lambda_{ij} = \Lambda\left(t_{ij} + \tau\right) - \Lambda\left(t_{i(j-1)} + \tau\right) \tag{28}$$

Meanwhile, the first degenerate increment ΔY_{i1} obeys the ED distribution, when $\Delta\Lambda_{i1} = \Lambda(t_{i1} + \tau)$.

In addition, Z is an indicator function that indicates the selection of the initial value of the test performance for each piece, indicating that σ needs to be considered when the data is taken to the initial degradation performance parameter value, which is also accompanied by the appearance of σ in D. The expression is as follows:

$$Z = \begin{cases} 1, j = 1 \\ 0, j > 1 \end{cases} \tag{29}$$

Further, it is necessary to consider the solution of the heterogeneity parameters μ, σ of the degradation process parameter η. Since at this point η satisfies the Gaussian distribution at the same time, it needs to be converted to hyperparameter estimation, and thus the Bayesian estimation is introduced to obtain the log-likelihood function of the BS-EDP as:

$$L(\Theta) = \sum_{i=1}^{n} \int_0^{+\infty} \sum_{j=1}^{m} ln\left[f_{\Delta y}\left(\Delta y_{ij} \mid \eta_i\right) f(\eta_i)\right] d\eta_i \tag{30}$$

where $\Theta = (\mu, \sigma, \lambda, \rho, \tau, \varepsilon, q)$ is the vector representation of the unknown parameter to be solved and η_i denotes the heterogeneity degradation rate parameter for each of the n products, and one has:

$$\begin{aligned} L(\Theta) &= \sum_{i=1}^{n}\int_0^{+\infty}\sum_{j=1}^{m} \ln\left[\frac{1}{2\pi\sigma}\sqrt{\frac{\lambda\cdot\Delta\Lambda_{ij}^{\rho-1}}{\left(\Delta Y_{ij}-\varepsilon Z\right)^{\rho}}} \cdot \exp\left(\lambda\cdot\Delta\Lambda_{ij}D - \frac{(\eta_i-\mu)^2}{2\sigma^2}\right)\right] d\eta_i \\ &= mn\cdot\ln\left(\frac{\sqrt{\lambda}}{2\pi\sigma}\right) + \frac{1}{2}\sum_{i=1}^{n}\int_0^{+\infty}\sum_{j=1}^{m}\begin{bmatrix}(\rho-1)\cdot\ln\left(\Delta\Lambda_{ij}\right) + 2\lambda\cdot\Delta\Lambda_{ij}D \\ -\rho\cdot\ln\left(\Delta Y_{ij}-\varepsilon Z\right) - \frac{(\eta_i-\mu)^2}{\sigma^2}\end{bmatrix} d\eta_i \\ &= \frac{mn}{2}\ln\left(\frac{\lambda}{2\pi\sigma}\right) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\begin{bmatrix}(\rho-1)\cdot\ln\left(\Delta\Lambda_{ij}\right) \\ -\rho\cdot\ln\left(\Delta Y_{ij}-\varepsilon Z\right)\end{bmatrix} + \sum_{i=1}^{n}\int_0^{+\infty}\sum_{j=1}^{m}\left[\lambda\cdot\Delta\Lambda_{ij}D - \frac{(\eta_i-\mu)^2}{2\sigma^2}\right] d\eta_i \end{aligned} \tag{31}$$

Furthermore, the log-likelihood function of BS-EDP in the accelerated case is:

$$L(\Theta) \quad = \frac{mn}{2}\ln\left(\frac{\lambda}{2\pi(\alpha_\sigma \exp(\beta_\sigma S))}\right) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\begin{bmatrix}(\rho-1)\cdot\ln(\Delta\Lambda_{ij})\\ -\rho\cdot\ln(\Delta Y_{ij}-\varepsilon Z)\end{bmatrix}$$
$$+\sum_{i=1}^{n}\int_{0}^{+\infty}\sum_{j=1}^{m}\left[\lambda\cdot\Delta\Lambda_{ij}D-\frac{\left(\eta_i-\alpha_\mu \exp(\beta_\mu S)\right)^2}{2\left(\alpha_\sigma \exp(\beta_\sigma S)\right)^2}\right]d\eta_i \tag{32}$$

In the parameter solving process, the optimization model can be constructed based on the fundamental property that the maximum point derivative of MLE is 0, so that the partial derivative of each parameter is 0.

However, since η is unobservable, direct constrained optimization of the above log-likelihood function is computationally difficult to implement and usually fails to converge to a solution. Moreover, for the likelihood function in Eq.(31), the integral operation is too complicated to directly solve all parameters as a whole. It is necessary to introduce the EM algorithm to solve the hyperparameter solution problem.

3.2. The Improved stochastic step-by-step EM algorithm of BS-EDP

The EM algorithm provides a feasible method for solving unobservable and random effect parameter solutions by finding the MLE of the potential distribution parameter with the given dataset and utilizing the conditional expectation of the observable data in place of the hidden variables. But the complex integral is difficult to compute, and the results obtained by the EM algorithm are easily affected by the initial value setting when applied to real situations. To avoid this problem, a stochastic process is introduced into the EM algorithm, and an improved EM algorithm is proposed that uses Markov chain Monte Carlo (MCMC) instead of estimating the missing values. The advantages of this method are utilized to improve the missing value estimation accuracy.

The overall idea of the MCMC method is to construct a Markov chain so that its smooth distribution is the posterior distribution of the parameter to be estimated, generate samples of the posterior distribution through this Markov chain, and perform Monte-Carlo integration on the samples (i.e., the valid samples) based on the Markov chain when it reaches the smooth distribution. The specific steps of the improved procedure are as follows:

Step 1: Initialize the parameters, mainly including global parameters, local parameters and iteration number.

a. Perform overall MLE based on global data to obtain initialized global parameters to be sought $\Theta_0 = (\lambda_0, \rho_0, \tau_0, \varepsilon_0, q_0)$.

b. Initialize the local pending parameters $\Theta_0^{'} = (\mu_0, \sigma_0)$, and $\eta_0 \sim N(\mu_0, {\sigma_0}^2)$, construct the initial Markov chain.

c. Set the number of iterations v and initially $v = 0$.

Step 2: In E-step, the posterior distribution of η is obtained by MCMC sampling due to the difficulty of the integral operation, and (μ_v, σ_v) is substituted into $f(\eta_{v+1}) = \frac{1}{\sigma_v\sqrt{2\pi}} exp\left(-\frac{(\eta_{v+1}-\mu_v)^2}{2{\sigma_v}^2}\right)$ to obtain η_{v+1}, which serves as the new Markov chain, and the new parameters are constructed as follows:

$$\begin{cases}\mu_{v+1} = \frac{1}{n-m}\sum_{v=m+1}^{n} f(x_v) = \eta_v \cdot \Lambda(t+\tau_v)\\ {\sigma_{v+1}}^2 = \frac{{\eta_v}^{\rho_0}{\sigma_v}^2}{\lambda_v{\sigma_v}^2\cdot[(1-\rho_0)\cdot\Lambda(t+\tau_v)+\rho_0\cdot\eta_v\cdot(Y-\varepsilon)]+{\eta_v}^{\rho_0}}\end{cases} \tag{33}$$

where the first $\eta_{v=0}$ can be obtained by global overall MLE, the next μ_{v+1} is obtained by Monte-Carlo integration to compute the expectation, and $f(x_v)$ is the MC simulation value of the distribution function of η_v; the ${\sigma_{v+1}}^2$ is obtained by Markov chain a priori distribution samples and the global to-be-demanded parameters from the previous step.

Step 3: M-step, based on the new $\eta_{v+1} \sim N(\mu_{v+1}, {\sigma_{v+1}}^2)$, maximize the MLE by combining the MLE formula of BS-EDP to update $\Theta_{v+1} = (\lambda_{v+1}, \tau_{v+1}, \varepsilon_{v+1}, q_{v+1})$, where ρ_0 will not be changed under the same working condition due to the product's own fixed characteristics.

Step 4: Set v+1 = v+2 and repeat steps 2 and 3 until the global MLE is maximum and the local parameter (μ_v, σ_v) converges, convergence is defined as the difference between two neighboring estimates being less than a given threshold.

In addition, for the BS-EDP considering accelerated stress covariates, the local parameter to be solved is $\Theta_0^{'} = (\alpha_{\mu(0)}, \alpha_{\sigma(0)}, \beta_{\mu(0)}, \beta_{\sigma(0)})$, and the accelerated covariate parameter construction method for the E-step is based on the improved stochastic EM step-by-step algorithm that does not consider acceleration:

$$\begin{cases}\alpha_{\mu(v+1)} = \frac{\eta_v\cdot\Lambda(t+\tau_v)}{exp(\beta_{\mu(v)}S)}\\ \beta_{\mu(v+1)} = \frac{ln\,\mu_v - ln\,\alpha_{\mu(v)}}{S}\end{cases} \tag{34}$$

$$\begin{cases}\alpha_{\sigma(v+1)} = \alpha_{\sigma(v)} {\eta_v}^{\frac{\rho_0}{2}} \left(\lambda_v \sigma_v^2 \begin{bmatrix}(1-\rho_0)\Lambda(t+\tau_v) \\ +\rho_0 \eta_v (Y-\varepsilon_v)\end{bmatrix} + {\eta_v}^{\rho_0}\right)^{-2} \\ \beta_{\sigma(v+1)} = \frac{\ln \sigma_v - \ln \alpha_{\sigma(v)}}{S}\end{cases} \quad (35)$$

M-Step will maximally update the global parameters to be solved based on the new parameters $\eta_{v+1} \sim N\left[\alpha_{\mu(v+1)}\, exp\left(\beta_{\mu(v+1)} S\right), \alpha_{\sigma(v+1)}^2\, exp^2\left(\beta_{\sigma(v+1)} S\right)\right]$.

The key to the EM algorithm is to alternate iterations between *E*-steps and *M*-steps until the parameters converge. This will gradually improve the accuracy of the parameter estimates. By improving the stochastic step-by-step EM algorithm, the difficulties in integral solving can be effectively solved to maximize the use of a priori data to obtain point estimates of the parameters.

3.3. An interval estimation method based on information matrix and *Ito* calculus

For the unknown parameters of the model obtained by the overall MLE and EM algorithms, they are generally approximations of the distributed parameters, and the characteristics of statistical theory make it impossible to find their true values. Therefore, the interval estimation, as part of the parameter estimation task, can determine the range that exists for the true values of the distributional parameters and determine the accuracy and confidence of the parameter estimates.

Since the EM algorithm is used in this paper to solve for the unknown parameters, the interval estimates of the parameters can be derived based on the normal asymptotic nature by solving the inverse matrix by observing the Fisher information matrix and obtaining the estimated asymptotic covariance to derive the interval estimates of the parameters of interest.

