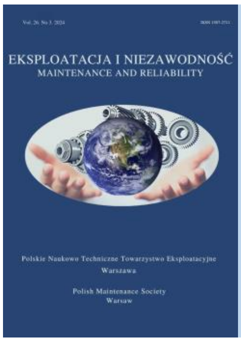

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Bayesian inference for the inverse Weibull distribution based on symmetric and asymmetric balanced loss functions with application

Indexed by:

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Highlights

- Statistical inference methods.
- Unified hybrid censoring scheme.
- Maximum Likelihood estimation.
- Bayesian estimation.

Abstract

In this study, the unified hybrid censored approach is employed to estimate the parameters of the inverse Weibull distribution, as well as the survival and hazard rate functions. Parameter estimates are obtained using both Bayesian and maximum likelihood approaches, with Bayesian estimates acquired through Lindley's approximation method using three distinct balanced loss functions. These encompass both symmetric and asymmetric balanced loss functions, specifically the balanced squared error (BSE) loss function, the balanced linear exponential (BLINEX) loss function, and the balanced general entropy (BGE) loss function. We conduct a simulation study to compare the effectiveness of various estimators, and a real-world data analysis is presented to illustrate practical implementation. Ultimately, our findings indicate that Bayesian parameter estimates consistently outperform their maximum likelihood counterparts across all methods..

Keywords

bayesian estimation, Lindley's approximation, inverse Weibull distribution, maximum likelihood estimation, unified hybrid censoring schemes, symmetric and asymmetric balanced loss functions

1. Introduction

In the field of reliability analysis, data often undergo censoring due to cost and time constraints. Two commonly employed censoring schemes are Type-I and Type-II. Type-I censoring concludes the experiment at a predetermined time T, recording the observed number of failures. Conversely, Type-II censoring entails waiting until a specified number d of failures occurs. Epstein [23] introduced a hybrid censoring scheme that combines Type-I and Type-II to address the limitations of each. In this Type-I hybrid scheme, the experiment concludes at $T^* = \min\{T, Y_{d:n}\}$, where $Y_{d:n}$ represents the failure time of the d-th unit. Similarly, Childs et al. [18] proposed a Type-II hybrid censoring scheme, concluding the experiment at $T^* = \max\{T, Y_{d:n}\}$. However, it was discovered that the hybrid censoring schemes also had limitations similar to their Type-I and Type-II counterparts. To overcome these drawbacks, Chandrasekar et al. [16] introduced generalized hybrid censoring schemes. In generalized Type-I hybrid censoring, predefined integers k and d ($d < n$) and a fixed time T are chosen. The experiment concludes at $T^* = \min\{T, Y_{d:n}\}$ if the k-th failure occurs before T, or at $Y_{k:n}$ if the k-th failure occurs after T. Similarly, the generalized Type-II hybrid censoring involves specifying

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d, time thresholds T_1 and T_2 ($T_1 < T_2$), and concludes at $T_1, Y_{d:n}$, or T_2 based on the timing of the d-th failure.

Despite the generalized schemes aiming to overcome previous disadvantages, they also have limitations. In generalized Type-I hybrid censoring, there is a possibility of not obtaining the d-th failure within the pre-fixed time. In the case of generalized Type-II hybrid censoring, obtaining an effective sample size of zero or a very small value is possible. To address these issues, a unified hybrid censoring scheme (UHCS) was proposed by Balakrishnan et al. [12].