If it is an MLE of the parameter in the overall density function, then it asymptotically obeys a normal distribution:

$$\hat{\theta} \sim N\left(\theta, \left[nE\left(\frac{\partial \ln f}{\partial \theta}\right)^2\right]^{-1}\right) \quad (36)$$

Thus, under asymptotic theory, the parameters $\hat{\Theta} = (\hat{\mu}, \hat{\sigma}, \hat{\lambda}, \hat{\rho}, \hat{\tau}, \hat{\varepsilon}, \hat{q})$ follow a multidimensional normal distribution with mean $\Theta = (\mu, \sigma, \lambda, \rho, \tau, \varepsilon, q)$. the observed Fisher matrix is:

$$I_0(\Theta) = \left.\begin{pmatrix} -\frac{\partial^2 L}{\partial \mu^2} & -\frac{\partial^2 L}{\partial \mu \partial \sigma} & -\frac{\partial^2 L}{\partial \mu \partial \lambda} & -\frac{\partial^2 L}{\partial \mu \partial \rho} & -\frac{\partial^2 L}{\partial \mu \partial \tau} & -\frac{\partial^2 L}{\partial \mu \partial \varepsilon} & -\frac{\partial^2 L}{\partial \mu \partial q} \\ -\frac{\partial^2 L}{\partial \sigma \partial \mu} & -\frac{\partial^2 L}{\partial \sigma^2} & -\frac{\partial^2 L}{\partial \sigma \partial \lambda} & -\frac{\partial^2 L}{\partial \sigma \partial \rho} & -\frac{\partial^2 L}{\partial \sigma \partial \tau} & -\frac{\partial^2 L}{\partial \sigma \partial \varepsilon} & -\frac{\partial^2 L}{\partial \sigma \partial q} \\ -\frac{\partial^2 L}{\partial \lambda \partial \mu} & -\frac{\partial^2 L}{\partial \lambda \partial \sigma} & -\frac{\partial^2 L}{\partial \lambda^2} & -\frac{\partial^2 L}{\partial \lambda \partial \rho} & -\frac{\partial^2 L}{\partial \lambda \partial \tau} & -\frac{\partial^2 L}{\partial \lambda \partial \varepsilon} & -\frac{\partial^2 L}{\partial \lambda \partial q} \\ -\frac{\partial^2 L}{\partial \rho \partial \mu} & -\frac{\partial^2 L}{\partial \rho \partial \sigma} & -\frac{\partial^2 L}{\partial \rho \partial \lambda} & -\frac{\partial^2 L}{\partial \rho^2} & -\frac{\partial^2 L}{\partial \rho \partial \tau} & -\frac{\partial^2 L}{\partial \rho \partial \varepsilon} & -\frac{\partial^2 L}{\partial \rho \partial q} \\ -\frac{\partial^2 L}{\partial \tau \partial \mu} & -\frac{\partial^2 L}{\partial \tau \partial \sigma} & -\frac{\partial^2 L}{\partial \tau \partial \lambda} & -\frac{\partial^2 L}{\partial \tau \partial \rho} & -\frac{\partial^2 L}{\partial \tau^2} & -\frac{\partial^2 L}{\partial \tau \partial \varepsilon} & -\frac{\partial^2 L}{\partial \tau \partial q} \\ -\frac{\partial^2 L}{\partial \varepsilon \partial \mu} & -\frac{\partial^2 L}{\partial \varepsilon \partial \sigma} & -\frac{\partial^2 L}{\partial \varepsilon \partial \lambda} & -\frac{\partial^2 L}{\partial \varepsilon \partial \rho} & -\frac{\partial^2 L}{\partial \varepsilon \partial \tau} & -\frac{\partial^2 L}{\partial \varepsilon^2} & -\frac{\partial^2 L}{\partial \varepsilon \partial q} \\ -\frac{\partial^2 L}{\partial q \partial \mu} & -\frac{\partial^2 L}{\partial q \partial \sigma} & -\frac{\partial^2 L}{\partial q \partial \lambda} & -\frac{\partial^2 L}{\partial q \partial \rho} & -\frac{\partial^2 L}{\partial q \partial \tau} & -\frac{\partial^2 L}{\partial q \partial \varepsilon} & -\frac{\partial^2 L}{\partial q^2} \end{pmatrix}\right|_{\theta_k = \hat{\theta}_k} \quad (37)$$

Since the Fisher matrix is a symmetric matrix, $I_{ij} = I_{ji}$, the elements in the observed Fisher matrix under the BS-ED model can be obtained:

$$-\frac{\partial^2 L}{\partial \mu^2} = \frac{1}{\sigma^2}$$

$$-\frac{\partial^2 L}{\partial \lambda^2} = \frac{mn}{2\lambda^2}$$

$$-\frac{\partial^2 L}{\partial \sigma^2} = -\frac{mn}{2\sigma^2} + \sum_{i=1}^{n} \int_0^{+\infty} \left[\frac{3(\eta_i - \mu)^2}{\sigma^4}\right] d\eta_i$$

$$-\frac{\partial^2 L}{\partial \rho^2} = -\sum_{i=1}^{n} \int_0^{+\infty} \sum_{j=1}^{m} \left(\lambda \cdot \Delta\Lambda_{ij} \frac{\partial^2 D}{\partial \rho^2}\right) d\eta_i$$

$$-\frac{\partial^2 L}{\partial \tau^2} = \frac{1}{2}(\rho - 1) \sum_{i=1}^{n} \sum_{j=1}^{m} \begin{bmatrix} \left(\Delta\Lambda_{ij}\right)^{-2} \cdot \left(\frac{\partial \Delta\Lambda_{ij}}{\partial \tau}\right)^2 \\ -\left(\Delta\Lambda_{ij}\right)^{-1} \cdot \frac{\partial^2 \Delta\Lambda_{ij}}{\partial \tau^2} \end{bmatrix}$$
$$-\lambda \sum_{i=1}^{n} \int_0^{+\infty} \sum_{j=1}^{m} \left[\left(D \frac{\partial^2 \Delta\Lambda_{ij}}{\partial \tau^2} + 2 \frac{\partial \Delta\Lambda_{ij}}{\partial \tau} \frac{\partial D}{\partial \tau} + \Delta\Lambda_{ij} \frac{\partial^2 D}{\partial \tau^2}\right)\right] d\eta_i$$

$$-\frac{\partial^2 L}{\partial \varepsilon^2} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \left[\frac{\rho Z^3}{\left(\Delta Y_{ij} - \varepsilon Z\right)^2}\right] - \sum_{i=1}^{n} \int_0^{+\infty} \sum_{j=1}^{m} \left[\lambda \cdot \Delta\Lambda_{ij} \frac{\partial^2 D}{\partial \varepsilon^2}\right] d\eta_i$$

$$-\frac{\partial^2 L}{\partial q^2} = \frac{1}{2}(\rho - 1) \sum_{i=1}^{n} \sum_{j=1}^{m} \begin{bmatrix} \left(\Delta\Lambda_{ij}\right)^{-2} \cdot \left(\frac{\partial \Delta\Lambda_{ij}}{\partial q}\right)^2 \\ -\left(\Delta\Lambda_{ij}\right)^{-1} \frac{\partial^2 \Delta\Lambda_{ij}}{\partial q^2} \end{bmatrix}$$
$$-\lambda \sum_{i=1}^{n} \int_0^{+\infty} \sum_{j=1}^{m} \left[\left(D \frac{\partial^2 \Delta\Lambda_{ij}}{\partial q^2} + 2 \frac{\partial \Delta\Lambda_{ij}}{\partial q} \frac{\partial D}{\partial q} + \Delta\Lambda_{ij} \frac{\partial^2 D}{\partial q^2}\right)\right] d\eta_i$$

$$-\frac{\partial^2 L}{\partial \mu\, \partial \sigma} = 2 \sum_{i=1}^{n} \int_0^{+\infty} \sum_{j=1}^{m} \left(\frac{\eta_i - \mu}{\sigma^3}\right) d\eta_i$$

$$-\frac{\partial^2 L}{\partial \lambda\, \partial \rho} = -\sum_{i=1}^{n} \int_0^{+\infty} \sum_{j=1}^{m} \left(\Delta\Lambda_{ij} \frac{\partial D}{\partial \rho}\right) d\eta_i$$

$$-\frac{\partial^2 L}{\partial \rho\, \partial \tau} = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \left[\frac{1}{\Delta\Lambda_{ij}} \frac{\partial \Delta\Lambda_{ij}}{\partial \tau}\right]$$
$$-\lambda \sum_{i=1}^{n} \int_0^{+\infty} \sum_{j=1}^{m} \left[\frac{\partial \Delta\Lambda_{ij}}{\partial \tau} \frac{\partial D}{\partial \rho} + \Delta\Lambda_{ij} \frac{\partial^2 D}{\partial \rho\, \partial \tau}\right] d\eta_i$$

$$-\frac{\partial^2 L}{\partial \varepsilon\, \partial q} = -\lambda \cdot \sum_{i=1}^{n} \int_0^{+\infty} \sum_{j=1}^{m} \left[\frac{\partial \Delta\Lambda_{ij}}{\partial q} \frac{\partial D}{\partial \varepsilon} + \Delta\Lambda_{ij} \frac{\partial^2 D}{\partial \varepsilon\, \partial q}\right] d\eta_i$$

$$-\frac{\partial^2 L}{\partial \varepsilon\, \partial q} = -\lambda \cdot \sum_{i=1}^{n} \int_0^{+\infty} \sum_{j=1}^{m} \left[\frac{\partial \Delta\Lambda_{ij}}{\partial q} \frac{\partial D}{\partial \varepsilon} + \Delta\Lambda_{ij} \frac{\partial^2 D}{\partial \varepsilon\, \partial q}\right] d\eta_i$$

$$-\frac{\partial^2 L}{\partial \lambda\, \partial \tau} = -\sum_{i=1}^{n} \int_0^{+\infty} \sum_{j=1}^{m} \left[D \frac{\partial \Delta\Lambda_{ij}}{\partial \tau} + \Delta\Lambda_{ij} \frac{\partial D}{\partial \tau}\right] d\eta_i$$

$$-\frac{\partial^2 L}{\partial\tau\,\partial\varepsilon} = -\lambda\cdot\sum_{i=1}^{n}\int_0^{+\infty}\sum_{j=1}^{m}\left[\frac{\partial\Delta\Lambda_{ij}}{\partial\tau}\frac{\partial D}{\partial\varepsilon} + \Delta\Lambda_{ij}\frac{\partial^2 D}{\partial\tau\,\partial\varepsilon}\right]d\eta_i$$

$$-\frac{\partial^2 L}{\partial\rho\,\partial\varepsilon} = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\frac{Z}{\Delta Y_{ij}-\varepsilon Z}\right] - \lambda\sum_{i=1}^{n}\int_0^{+\infty}\sum_{j=1}^{m}\left[\Delta\Lambda_{ij}\frac{\partial^2 D}{\partial\rho\,\partial\varepsilon}\right]d\eta_i$$

$$-\frac{\partial^2 L}{\partial\tau\,\partial q} = \frac{1}{2}(\rho-1)\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\left(\frac{1}{(\Delta\Lambda_{ij})^2} - \frac{1}{\Delta\Lambda_{ij}}\right)\frac{\partial^2\Delta\Lambda_{ij}}{\partial q\,\partial\tau}\right]$$
$$-\lambda\sum_{i=1}^{n}\int_0^{+\infty}\sum_{j=1}^{m}\left[D\frac{\partial^2\Delta\Lambda_{ij}}{\partial\tau\,\partial q} + \frac{\partial\Delta\Lambda_{ij}}{\partial\tau}\frac{\partial D}{\partial q} + \frac{\partial\Delta\Lambda_{ij}}{\partial q}\frac{\partial D}{\partial\tau} + \Delta\Lambda_{ij}\frac{\partial^2 D}{\partial\tau\,\partial q}\right]d\eta_i$$

$$-\frac{\partial^2 L}{\partial\lambda\,\partial\varepsilon} = -\sum_{i=1}^{n}\int_0^{+\infty}\sum_{j=1}^{m}\left[\Delta\Lambda_{ij}\frac{\partial D}{\partial\varepsilon}\right]d\eta_i$$

$$-\frac{\partial^2 L}{\partial\rho\,\partial q} = -\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left[\frac{1}{\Delta\Lambda_{ij}}\frac{\partial\Delta\Lambda_{ij}}{\partial q}\right]$$
$$-\lambda\sum_{i=1}^{n}\int_0^{+\infty}\sum_{j=1}^{m}\left[\frac{\partial\Delta\Lambda_{ij}}{\partial q}\frac{\partial D}{\partial\rho} + \Delta\Lambda_{ij}\frac{\partial^2 D}{\partial\rho\,\partial q}\right]d\eta_i$$

$$-\frac{\partial^2 L}{\partial\lambda\,\partial q} = -\sum_{i=1}^{n}\int_0^{+\infty}\sum_{j=1}^{m}\left[D\frac{\partial\Delta\Lambda_{ij}}{\partial q} + \Delta\Lambda_{ij}\frac{\partial D}{\partial q}\right]d\eta_i$$

$$-\frac{\partial^2 L}{\partial\sigma\partial\varepsilon} = -\frac{\partial^2 L}{\partial\mu\partial\tau} = -\frac{\partial^2 L}{\partial\mu\partial\varepsilon} = -\frac{\partial^2 L}{\partial\sigma\partial q} = -\frac{\partial^2 L}{\partial\mu\partial q} = -\frac{\partial^2 L}{\partial\sigma\partial\lambda} = -\frac{\partial^2 L}{\partial\mu\partial\lambda}$$

$$= -\frac{\partial^2 L}{\partial\sigma\partial\rho} = -\frac{\partial^2 L}{\partial\mu\partial\rho} = -\frac{\partial^2 L}{\partial\sigma\partial\tau} = 0$$

In summary, the partial derivatives of the second layer need to be solved; the first partial derivatives of the second layer have already been solved in Section 3.1, and the specific solution for the second partial derivatives is given in Appendix A.

Based on the results of L second order partial derivatives, it is clear that the calculation contains many difficult integrals of random variables and integrals in non-closed form, which cannot be solved by the traditional methods. In this paper, the *Ito* calculus is used to deal with the complex stochastic integrals.

First, given the product function $f\ (\eta,\ t)$, where η is a normally distributed random variable. We can write $f\,(\eta,\ t)$ in the form of an *Ito* process:

$$dY_t = f(\eta, t)d\eta \tag{38}$$

Next, using *Ito* integral processing, define a new procedure Z_s that satisfies the following conditions:

$$Z_s = \int_0^t f(\eta, s)d\eta_s \tag{39}$$

where $f\,(\eta,\ s)$ is a measurable function with respect to time and stochastic processes arising from a second partial derivative computation, and $d\eta_s$ in the *Ito* integral introduces stochasticity in η and satisfies Brownian motion. However, since $f\,(\eta,\ s)$ is a function of η satisfying a stochastic process, after expansion by Taylor's formula, there are:

$$\Delta f = f(\eta_s + \Delta\eta_s) - f(\eta_s) = f'(\eta_s)(\Delta\eta_s) + \frac{f''(\eta_s)}{2}(\Delta\eta_s)^2 + \frac{f'''(\eta_s)}{6}(\Delta\eta_s)^3 + \tag{40}$$

We find that since the Brownian motion itself has a quadratic variation that is not 0 and is $(d\eta_s)^2 = ds$. Therefore, the second term of the Taylor expansion is the same order as the first term, and cannot be omitted, and the degeneracy parameter η_s is not trivial by itself. We can obtain:

$$df(\eta_s) = f'(\eta_s)d\eta_s + \frac{1}{2}f''(\eta_s)ds \tag{41}$$

Further, more generally, if η satisfies Brownian motion η_s, we have:

$$df = \frac{\partial f}{\partial s}ds + \frac{\partial f}{\partial\eta}d\eta_s + \frac{1}{2}\frac{\partial^2 f}{\partial\eta^2}(d\eta_s)^2 = \left(\frac{\partial f}{\partial s} + \frac{1}{2}\frac{\partial^2 f}{\partial\eta^2}\right)ds + \frac{\partial f}{\partial\eta}d\eta_s \tag{42}$$

The next step is to solve the specific form of complex calculus based on the above method. We can see that if Z_s is a stochastic process satisfying a stochastic integral equation as Eq. (38), where f and η can be obtained by data estimation or measurement, then for η, for any s on η_s and a small increment Δs, one has:

$$Z_{s+\Delta s} - Z_s = \frac{\partial f}{\partial s}\Delta s + \frac{\partial f}{\partial\eta}\Delta\eta_s + \frac{1}{2}\frac{\partial^2 f}{\partial\eta^2}(\Delta\eta_s)^2 \tag{43}$$

The above process involves the product of $f(\eta,s)$ with $d\eta_s$ and will accumulate over the entire η interval, thus avoiding direct integration by approximating the area and ultimately obtaining a numerical solution for each element of the observation information matrix that involves a complex non-closed integral.

Since the inverse matrix of matrix I_0 is the variance-covariance matrix of parameter Θ, the two-sided confidence interval of the parameter at confidence level γ can be expressed as:

$$[\theta_L, \theta_U] = \left[\theta - k\sqrt{var(\hat{\theta})}, \theta + k\sqrt{var(\hat{\theta})}\right] \tag{44}$$

where k is the quantile of the confidence band 1-γ under the standard normal distribution, $var(\hat{\theta})$ is the diagonal element of the variance-covariance matrix, and is the asymptotic covariance of each parameter.

In summary, the overall process of reliability assessment and life prediction based on BS-EDP is shown in Fig. 1.

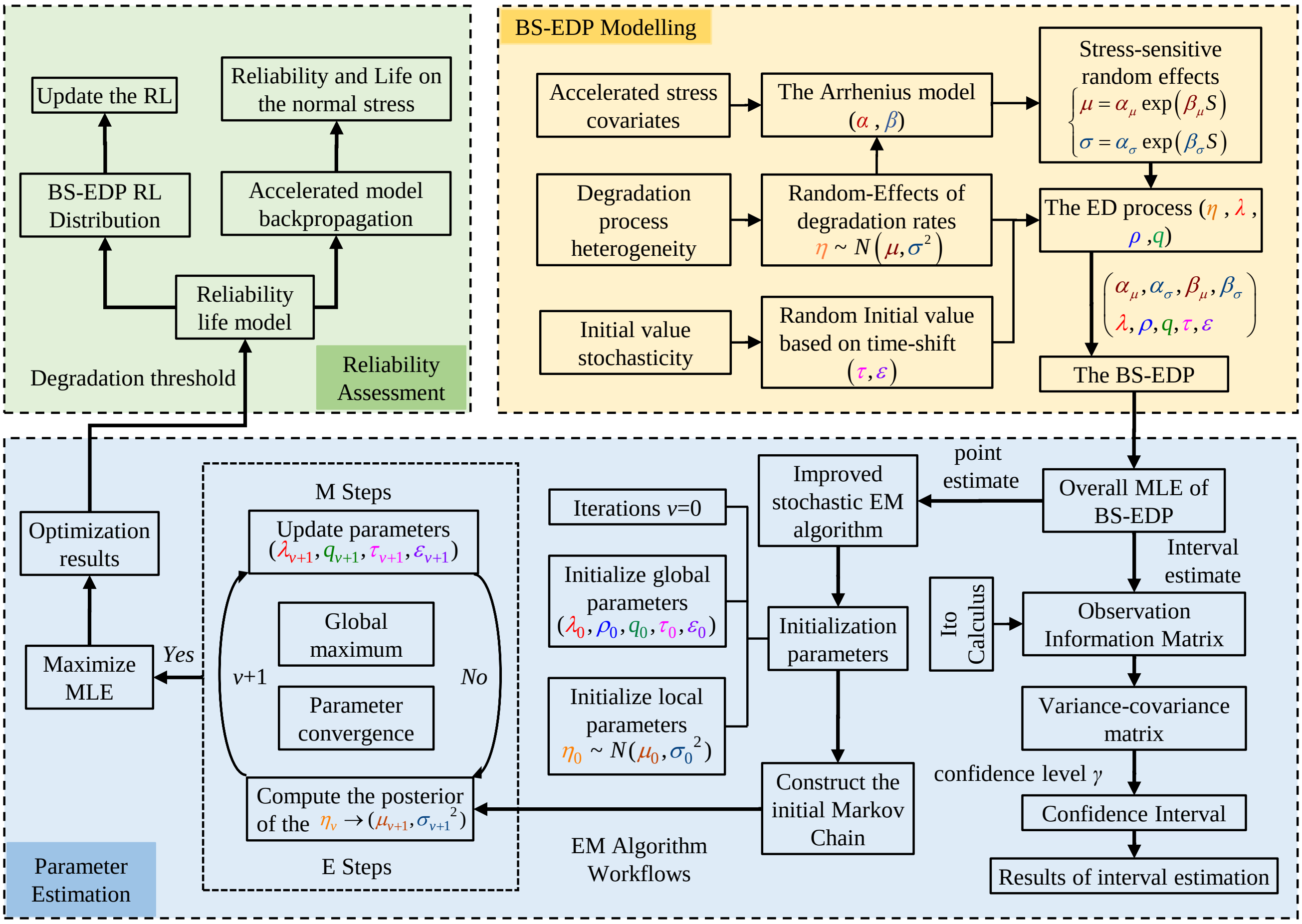


Fig. 1. The pipeline of the proposed BS-EDP-based degradation modelling framework.

4. Simulation case studies

In this paper, multiple sets of Monte Carlo simulation tests are performed to show the correctness, stability and convergence of the method. The simulation data for two sets of degradation, n=50 and n=200, are first analyzed using the model proposed in this paper to give the estimation results and to compare the errors in parameter estimation. Next, real accelerated degradation tests are simulated to assess the time-shift-based random initial value and degradation heterogeneity under accelerated stress covariates. The parameters of the BS-EDP are also estimated to extrapolate reliability metrics under the normal stress, which are ultimately compared with the original real values.

4.1. Simulation and parameter estimation validation without considering acceleration

This part introduces the simulation results of the BS-ED model that does not consider the random effects of accelerated stress. The prior model and parameters of the model are as follows:

$$Y(t) \sim TED(N(1,0.1^2) \cdot t^{1.2}, 10,2.5,0,0.25) \tag{45}$$

where $N(1,0.1^2)$ indicates that the heterogeneity of the degradation rate satisfies a Gaussian process with a mean of 1 and a standard deviation of 0.1.

The two parameters considering the time shift are realized by data interception after simulating the trials. Firstly, 10 sets of degradation data of 5 products and 20 sets of degradation data of 10 products, i.e., two sets of tests with n=50 and n=200, were simulated respectively, and τ = 0.25 and ε = 0 were intercepted as the true values of the time shifts. After obtaining the simulation data, parameter estimation is performed using our proposed method.

In order to ensure the stability of the model parameter solution, it is necessary to compare the results of the parameter estimation with the true value, and the commonly used error measurement indexes are relative error (RE), mean absolute error (MAE), and mean square error (MSE). This paper mainly compares these three types of error indexes, and the results are shown in Table 3.

Table 3. Parameter estimation results and errors from true values.

n	parameters	$\rho = 2.5$	$\lambda = 10$	$q = 1.2$	$\tau = 0.25$	$\varepsilon = 0$	$\mu = 1$	$\sigma = 0.1$
50	estimated value	2.0827	13.5148	1.0008	0.3681	0.1186	1.2221	0.1503
	RE	0.1669	0.3515	0.1660	0.4724	0.1186	0.2221	0.5035
	MAE	0.4173	3.5148	0.1992	0.1181	0.1186	0.2221	0.0503
	MSE	3.48E-03	2.47E-01	7.94E-04	2.79E-04	2.81E-04	9.86E-04	5.07E-05
200	estimated value	2.3908	10.7369	1.1677	0.2550	0.0109	1.1001	0.1132
	RE	0.0437	0.0737	0.0269	0.0200	0.0109	0.1001	0.1320
	MAE	0.1092	0.7369	0.0323	0.0050	0.0109	0.1001	0.0132
	MSE	5.96E-05	2.72E-03	5.21E-06	1.25E-07	5.94E-07	5.01E-05	8.71E-07

It can be seen that with the increase in the number of samples, the errors of the estimated values of each parameter compared with the true values are significantly reduced, and the MSEs at n = 200 are all controlled within 1.00E-2, which indicates that the model has good convergence and stability.