In the context of reliability analysis, data often encounter censoring due to cost and time constraints. Two widely used censoring schemes are Type-I and Type-II. In Type-I censoring, the experiment terminates at a predetermined time T, and the number of failures observed is recorded. Conversely, Type-II censoring involves waiting until a specified number (d) of failures occur. Epstein [23] introduced a hybrid censoring scheme combining Type-I and Type-II to address the disadvantages of each. In this Type-I hybrid scheme, the experiment terminates at $T^* = min\{T, Y_{d:n}\}$, where $Y_{d:n}$ denotes the failure time of the mth unit. Similarly, Childs et al. [18] proposed a Type-II hybrid censoring scheme, where the experiment ends at $T^* = max\{T, Y_{d:n}\}$. However, it was discovered that the hybrid censoring schemes also had limitations similar to their Type-I and Type-II counterparts. To overcome these drawbacks, Chandrasekar et al. [16] introduced generalized hybrid censoring schemes. In the generalized Type-I hybrid censoring, pre-defined integers k and $d(< n)$ and a fixed time T are chosen. The experiment terminates at $T^* = min\{T, Y_{d:n}\}$ if the kth failure occurs before T, or at $Y_{k:n}$ if the kth failure occurs after T. Similarly, the generalized Type-II hybrid censoring involves specifying d, time thresholds T_1 and T_2 ($T_1 < T_2$), and terminates at $T_1, Y_{d:n}$, or T_2 based on the timing of the dth failure. Although the generalized schemes aimed to avoid previous disadvantages, they too had their limitations. The possibility of not obtaining the dth failure within the pre-fixed time exists in generalized Type-I hybrid censoring. In the case of generalized Type-II hybrid censoring, obtaining an effective sample size of zero or a very small value is possible. To address these issues, unified hybrid censoring scheme (UHCS) were proposed by Balakrishnan et al. [12].

In the UHCS, an experimenter establishes predetermined parameters for a life testing experiment involving n items. Within this framework, specific values, such as k and d in the range of $(0, \ldots, n)$ with $k < d < n$, and $T_1 < T_2$, are pre-defined. If the k-th failure occurs before T_1, the experiment concludes at $min\{max\{Y_{d:n}, T_1\}, T_2\}$. In the scenario where the k-th failure takes place between T_1 and T_2, the experiment is terminated at $min\{Y_{d:n}\}$. If the k-th failure occurs after T_2, the experiment concludes at Y_k. Under the UHCS framework, specified integers k and d and time thresholds T_1 and T_2 (with $T_1 < T_2$ and $k < d < n$) are employed. The UHCS outlines various cases of observation as follows

1. Experiment ending at T_1 if $0 < y_{k:n} < y_{d:n} < T_1 < T_2$.
2. Experiment ending at $y_{d:n}$ if $0 < y_{k:n} < T_1 < y_{d:n} < T_2$.
3. Experiment ending at T_2 if $0 < y_{k:n} < T_1 < T_2 < y_{d:n}$.
4. Experiment ending at $y_{d:n}$ if $0 < T_1 < y_{k:n} < y_{d:n} < T_2$.
5. Experiment ending at T_2 if $0 < T_1 < y_{k:n} < T_2 < y_{d:n}$.
6. Experiment ending at $y_{k:n}$ if $0 < T_1 < T_2 < y_{k:n} < y_{d:n}$.

In this context, $y_{k:n}$ and $y_{d:n}$ represent the kth and dth order statistics of the sample, respectively, while T_1 and T_2 denote predetermined stopping times.

The significance of the UHCS has motivated numerous researchers to explore estimation challenges across various statistical models employing this censoring scheme. For instance, Red and Izanlo [48] conducted a study concentrating on obtaining MLEs for the parameters of a generalized exponential distribution, utilizing the UHCS. Additionally, they derived asymptotic confidence intervals using the observed Fisher information matrix. In a similar vein, Panahi and Sayyareh [47] delved into the statistical inference of the Burr-XII distribution, employing the UHCS. Furthermore, Jeon and Kang [32] proposed point and interval estimates for the parameters of a Rayleigh distribution within the context of the unified hybrid censored sample. Sen et al. [50] explored inferential procedures and Bayesian optimal lifetesting issues under the UHCS.

In recent times, numerous researchers have explored diverse schemes and various lifetime models across multiple applications. For more detailed information, refer to the works by Ferreira and Silva [24], Celik and Guloksuz [15], Sultan and Emam [54], Emam and Sultan [22], Chiou and Chen [17], Lone et al. [40, 41], Sindhu et al. [53] Lone and Panahi [39], Sarkar [49], Ateya [10], Dutta and Kayal [19], Yan et al. [55], Asadi et al. [11], Hasaballah et al. [26], Dutta et al. [20], Abo-Kasem et al. [3], Elshahhat et al. [21] and Alrashidi et al. [6].

2. Review of Related Literature

In life testing and reliability theory, the inverse Weibull distribution (IWD) is a frequently employed and well-liked model for examining failure time. It has been demonstrated to be effective for modeling and analyzing lifetime data in various fields such as medical, biological, and engineering sciences. Its usefulness was first explored by Keller and Kanath [35] for investigating mechanical component decay. Nelson [45] used the IWD to model survival data for determining the time to breakdown of insulating fluid under constant tension. Maximum likelihood (ML) and least squares methods for calculating the IWD parameters were described by Calabria and Pulcini [14]. Recurrence relations for the moments of order statistics for both non-truncated and truncated IWD were developed by Mahmoud et al. [44]. The generalized IWD was proposed by Gusmão et al. [25] Shahbaz et al. [51] constructed the Kumaraswamy IWD. Abbas et al. [2] established the Topp Leone IWD, while Shuaib et al. [52] generalised the IWD for lifespan modelling. IWD with Marshall-Olkin alpha power was proposed by Basheer et al. [13]. For the bivariate IWD with progressive type-II censoring, Muhammed and Almetwally [42] provided both Bayesian and non-Bayesian estimate techniques. Alshaikh and Baklizi [8] investigated maximum likelihood estimation in the context of the inverse Weibull distribution with type II censored data. Jana and Bera [31] explored the estimation of parameters in the IWD and its application to a multi-component stress-strength model. The DUS-neutrosophic multivariate IWD was recently studied by Hassan and Aslam [28]. Several authors have explored the IWD, as evidenced in the works of Andrzejczak and Bukowski [9], Ilori1 et al. [30], Mahmoud et al. [43], Al-Essa et al. [5], Aljeddani and Mohammed [7], Hussam et al. [29], Abbas et al. [1], Khalaf et al. [34] and Abo-Kasem et al. [4]. The cumulative distribution function (CDF) of the IWD is expressed as follows:

$$F(y) = e^{-\alpha y^{-\beta}}, x > 0, \alpha, \beta > 0, \quad (1)$$

and its corresponding probability density function (PDF) is given by:

$$f(y) = \alpha\beta y^{-(\beta+1)} e^{-\alpha y^{-\beta}}, x > 0, \alpha, \beta > 0. \quad (2)$$

The survival function, denoted by $S(t)$, is defined as:

$$S(t) = 1 - e^{-\alpha t^{-\beta}}, \quad (3)$$

The hazard rate function, denoted by $h(t)$, is given by:

$$h(t) = \frac{\alpha\beta t^{-(\beta+1)} e^{-\alpha t^{-\beta}}}{1 - e^{-\alpha t^{-\beta}}}. \quad (4)$$

In these expressions, α represents the scale parameter, and β denotes the shape parameter.

Figures 1–4 display the PDF, CDF, survival function, and hazard rate function plots.