According to the parameter estimation results, the time-shift parameters $\tau = 0.2550$ and $\varepsilon = 0.0109$ at $n = 200$ are very close to the real values of the simulated interception, which can be a good measure of the initial randomness. After calculation, the heterogeneity of the degradation rate is expressed as $N(1.1001, 0.1132^2)$, which is close to the true value.

4.2. Validation of simulation and parameter estimation under accelerated test

A product is subjected to accelerated degradation test, the normal operating temperature is 298K, the accelerated temperature stress is 333K and 353K, the limiting temperature is 383K.

As $Y(\Lambda(t)) \sim TED\left(N\left(\alpha_\mu exp(\beta_\mu S), \alpha_\sigma^2 exp(2\beta_\sigma S)\right) \cdot t^q, \lambda, \rho\right)$, where the initial value is random:

$$Y(0) - \varepsilon = X(\tau^q) \sim TED\left(N\left(\alpha_\mu exp(\beta_\mu S), \alpha_\sigma^2 exp(2\beta_\sigma S)\right) \cdot \tau^q, \lambda, \rho\right) \tag{46}$$

Therefore, the parameters to be solved are $\alpha_\mu, \beta_\mu, \alpha_\sigma, \beta_\sigma, \lambda, \rho, q, \tau, \varepsilon$.

Monte Carlo simulations of accelerated degradation tests are performed, with 20 sets of degradation data for each of 10 products randomly selected, for a total of 200 sets, with a test interval of 0.25 units time. When the amount of degradation reaches 20%, we consider the product to be failure.

The simulation parameters are set as follows: global parameter $\lambda, \rho, q, \tau, \varepsilon = (8, 2.2, 1.2, 1.25, 0.1)$, random effect local parameter $\alpha_\mu, \beta_\mu, \alpha_\sigma, \beta_\sigma = (3, 0.1, 0.4, 0.6)$. The degradation trajectory of the simulated data is shown in Fig. 2, and for the equivalent initial performance randomization, the data before t = 1.25 is deleted, and the upper side x-axis coordinate is the new degradation time axis.

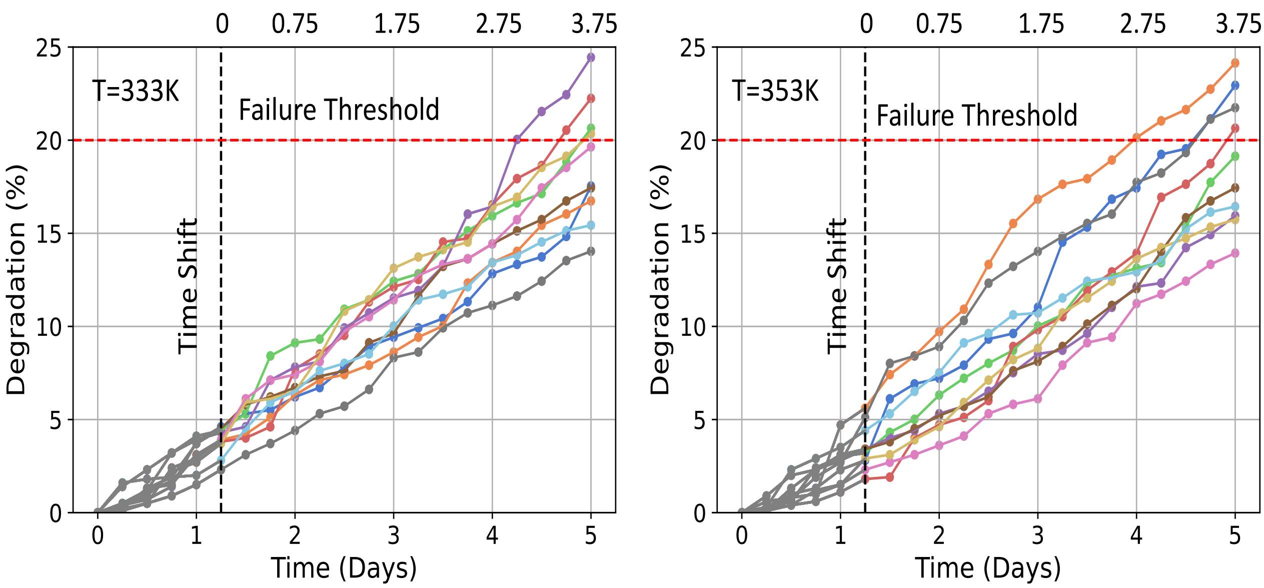


Fig. 2. Simulation data and degradation trajectories under different accelerated degradation test.

The estimation is carried out using the method proposed in this paper and the results of parameter estimation are shown in

Table 4.

Table 4. Parameter estimation results and errors from true values under simulated accelerated degradation tests.

Parameters	results	RE	MAE	MSE
$\rho = 2.2$	2.1570	1.95E-02	4.30E-02	9.24E-06
$\lambda = 8$	8.0029	3.62E-04	2.89E-03	4.18E-08
$q = 1.2$	1.1516	4.04E-02	4.84E-02	1.17E-05
$\alpha_\mu = 3$	3.0240	8.01E-03	2.40E-02	2.89E-06
$\alpha_\sigma = 0.1$	0.1152	1.52E-02	1.52E-02	1.16E-06
$\beta_\mu = 0.4$	0.3858	3.56E-02	1.42E-02	1.01E-06
$\beta_\sigma = 0.6$	0.6629	1.05E-01	6.29E-02	1.98E-05
$\tau = 1.25$	1.2455	3.60E-03	4.50E-03	1.01E-07
$\varepsilon = 0.1$	0.0785	2.15E-01	2.15E-02	2.31E-06

Based on the estimated results, it can be seen that RE , MAE and MSE are controlled within 25%, 10% and 1%, respectively, which indicates that the model has good stability.

In addition, even under the condition of accelerated test, the degradation process still exists heterogeneity, σ and μ still satisfy the Gaussian distribution, as shown in Fig. 3. Compared with the case of normal stress, the mean and variance of η under acceleration are larger, the specific relationship is:

$$\begin{cases} \eta_1 \sim N(3.1936, 0.5280^2) = \\ N\left(3.0240e^{0.1152\times0.4736}, (0.3857e^{0.6629\times0.4736})^2\right) \\ \eta_2 \sim N(3.2788, 0.6144^2) = \\ N\left(3.0240e^{0.1152\times0.7020}, (0.3857e^{0.6629\times0.7020})^2\right) \end{cases} \quad (47)$$

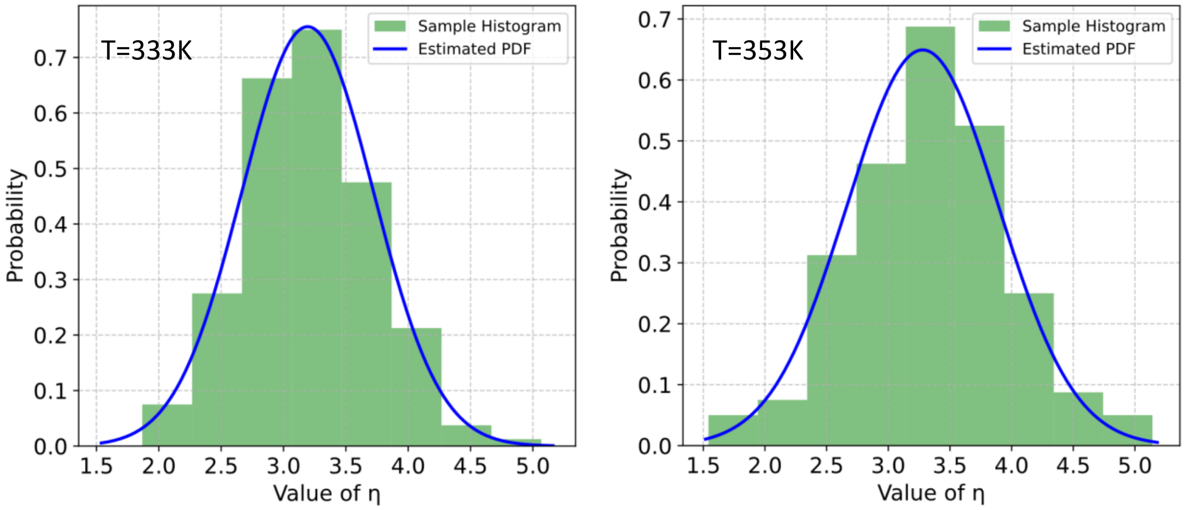


Fig. 3. Heterogeneity distribution of η under accelerated degradation test.

For the acceleration test data, the reliability metrics under the normal stress are extrapolated, and shown in Fig. 4.

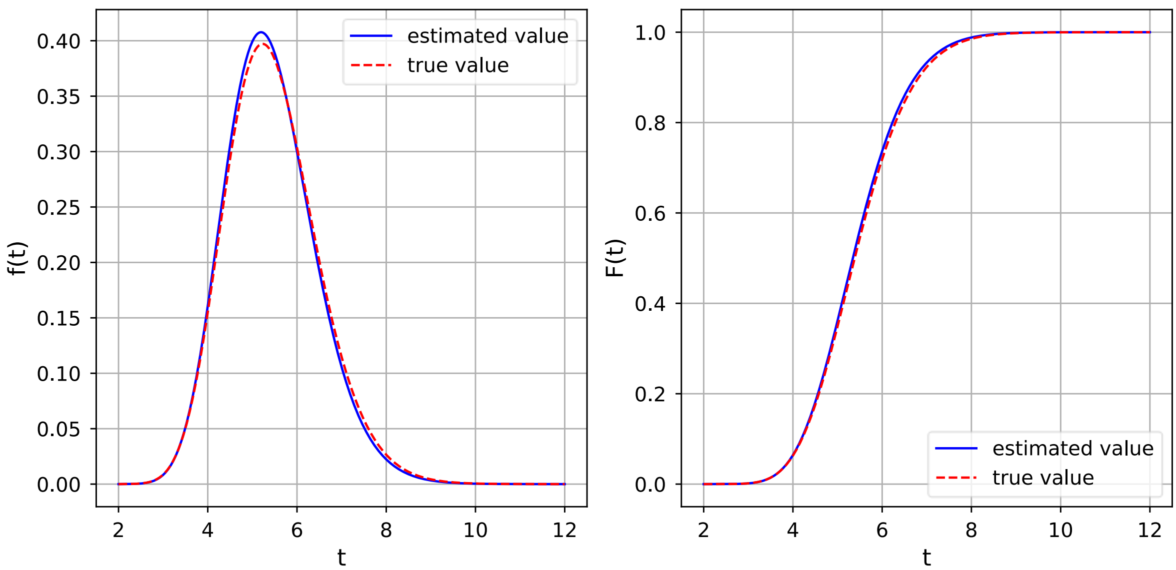


Fig. 4. CDF and PDF of life under normal operating conditions.

From the CDF and PDF, it can be seen that the reliability and life estimation results have less error compared to the true values, and the estimated median life, characteristic life, and the errors, as tabulated in Table 5. It can be seen that the extrapolated life error is controlled within 1%.

Table 5. Lifetime and errors for extrapolation.

	actual value	estimated value	RE	MAE	MSE
Median lifetime	5.3833	5.3422	7.63E-03	4.11E-02	8.45E-06
Characteristic lifetime	5.7349	5.6844	8.81E-03	5.05E-02	1.28E-05

5. Practical case studies

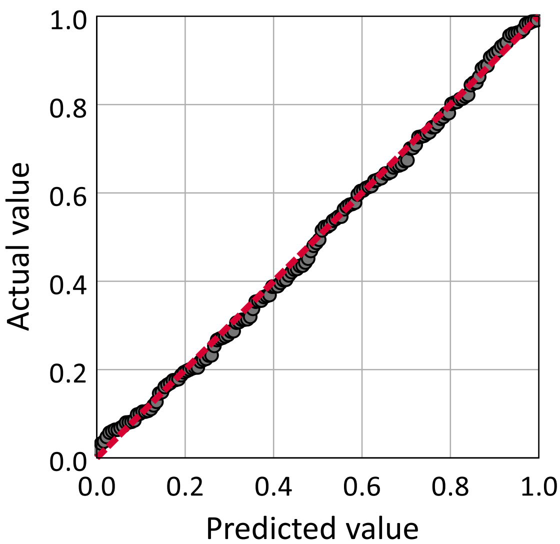
In this section, we analyze real cases of GaAs laser and LED chips for validation, showing that when the initial degradation level is unknown, the BS-ED can describe the degradation more accurately and realistically. One of the LED chip cases was also used to verify the accuracy of the extrapolation of accelerated degradation to normal operating life.