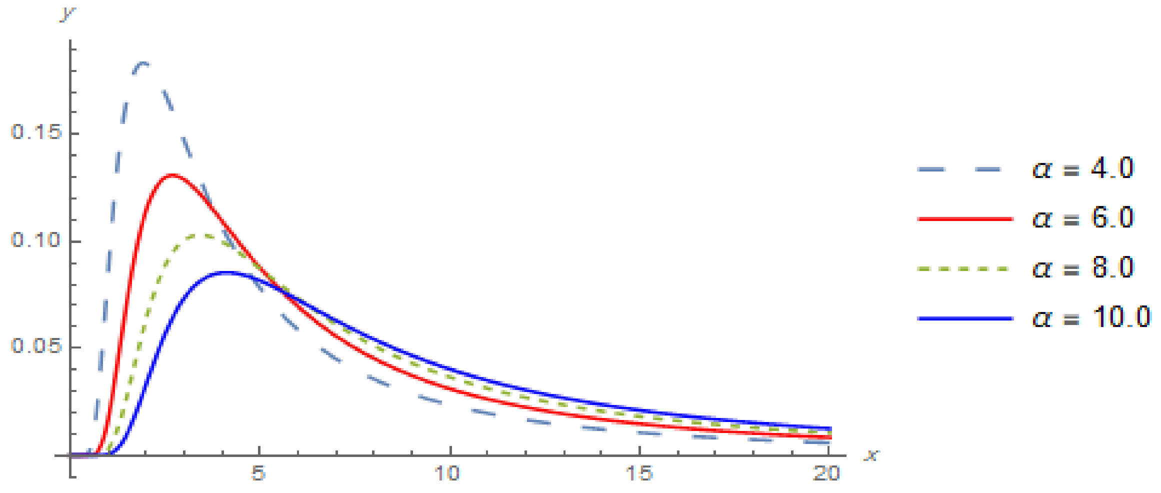

Figure 1. The PDF graph for the IWD.

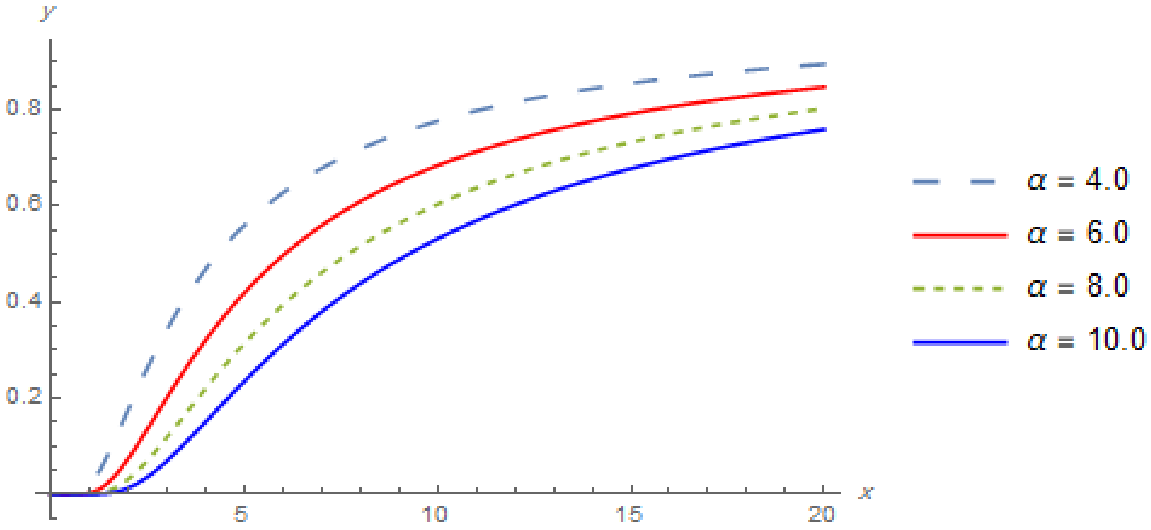

Figure 2. The CDF graph for the IWD.