5.1. GaAs laser

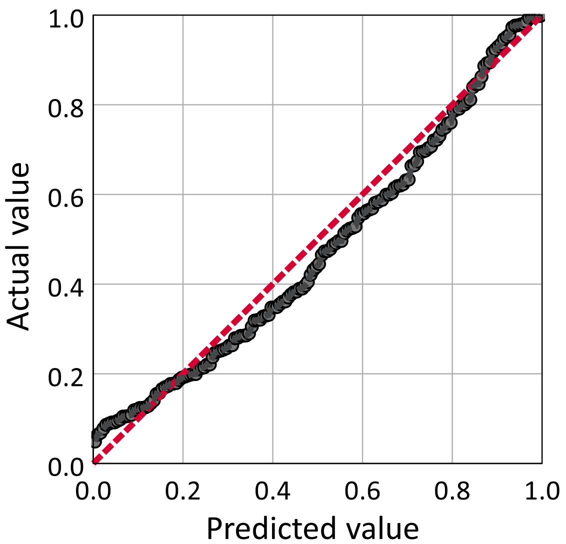
We selected the laser device degradation set proposed by Meeker at a temperature stress of 80 °C and 110 °C 36, the product measured the operating current every 250 hours interval until the test was stopped at 4000 hours, a total of 16 times. The failure criterion of GaAs lasers is the operating current, when the product laser intensity remains consistent, the operating current exceeds 10mA judged as failure. We randomly select the data of 10 sets of these products to carry out the degradation trajectory analysis, model comparison and parameter estimation as well as life extrapolation.

In the modelling process, we need to pay attention to the actual degradation trajectory of the product, so before carrying out the analysis, we need to determine the form of Λ(t). Through the analysis, it can be obtained that the RMSE using the power function and exponential function is 1.1724 and 1.2602, respectively, and the former is smaller, so the fit is better using the power function, and at this time the shape function is $\Lambda(t)=t^{q}$.

Meanwhile, Fig. 5 gives the normality P-P plot and skewness P-P plot (ED process for the assumed distribution) for these 10 sets of data, which can be seen that the data is asymmetric as a whole, which is further evidence that the data satisfies the skewness distribution.



(a) ED process bias test



(b) normality test

Fig. 5. P-P plot for 10 sets of data.

In addition, in order to prove the optimality of considering the BS-ED model, it is necessary to compare it with other models. Akaike Information Criterion (*AIC*), Bayesian Information Criterion (*BIC*) and *AICc* are introduced to prove the goodness of fit and the best fitting stochastic process is selected. Notably, (1) Both *AIC* and *BIC* consider fit and complexity, but penalize complexity differently, with *BIC* being stricter on parameters. (2) *AICc* corrects *AIC*, suitable for small samples to prevent overfitting. (3) With large samples, *AIC*, *BIC*, and *AICc* give similar results. But for small samples, *AICc* is preferred 37. Therefore, this paper concurrently assesses three information criteria for model fit, rigorously considering complexity and preventing overfitting in small or limited samples. The goodness of fit is defined as follows:

$$\begin{cases} AIC = 2k - 2L \\ AICc = AIC + \frac{2k(k+1)}{(n-k-1)} \\ BIC = k\ln(n) - 2L \end{cases} \tag{48}$$

where L is the log likelihood, k is the number of parameters, and n is the number of samples.

After determining the degradation trajectories, we calculate the parameter values and information criterion scales of the Wiener process, the Gamma process, the IG process, and the generalized trajectory ED process, respectively, and use them to compare the relative goodness of fit of the models, and the results are shown in Table 6.

Table 6. Parameter values and Information Criterion for different models of the same GaAs lasers.

	Wiener	Gamma	IG	ED
λ	6.5540	30.2979	59.8611	42.0377
η	2.1181	2.1181	2.1181	2.1181
ρ	0	2	3	2.4703
MLE	37.0616	50.2981	50.1522	50.8073
AIC	-70.1232	-96.5961	-96.3045	-97.6145
BIC	-63.9728	-90.4458	-90.1541	-91.4642
AICc	-70.0468	-96.5197	-96.2281	-97.5381

Through further calculations, it is found that the ED model has a smaller value of the information criterion scale than other conventional models, and therefore more accurately describes the degradation process of GaAs lasers.

As can be seen from the original data, the initial degradation level is 0. In order to illustrate the use scenario and performance of the model, we delete the first 4 sets of incremental data to assume that the product has been degraded for 1000 hours at the beginning of the test and consider this moment to be 0, and to obtain a set of data with a random initial performance. The degradation trajectory can be represented in Fig. 6, with the gray color indicating the deleted data and the horizontal coordinates on the top side indicating the true time of the new degradation trajectory.

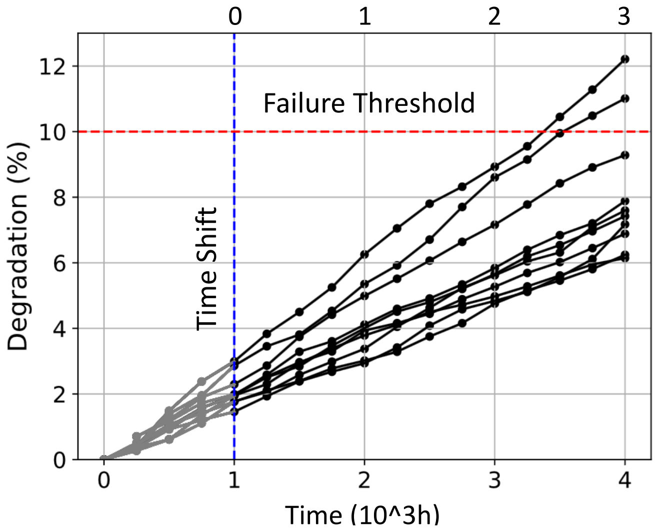


Fig. 6. Degenerate trajectories considering initial value randomization.

In this paper, we simultaneously consider the initial degradation random, degradation process heterogeneity separately, and compare it with the generalized ED model without component heterogeneity to illustrate the accuracy of the description of the BS-ED model when there is a case of initial value randomness, as shown in Table 7. Where the generalized trajectory ED model is M1[8], the ED process considering initial value randomness is M2[20], the ED process considering degradation heterogeneity is M3[7], and the BS-ED proposed in this paper is M0. Parameter estimation is carried out and the results are shown in Table 7.

Table 7. statistical inference results from different models.

parameters	M1	M2	M3	M0
ρ	2.3844	2.4814	2.3557	2.4703
λ	39.4591	40.4285	39.7228	42.0377
q	0.9802	0.9999	0.9549	0.9751
$\mu\,(\eta)$	2.0727	2.0463	2.1311	2.1181
σ	-	-	0.5319	0.5202
τ	-	1.1234	-	0.9628
ε	-	0.0267	-	0.0229
log-likelihood	40.0108	40.2800	64.3890	65.0974
AIC	-72.0216	-72.5600	-118.7781	-120.1949
BIC	-60.8717	-61.4100	-103.4022	-104.8190
AICc	-71.6738	-72.2122	-118.3859	-119.8027

It can be seen that the value of the information criterion decreases significantly after considering the randomness of the initial values and the heterogeneity of the degradation process, indicating a more accurate description of the true situation. τ = 0.9628 estimated by the BS-ED model indicates that at the beginning of the test, the products of the 10 groups on average had already produced degradation equivalent to about 962.8 hours or so at the same level of stress, which is close to the true value of 1000 hours.

In addition, in the process of EM algorithm, the *E*-step obtained the posterior distribution of heterogeneity of η as $N(2.1181, 0.5202^2)$ by MCMC sampling, as shown in Fig. In addition, the P-P plot of the goodness-of-fit of the independent heterogeneity η of each test piece with the posterior distribution is shown in Fig. 7, which shows that the heterogeneity distribution obtained by the *E*-step is very close to the real situation.

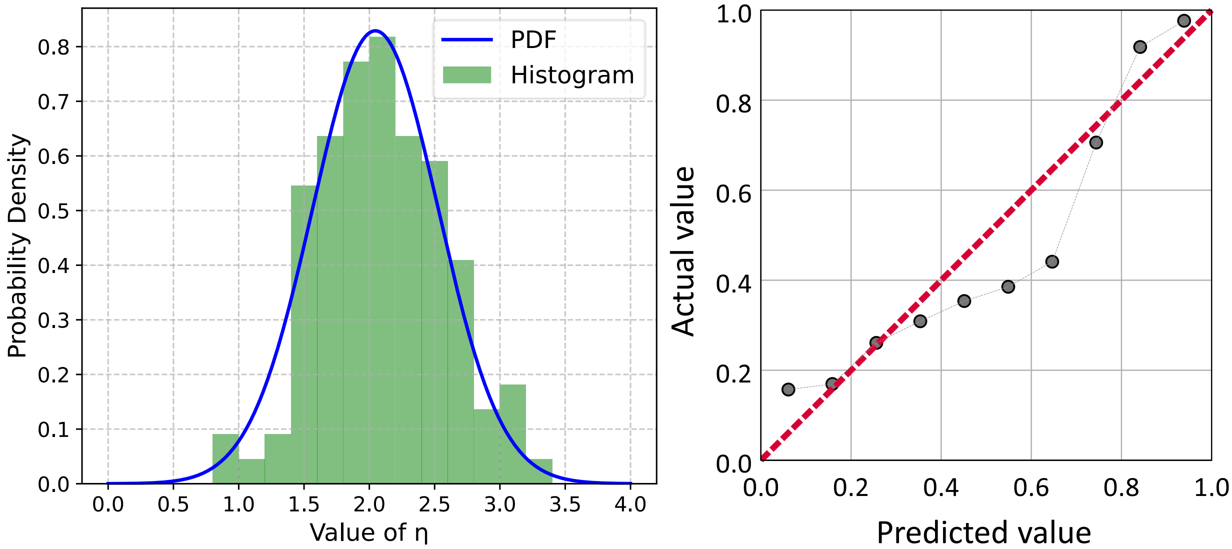


Fig. 7. Heterogeneity distribution of η.

Therefore, the model considers the degradation before the test and incorporates the uncertainty of the initial performance of the product and the degradation process, which helps to retrieve the real operation and degradation time of the product and makes the degradation modeling more reasonable. The reliability assessment of the final solution accelerated stress case considering double random variables and without is shown in Fig. 8.

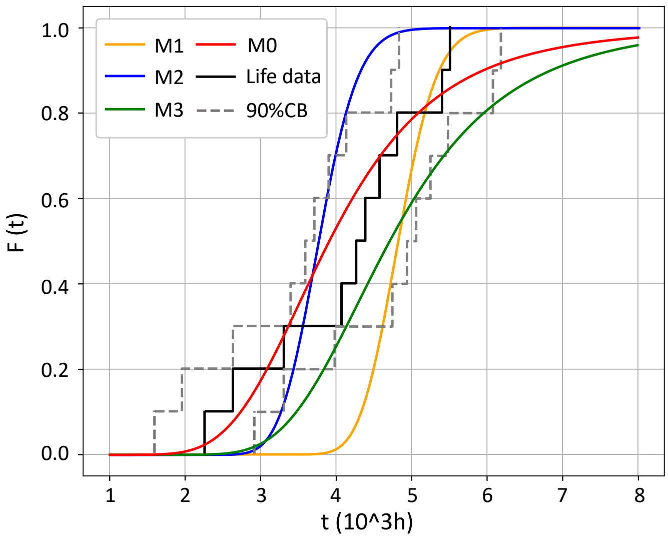


Fig. 8. Results of reliability assessment.

From the above results, it can be concluded that describing the degradation behavior with the ED process considering the initial value stochastic is closer to the real situation, and the BS-ED process fits better. Finally, the median lifetime of BS-ED can be obtained as 3747.6 h with a relative error of 8.60%, and the median lifetime of the traditional ED model is 4824.7 h with a relative error of 17.68%.The error of BS-ED is smaller and the lifetime is shorter compared to the traditional model, which is in line with the actual situation where the time-shift of about 1000 h is considered.

5.2. LED chips

LED chips are very common in aircraft, mainly used for illumination and signal indication, and failure may lead to improper operation by the crew during flight. Therefore, it is necessary to analyze the reliability of LED chips. LED chips are generally judged by the decay value of luminous flux to obtain the test data, the conventional method requires a long test time, and the traditional reliability test method is long and costly, which seriously affects the development progress of products. In order to illustrate the validity of the proposed model, the accelerated degradation test of LED chips is carried out with temperature as the accelerating variable.

Tests were carried out with LED chips from GREE, and the product-related indexes are shown in Table 8.

Table 8. Parameters of LED chips under test.

Performance indicators	Values
Rating	3W
Rated voltage	3-3.6V
Rated current	0.6-1A
Color temperatures	6500K White
Junction temperature under normal operation	85℃
Theoretical maximum ambient temperature	150℃
Theoretical light intensity at 15cm	560Lux
Actual life span	20000h-25000h

To ensure that the failure mechanism is not changed under accelerated stress, the test needs to be carried out under the requirements of thermal design. The relationship between its ambient temperature and maximum current is shown in Fig. 9.

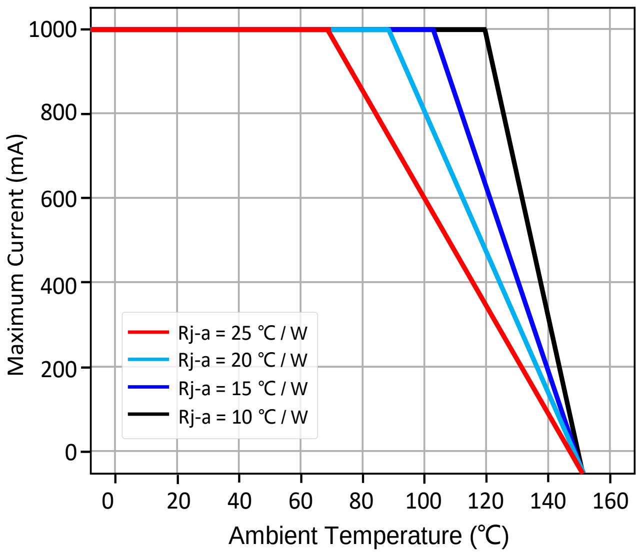


Fig. 9. Thermal design curve of LED chip.

The daily working stress of the product is 25°C. In order to ensure that the failure mechanism does not change, the product is operated at a rated current of 0.6A, and accelerated stresses of 60°C and 100°C are applied, with an ultimate stress of 150°C. The product is then subjected to an accelerated test with a constant current source. The test platform is shown in fig. 10, the tested parts is placed in the accelerated test chamber, powered by a constant current source and 5 groups of LED chips are connected in series to ensure a consistent operating current.

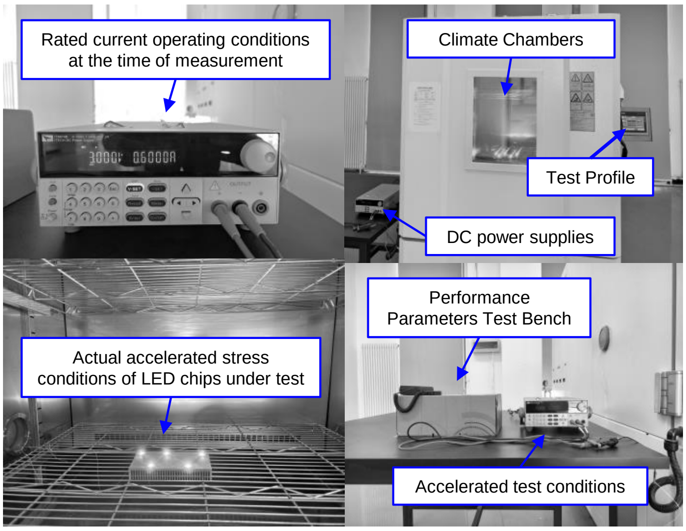


Fig. 10. Accelerated Degradation Testbed.

The luminous flux was measured once every 100 hours, and the light intensity of each tested piece at the vertical 15cm was collected separately by using an illuminance meter during the test, which lasted for 2000h, and the failure threshold was defined as 70% of the initial luminance, and the results of the test are shown in Fig. 11. It can be seen that there are obvious differences in the initial state and heterogeneity in the degradation process. The data are used for model building and parameter estimation.

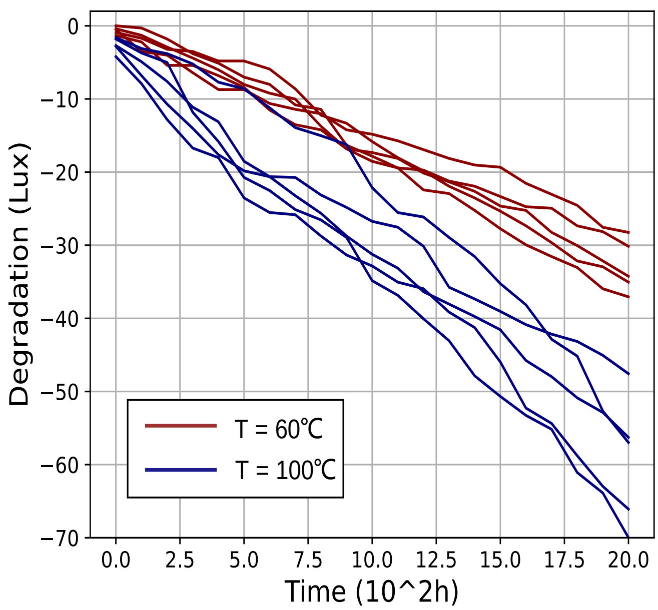


Fig. 11. Actual degradation data of LED chips under accelerated test.

We first compare the goodness of fit of different degradation trajectories, the degradation trajectory is a power function with an RMSE of 8.5161, and when it is an exponential function, the

RMSE is 8.8920, which is a better fit for the power function. It is also important to compare the goodness of fit of different stochastic degradation processes, as shown in Table 9, which lists all the parameter estimation results and Information Criterion.

Table 9. Parameter values and Information Criterion for different models of the same LED chips.

	Wiener	Gamma	IG	ED
λ	0.5129	1.6573	0.0655	1.5425
η	2.2273	2.2272	2.2272	2.2272
ρ	0	2	3	1.4387
MLE	-350.5544	-333.0783	-353.0723	-312.2451
AIC	705.1087	670.1566	710.1446	630.4902
BIC	711.7054	676.7533	716.7413	640.3851
AICc	705.1696	670.2175	710.2055	630.6126

It can be seen that the ED process better meets the actual degradation situation of LED chips. Based on this, the proposed BS-ED model is substituted to solve the parameters and compared with the ED model. The results of the parameter estimation for the two sets of experimental data are also given, as shown in Table 10.

Table 10. The parameter estimations.

	nonlinear ED (overall)	nonlinear ED (60℃)	nonlinear ED (100℃)	BS-ED (overall)	BS-ED (60℃)	BS-ED (100℃)
ρ	1.4395	1.2662	1.4916	1.4400	1.2820	1.4941
λ	1.6394	1.8609	2.1915	1.4274	1.6412	1.9375
q	0.9497	1.0184	0.9101	1.0547	1.1275	1.0128
$\mu(\eta)$	2.5894	1.5301	3.7150	1.8907	1.1032	2.7311
σ	-	-	-	0.5774	0.1318	0.3795
τ	-	-	-	0.0001	0.0001	0.0001
ε	-	-	-	0.3012	0.3013	0.2710
MLE	-312.8726	-117.9886	-174.5451	-290.8251	-118.6096	-169.8676
AIC	631.7452	241.9772	355.0902	591.6502	247.2191	349.7353
BIC	641.6402	249.7927	362.9057	608.1417	260.2450	362.7611
AICc	631.8677	242.2272	355.3402	591.9594	247.8507	350.3668

The overall estimated BS-ED model has a smaller scale of information criterion and fits the actual data better. It is further found that the time-shift parameter of the LED chip of this model is close to 0, about 0.01h, which can be learnt that the factory testing and transport almost did not affect the performance of the product. However, the parameter of the initial value of the quality characteristics is larger, and the initial value of the performance has a large discrepancy when it is shipped out of the factory, probably due to imperfections in the process and quality control. This is also reflected in the fact that the light intensity of the different test pieces was significantly different during the first test before the test.

In addition, the overall estimation ensures that the failure mechanisms of the product under different stresses are statistically consistent, which is conducive to accelerated life extrapolation, which can also be derived from the information criterion that the BS-ED model fits better for the overall estimation but not necessarily for the separate estimation. Therefore, when extrapolating the life, the accuracy of the overall and separate estimates needs to be concerned.

Next, the lifetimes under accelerated tests are evaluated and compared simultaneously with the conventional ED model, the overall estimated BS-ED model, and the separately estimated BS-ED model, with the degree of fit to the real soft failure data, as shown in Fig. 12.

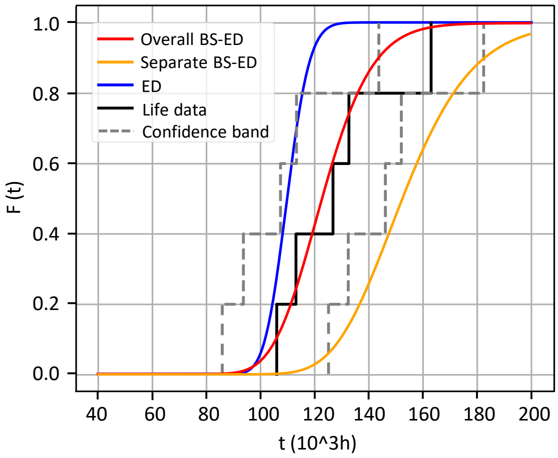


Fig. 12. Reliability assessment results.

It can be seen that the BS-ED process fits better compared to the conventional ED process, and compared to the separate estimation, the overall estimation of the results at 60°C has a better fit, and both fit better at 100°C.

Next, the lifetime under normal use is extrapolated based on the statistical extrapolation of the BS-ED results. The model is very complex due to the large number of local parameters satisfied by the degradation parameters of the model after considering acceleration. Therefore, the MCMC method is used to perform Metropolis-Hasting sampling based on the posterior distribution of η-heterogeneity in the accelerated case, which is equivalent to performing the E-step in the EM algorithm. After that, the *M*-step is executed to estimate the parameter $\Theta_\eta = (\alpha_\mu, \beta_\mu, \alpha_\sigma, \beta_\sigma)$ in $\eta \sim N\left(\alpha_\mu \exp(\beta_\mu S), \alpha_\sigma^2 \exp(2\beta_\sigma S)\right)$. the final result of parameter estimation is obtained as shown in Table 11.

Table 11. Parameter estimation results for stress covariates.