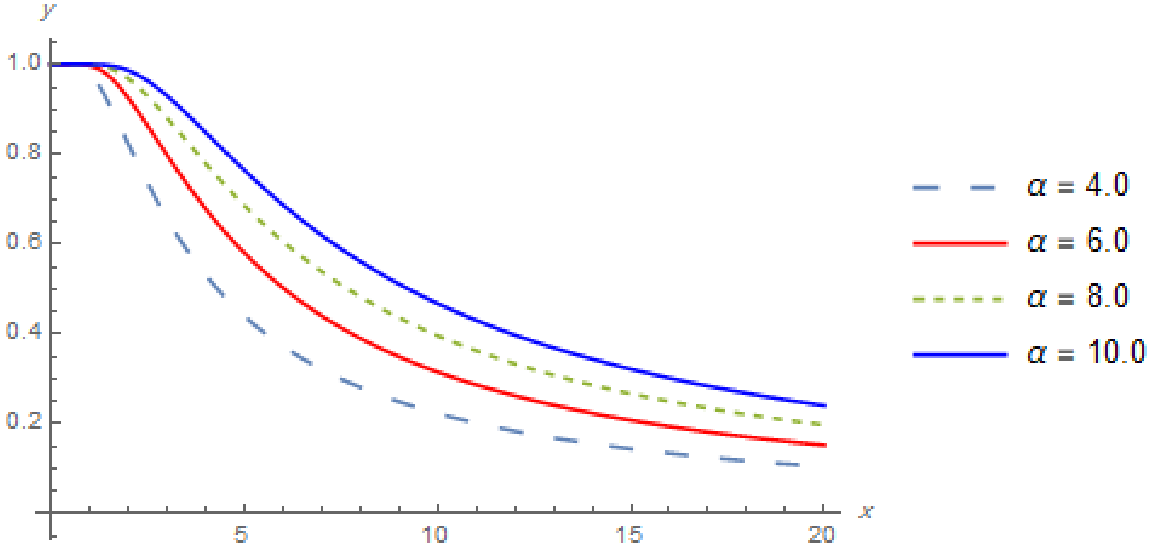

Figure 3. The survival function graph for the IWD.

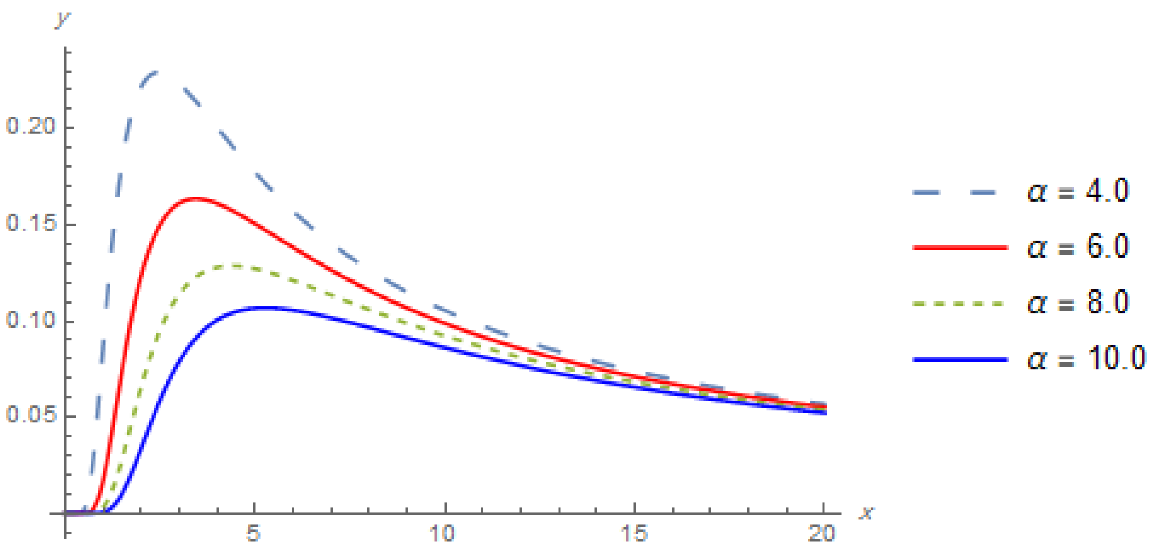

Figure 4. The hazard rate function graph for the IWD.

We use the inverse Weibull distribution for the following reasons:

• Reliability Analysis: The IWD is commonly used in reliability analysis, where data often encounter censoring due to cost and time constraints.

• Lifespan Modeling: The IWD can be applied in lifespan modeling, which involves studying the duration or lifetime of a particular event or phenomenon.

• Stress-Strength Models: The IWD can be utilized in stress-strength models, which involve assessing the reliability or failure probability of a system or component based on the comparison of stress and strength variables.

• Multivariate Distributions: The IWD can be extended to multivariate distributions, where multiple variables are considered simultaneously, allowing for the analysis of complex systems or processes involving multiple factors.

• Other Applications: The IWD may have potential applications in various other fields, such as actuarial science, finance, engineering, and environmental studies, depending on the specific research or analysis requirements.

In this manuscript, we tackle the challenges associated with estimating the IWD. Utilizing the UHCS, we derive MLEs and Bayesian estimators for the IWD. The Bayesian estimators are formulated concerning BSE, BLINEX, and BGE loss functions, all under Lindley's approximation. We illustrate the practical application of our approach using real-life datasets. To evaluate the performance of these proposed estimators, we conduct a simulation study employing both Lindley's approximation and MLE methods, comparing the estimates based on their average values and root mean square error (RMSE).