Parameters	Estimated results
α_μ	0.7411
β_μ	1.7327
α_σ	0.0475
β_σ	2.8715

In addition, we give a comparison of the independent degradation heterogeneity of the same product under accelerated case and normal stress, as shown in Fig. 13. It can be seen that the mean and variance of the degradation related parameters become larger as the applied stress level increases. It can be seen that at higher stress levels, not only the degradation rate of the product increases and the lifetime decreases, but also the heterogeneity of the degradation process of different test pieces is magnified. This conclusion is particularly evident in the case of temperature-sensitive LED chips.

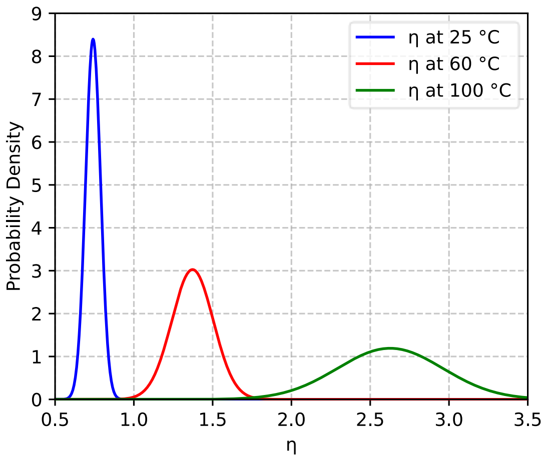


Fig. 13. Heterogeneity of degradation parameters under accelerated vs. normal stresses.

For this LED chip, accelerated degradation test is carried out, and the results show that the median life of the BS-ED model extrapolated to normal working conditions is 22628.35 h, while the median life of the traditional ED model extrapolated to 16552.93 h. The extrapolated life of the BS-ED model is in line with the theoretical life, but it is not greater than the maximum theoretical life, which shows that the initial loss or the process level will affect the actual life. In summary, the proposed method can accurately extrapolate the reliability index under normal working conditions.

6. Conclusion

Stochastic models for product degradation are limited in handling heterogeneity in initial values and processes, affecting reliability estimates. The BS-ED model incorporating heterogeneity addresses this. Simulation and case studies reveal that: (1) ED outperforms traditional models in describing degradation. (2) It considers non-zero, varying initial degradation for missing early data, enhancing test confidence. (3) For heterogeneous degradation, BS-ED provides more accurate measures, especially under accelerated stress. (4) BS-ED more realistically assesses reliability and lifetime metrics for heterogeneous products. (5) Considering heterogeneity in reliability metrics helps industrial users with maintenance strategies, preventing premature stoppage or late maintenance risks.

In the future work, we will focus on the following aspects: first, the BS-ED process only models one kind of performance degradation, and the multivariate performance degradation can be established for some products with multiple performance parameters. In addition, this paper only considers the degradation rate parameter and its random effect parameter to satisfy the accelerated process, and does not consider in depth that the other parameters, such as scale parameter, also have random effects, and may be affected by accelerating stresses as well.

In addition, the proposed model is also applicable to modeling the degradation of mechanical products, quantifying the initial performance variations, for instance, in machining spindles, where varying degrees of intrinsic offsets may arise from manufacturing, transportation, and installation. Given the diversity of mechanical component failure modes, the selection of observed performance indicators and measurement of degradation vary significantly among products. This paper primarily focuses on data-driven modeling and statistical inference, excluding failure physics analysis. We envision that future research can integrate data-driven methods with failure physics modeling for degradation modeling and lifespan estimation of specific mechanical product failure modes, a promising research direction.

Appendix A

Firstly, we summarize the procedure for calculating first order partial derivatives as follows:

$$\frac{\partial D}{\partial \varepsilon}=\begin{cases}\frac{1}{\Delta\Lambda_{ij}}\left(\frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}}-\eta_i\right),\rho=0\\ \frac{1}{\Delta\Lambda_{ij}}\cdot ln\left(\frac{\Delta Y_{ij}-\varepsilon Z}{\eta_i\cdot\Delta\Lambda_{ij}}\right),\rho=1\\ \frac{1}{\eta_i\cdot\Delta\Lambda_{ij}}-\frac{1}{\Delta Y_{ij}-\varepsilon Z},\rho=2\\ \frac{1}{(1-\rho)(\Delta Y_{ij}-\varepsilon Z)}\cdot\left(\frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}}\right)^{2-\rho}-\frac{{\eta_i}^{1-\rho}}{(1-\rho)\Delta\Lambda_{ij}},others\end{cases} \tag{A.1}$$

$$\frac{\partial D}{\partial \tau}=\begin{cases}\frac{\Delta Y_{ij}-\varepsilon Z}{{\Delta\Lambda_{ij}}^2}\left(\frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}}-\eta_i\right)\cdot\frac{\partial\Delta\Lambda_{ij}}{\partial\tau},\rho=0\\ \frac{\Delta Y_{ij}-\varepsilon Z}{{\Delta\Lambda_{ij}}^2}\cdot ln\left(\frac{\Delta Y_{ij}-\varepsilon Z}{\eta_i\cdot\Delta\Lambda_{ij}}\right)\cdot\frac{\partial\Delta\Lambda_{ij}}{\partial\tau},\rho=1\\ \left(\frac{\Delta Y_{ij}-\varepsilon Z}{\eta_i\cdot{\Delta\Lambda_{ij}}^2}-\frac{1}{\Delta\Lambda_{ij}}\right)\cdot\frac{\partial\Delta\Lambda_{ij}}{\partial\tau},\rho=2\\ \left[\frac{1}{(1-\rho)\Delta\Lambda_{ij}}\left(\frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}}\right)^{2-\rho}-\frac{{\eta_i}^{1-\rho}(\Delta Y_{ij}-\varepsilon Z)}{(1-\rho){\Delta\Lambda_{ij}}^2}\right]\cdot\frac{\partial\Delta\Lambda_{ij}}{\partial\tau},others\end{cases} \tag{A.2}$$

Generally, when ρ is uncertain:

$$\begin{aligned}\frac{\partial D}{\partial\rho}=&\frac{\eta_i^{1-\rho}\cdot(\Delta Y_{ij}-\varepsilon Z)}{(1-\rho)^2\Delta\Lambda_{ij}}+\frac{\eta_i^{2-\rho}\cdot\ln\,\eta_i}{2-\rho}+\frac{\ln\left(\frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}}\right)\cdot\left(\frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}}\right)^{2-\rho}}{(1-\rho)(2-\rho)}\\ &-\frac{\eta_i^{1-\rho}\cdot\ln\,\eta_i\cdot(\Delta Y_{ij}-\varepsilon Z)}{(1-\rho)\Delta\Lambda_{ij}}-\frac{\left(\frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}}\right)^{2-\rho}}{(1-\rho)^2(2-\rho)}-\frac{\left(\frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}}\right)^{2-\rho}}{(1-\rho)(2-\rho)^2}-\frac{\eta_i^{2-\rho}}{(2-\rho)^2}\end{aligned} \tag{A.3}$$

where ρ is determined by the product characteristics and does not change with the test setup and stress level, and in the optimization problem the parameter is considered to be related only to the product and can be obtained by extrapolation from a priori information.

In the case of engineering practice, it can be considered as a constant in the BS-EDP parameter estimation.

In addition, the parameter solution of the time covariate requires attention to the solution method of $\frac{\partial D}{\partial q}$, specifically:

$$\frac{\partial D}{\partial q}=\begin{cases}\left[\frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}}-\eta\right]\frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}^{2}}\cdot\frac{\partial\Delta\Lambda_{ij}}{\partial q},\rho=0\\ \left[ln\left(\frac{\Delta Y_{ij}-\varepsilon Z}{\eta\cdot\Delta\Lambda_{ij}}\right)-1\right]\cdot\frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}^{2}}\cdot\frac{\partial\Delta\Lambda_{ij}}{\partial q}-\frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}^{2}},\rho=1\\ \left(\frac{\Delta Y_{ij}-\varepsilon Z}{\eta\cdot\Delta\Lambda_{ij}^{2}}-\frac{\Delta Y_{ij}-\varepsilon Z}{\Delta\Lambda_{ij}}\right)\cdot\frac{\partial\Delta\Lambda_{ij}}{\partial q},\rho=2\\ \left[\frac{(\Delta Y_{ij}-\varepsilon Z)^{2-\rho}}{(1-\rho)\Delta\Lambda_{ij}^{3-\rho}}-\frac{\eta^{1-\rho}(\Delta Y_{ij}-\varepsilon Z)}{(1-\rho)\Delta\Lambda_{ij}^{2}}\right]\cdot\frac{\partial\Delta\Lambda_{ij}}{\partial q},\rho\neq 1,2\end{cases}\tag{A.4}$$

When the degenerate trajectory is nonlinear and indeterminate, i.e., when it satisfies the power function $\Lambda(t)=t^{q}$ and the exponent $\Lambda(t)=exp(qt)-1$: $\Delta\Lambda_{ij}=(t_{ij}+\tau)^{q}-(t_{i(j-1)}+\tau)^{q}$; $\Delta\Lambda_{ij}=exp[q(t_{ij}+\tau)]-exp[q(t_{i(j-1)}+\tau)]$.

We have

$$\frac{\partial\Delta\Lambda_{ij}}{\partial\tau}=\begin{cases}\frac{q\cdot(t_{ij}+\tau)^{q}}{t_{ij}+\tau}-\frac{q\cdot(t_{i(j-1)}+\tau)^{q}}{t_{i(j-1)}+\tau},\Lambda=t^{q}\\ q\cdot exp[q(t_{ij}+\tau)]-q\cdot exp[q(t_{i(j-1)}+\tau)],\Lambda=exp(qt)\end{cases}\tag{A.5}$$

$$\frac{\partial\Delta\Lambda_{ij}}{\partial q}=\begin{cases}(t_{ij}+\tau)^{q}\cdot ln(t_{ij}+\tau)-(t_{i(j-1)}+\tau)^{q}\cdot ln(t_{i(j-1)}+\tau),\Lambda=t^{q}\\ (t_{ij}+\tau)\cdot exp[q(t_{ij}+\tau)]-(t_{i(j-1)}+\tau)\cdot exp[q(t_{i(j-1)}+\tau)],\Lambda=exp(qt)\end{cases}\tag{A.6}$$

As can be seen from the partial derivatives computational process, is mainly concerned with the second partial derivatives of the ED-distributed stochastic process deviation function *D* with respect to the parameters ρ, τ, ε, q and the second partial derivatives of the time covariate function $\Lambda(t)$ with respect to τ, q. The specific solution method is shown below:

(1) Second partial derivatives of the stochastic process deviation function *D*

When $\Lambda(t)$ satisfies the power function:

$$\frac{\partial^{2}D}{\partial q^{2}}=\begin{cases}\left(\frac{2\Delta_{ij}}{\Delta\Lambda_{ij}^{3}}-\frac{3\Delta_{ij}^{2}}{\Delta\Lambda_{ij}^{4}}\right)\left(\frac{\partial\Delta\Lambda_{ij}}{\partial q}\right)^{2}+\left(\frac{\Delta_{ij}^{2}}{\Delta\Lambda_{ij}^{3}}-\frac{\eta\Delta_{ij}}{\Delta\Lambda_{ij}^{2}}\right)\frac{\partial^{2}\Delta\Lambda_{ij}}{\partial q^{2}},\rho=0\\ \frac{\partial\Delta\Lambda_{ij}}{\partial q}\left(\frac{2\Delta_{ij}}{\Delta\Lambda_{ij}^{3}}+1\right)+\left[ln\left(\frac{\Delta_{ij}}{\eta\cdot\Delta\Lambda_{ij}}\right)-1\right]\frac{\Delta_{ij}}{\Delta\Lambda_{ij}^{2}}\frac{\partial^{2}\Delta\Lambda_{ij}}{\partial q^{2}}-\left[2\,ln\left(\frac{\Delta_{ij}}{\eta\cdot\Delta\Lambda_{ij}}\right)\frac{\partial\Delta\Lambda_{ij}}{\partial q}+1\right]\frac{\Delta_{ij}}{\Delta\Lambda_{ij}^{3}}\frac{\partial\Delta\Lambda_{ij}}{\partial q},\rho=1\\ \left(\frac{\Delta_{ij}}{\Delta\Lambda_{ij}^{2}}-\frac{2\Delta_{ij}}{\eta\cdot\Delta\Lambda_{ij}^{3}}\right)\frac{\partial\Delta\Lambda_{ij}}{\partial q}+\left(\frac{\Delta_{ij}}{\eta\cdot\Delta\Lambda_{ij}^{2}}-\frac{\Delta_{ij}}{\Delta\Lambda_{ij}}\right)\frac{\partial^{2}\Delta\Lambda_{ij}}{\partial q^{2}},\rho=2\\ \left[\frac{\Delta_{ij}^{2-\rho}}{(1-\rho)\Delta\Lambda_{ij}^{3-\rho}}-\frac{\eta^{1-\rho}\Delta_{ij}}{(1-\rho)\Delta\Lambda_{ij}^{2}}\right]\frac{\partial^{2}\Delta\Lambda_{ij}}{\partial q^{2}}+\left[\frac{(\rho-3)\Delta_{ij}^{2-\rho}}{(1-\rho)\Delta\Lambda_{ij}^{4-\rho}}+\frac{2\eta^{1-\rho}\Delta_{ij}}{(1-\rho)\Delta\Lambda_{ij}^{3}}\right]\left(\frac{\partial\Delta\Lambda_{ij}}{\partial q}\right)^{2},\rho\neq 1,2\end{cases}\tag{A.7}$$

$$\frac{\partial^{2}D}{\partial\tau^{2}}=\begin{cases}\left(\frac{2\eta_{i}\Delta_{ij}}{\Delta\Lambda_{ij}^{3}}-\frac{3\Delta_{ij}^{2}}{\Delta\Lambda_{ij}^{4}}\right)\left(\frac{\partial\Delta\Lambda_{ij}}{\partial\tau}\right)^{2}+\left(\frac{\Delta_{ij}^{2}}{\Delta\Lambda_{ij}^{3}}-\frac{\eta_{i}\Delta_{ij}}{\Delta\Lambda_{ij}^{2}}\right)\frac{\partial^{2}\Delta\Lambda_{ij}}{\partial\tau^{2}},\rho=0\\ \frac{\Delta_{ij}}{\Delta\Lambda_{ij}^{2}}ln\left(\frac{\Delta_{ij}}{\eta_{i}\cdot\Delta\Lambda_{ij}}\right)\frac{\partial^{2}\Delta\Lambda_{ij}}{\partial\tau^{2}}-\frac{\Delta_{ij}}{\Delta\Lambda_{ij}^{3}}\left[2\,ln\left(\frac{\Delta_{ij}}{\eta_{i}\cdot\Delta\Lambda_{ij}}\right)+1\right]\left(\frac{\partial\Delta\Lambda_{ij}}{\partial\tau}\right)^{2},\rho=1\\ \left(\frac{\Delta_{ij}}{\eta_{i}\cdot\Delta\Lambda_{ij}^{2}}-\frac{1}{\Delta\Lambda_{ij}}\right)\frac{\partial^{2}\Delta\Lambda_{ij}}{\partial\tau^{2}}+\left(\frac{1}{\Delta\Lambda_{ij}^{2}}-\frac{2\Delta_{ij}}{\eta_{i}\cdot\Delta\Lambda_{ij}^{3}}\right)\left(\frac{\partial\Delta\Lambda_{ij}}{\partial\tau}\right)^{2},\rho=2\\ \frac{\partial}{\partial\tau}\left[\frac{1}{(1-\rho)\Delta\Lambda_{ij}}\left(\frac{\Delta_{ij}}{\Delta\Lambda_{ij}}\right)^{2-\rho}-\frac{\eta_{i}^{1-\rho}\Delta_{ij}}{(1-\rho)\Delta\Lambda_{ij}^{2}}\right]\frac{\partial\Delta\Lambda_{ij}}{\partial\tau}+\left[\frac{1}{(1-\rho)\Delta\Lambda_{ij}}\left(\frac{\Delta_{ij}}{\Delta\Lambda_{ij}}\right)^{2-\rho}-\frac{\eta_{i}^{1-\rho}\Delta_{ij}}{(1-\rho)\Delta\Lambda_{ij}^{2}}\right]\frac{\partial^{2}\Delta\Lambda_{ij}}{\partial\tau^{2}},others\end{cases}\tag{A.8}$$

$$\frac{\partial^{2}D}{\partial\varepsilon^{2}}=\begin{cases}-Z^{2}(\Delta\Lambda_{ij})^{-2},\rho=0\\ -\frac{Z^{2}}{\Delta\Lambda_{ij}\Delta_{ij}},\rho=1\\ -Z^{2}(\Delta_{ij})^{-2},\rho=2\\ -Z^{2}(\Delta\Lambda_{ij})^{\rho-2}(\Delta_{ij})^{-\rho},others\end{cases}\tag{A.9}$$

$$\frac{\partial^2 D}{\partial \rho^2} = \frac{\eta_i^{1-\rho}\Delta_{ij}\ln(\eta_i)^2}{(1-\rho)\Delta\Lambda_{ij}} - \frac{2\eta_i^{1-\rho}\Delta_{ij}\ln(\eta_i)}{(1-\rho)^2\Delta\Lambda_{ij}} + \frac{2\eta_i^{1-\rho}\Delta_{ij}}{(1-\rho)^3\Delta\Lambda_{ij}} - \frac{2\left(\frac{\Delta_{ij}}{\Delta\Lambda_{ij}}\right)^{2-\rho}}{(1-\rho)^2(2-\rho)^2} - \frac{\eta_i^{2-\rho}\ln(\eta_i)^2}{2-\rho} + \frac{2\eta_i^{2-\rho}\ln(\eta_i)}{(2-\rho)^2} - \frac{2\eta_i^{2-\rho}}{(2-\rho)^3}$$
$$- \frac{\ln\left(\frac{\Delta_{ij}}{\Delta\Lambda_{ij}}\right)^2\left(\frac{\Delta_{ij}}{\Delta\Lambda_{ij}}\right)^{2-\rho}}{(1-\rho)(2-\rho)} + \frac{2\ln\left(\frac{\Delta_{ij}}{\Delta\Lambda_{ij}}\right)\left(\frac{\Delta_{ij}}{\Delta\Lambda_{ij}}\right)^{2-\rho}}{(1-\rho)^2(2-\rho)} + \frac{2\ln\left(\frac{\Delta_{ij}}{\Delta\Lambda_{ij}}\right)\left(\frac{\Delta_{ij}}{\Delta\Lambda_{ij}}\right)^{2-\rho}}{(1-\rho)(2-\rho)^2} - \frac{2\left(\frac{\Delta_{ij}}{\Delta\Lambda_{ij}}\right)^{2-\rho}}{(1-\rho)^3(2-\rho)} - \frac{2\left(\frac{\Delta_{ij}}{\Delta\Lambda_{ij}}\right)^{2-\rho}}{(1-\rho)(2-\rho)^3} \quad (A.10)$$

$$\frac{\partial^2 D}{\partial\rho\partial\varepsilon} = \frac{\partial^2 D}{\partial\varepsilon\partial\rho} = \frac{Z\eta_i^{1-\rho}\ln(\eta_i)}{(1-\rho)\Delta\Lambda_{ij}} - \frac{Z\eta_i^{1-\rho}}{(1-\rho)^2\Delta\Lambda_{ij}} - \frac{Z\ln\left(\frac{\Delta_{ij}}{\Delta\Lambda_{ij}}\right)\cdot\left(\frac{\Delta_{ij}}{\Delta\Lambda_{ij}}\right)^{2-\rho}}{(1-\rho)\Delta_{ij}} + \frac{Z\left(\frac{\Delta_{ij}}{\Delta\Lambda_{ij}}\right)^{2-\rho}}{(1-\rho)^2\Delta_{ij}} \quad (A.11)$$

$$\frac{\partial^2 D}{\partial\rho\partial q} = \frac{\partial^2 D}{\partial q\partial\rho} = \left[\frac{\eta_i^{1-\rho}\ln(\eta_i)\Delta_{ij}}{(1-\rho)\Delta\Lambda_{ij}^2} - \frac{\eta_i^{1-\rho}\Delta_{ij}}{(1-\rho)^2\Delta\Lambda_{ij}^2}\right]\frac{\partial\Delta\Lambda_{ij}}{\partial q} \quad (A.12)$$

$$\frac{\partial^2 D}{\partial\rho\partial\tau} = \frac{\partial^2 D}{\partial\tau\partial\rho} = \left[\frac{\eta_i^{1-\rho}\ln(\eta_i)\Delta_{ij}}{(1-\rho)\Delta\Lambda_{ij}^2} - \frac{\eta_i^{1-\rho}\Delta_{ij}}{(1-\rho)^2\Delta\Lambda_{ij}^2}\right]\frac{\partial\Delta\Lambda_{ij}}{\partial\tau} \quad (A.13)$$

where $\Delta_{ij} = \Delta Y_{ij} - \varepsilon Z$. the solution is similar when $\Lambda(t)$ satisfies the exponential function. Due to space limitations, only part of the second partial derivative solution is given.

(2) Second partial derivatives of the time covariate function $\Lambda(t)$

$$\frac{\partial^2\Delta\Lambda_{ij}}{\partial\tau^2} = \begin{cases} q\left[\frac{q(t_{ij}+\tau)^q}{(t_{ij}+\tau)^2} - \frac{(t_{ij}+\tau)^q}{(t_{ij}+\tau)^2} - \frac{q(t_{i(j-1)}+\tau)^q}{(t_{i(j-1)}+\tau)^2} + \frac{(t_{i(j-1)}+\tau)^q}{(t_{i(j-1)}+\tau)^2}\right], \Lambda = t^q \\ q^2\left[e^{q(t_{ij}+\tau)} - e^{q(t_{i(j-1)}+\tau)}\right], \Lambda = e^{qt} \end{cases} \quad (A.14)$$

$$\frac{\partial^2\Delta\Lambda_{ij}}{\partial q^2} = \begin{cases} (t_{ij}+\tau)^q \ln(t_{ij}+\tau)^2 - (t_{i(j-1)}+\tau)^q \ln(t_{i(j-1)}+\tau)^2, \Lambda = t^q \\ (t_{ij}+\tau)^2 e^{q(t_{ij}+\tau)} - (t_{i(j-1)}+\tau)^2 e^{q(t_{i(j-1)}+\tau)}, \Lambda = e^{qt} \end{cases} \quad (A.15)$$

$$\frac{\partial^2\Delta\Lambda_{ij}}{\partial\tau\partial q} = \frac{\partial^2\Delta\Lambda_{ij}}{\partial q\partial\tau} = \begin{cases} q\left[\frac{(t_{ij}+\tau)^q \ln(t_{ij}+\tau)}{t_{ij}+\tau} - \frac{(t_{i(j-1)}+\tau)^q \ln(t_{i(j-1)}+\tau)}{t_{i(j-1)}+\tau}\right] + \frac{(t_{ij}+\tau)^q}{t_{ij}+\tau} - \frac{(t_{i(j-1)}+\tau)^q}{t_{i(j-1)}+\tau}, \Lambda = t^q \\ e^{q(t_{ij}+\tau)}[1+q(t_{ij}+\tau)] - e^{q(t_{i(j-1)}+\tau)}[1+q(t_{i(j-1)}+\tau)], \Lambda = e^{qt} \end{cases} \quad (A.16)$$

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