The paper is organized as follows: In Section 3, we detail the derivation of MLEs for the unknown parameters of IWD. Section 4 presents approximate confidence intervals that are contingent upon the MLEs. In Section 5, we conduct Bayesian analysis, employing both symmetric and asymmetric balanced loss functions with Lindley's approximation. Section 6, is dedicated to the application of these estimation methods to real data sets for illustration purposes. We report the outcomes of simulations in Section 7. Results analysis and discussion are presented Section 8. Lastly, a concise conclusion is provided in Section 9.

3. Maximum Likelihood Estimation

Given a random sample $Y=(Y_1,Y_2,\ldots\ldots\ldots\ldots Y_n)$ of size n, the likelihood function of IWD based on UHCS can be constructed as follows:

$$L(y,\beta,\alpha)=\frac{n!}{(n-D)!}\left[\prod_{i=1}^{D} f(y_i)\right]\left[1-F(S)\right]^{n-D}, \qquad (5)$$

$$(D,S)=\begin{cases}(b,T_1), & for\ case(1),\\ (d,y_{d:n}), & for\ case(2) and\ case(4),\\ (b_2,T_2), & for\ case(3) and\ for\ case(5),\\ (k,y_{k:n}), & for\ case(6),\end{cases}$$

Here, the symbols D and S represent the number of total failures in the experiment up to the stopping time point, respectively. For case (1), the values are denoted as (b,T_1), for case (2) and case (4) as $(d,y_{d:n})$, for case (3) and case (5) as (b_2,T_2), and for case VI as $(k,y_{k:n})$. The variables b_1 and b_2 represent the number of failures occurring before time points T_1 and T_2, respectively, with the condition that $b=b_1=b_2$ for case I.

From (1), (2), and (5), we derive the likelihood function as follows:

$$L(y;\alpha,\beta)=Z\alpha^D\beta^D e^{-(\beta+1)\sum_{i=1}^{D}\ln y_i}e^{-\alpha\sum_{i=1}^{D}y_i^{-\beta}}\left[1-e^{-\alpha S^{-\beta}}\right]^{n-D}, \qquad (6)$$

where $Z=\frac{2^D n!}{(n-D)!}$.

Hence, the log-likelihood function is given by:

$$\begin{aligned}\ell(y;\alpha,\beta)= \quad & \ln(Z)+D\ln(\alpha)+D\ln(\beta)-(\beta+1)\sum_{i=1}^{D}\ln(y_i)-\alpha\sum_{i=1}^{D}y_i^{-\beta}\\ & +(n-D)\ln\left[1-e^{-\alpha S^{-\beta}}\right].\end{aligned} \qquad (7)$$

In order to obtain the MLEs for the unknown parameters α and β, we must solve a set of equations that are derived by taking the first derivatives of the log-likelihood function with respect to β and α, and then setting them to zero:

$$\frac{\partial\ell}{\partial\beta}=\frac{D}{\beta}+\sum_{i=1}^{D}\ln y_i+\alpha\sum_{i=1}^{D}y_i^{-\beta}\ln y_i-\frac{(n-D)\ \alpha\ S^{-\beta}\ln S\ e^{-\alpha S^{-\beta}}}{1-e^{-\alpha\ S^{-\beta}}}=0, \qquad (8)$$

$$\frac{\partial \ell}{\partial \alpha} = \frac{D}{\alpha} - \sum_{i=1}^{D} y_i^{-\beta} + \frac{(n-D)\ \ S^{-\beta} e^{-\alpha S^{-\beta}}}{1-e^{-\alpha\ \ S^{-\beta}}} = 0. \tag{9}$$

As (8) and (9) are complex and nonlinear, it can be challenging to solve them analytically. Therefore, numerical methods like the Newton-Raphson method may be used to compute the MLEs of the unknown parameters α and β.

4. Fisher Information Matrix

Given below is the log-likelihood function's second derivative with regard to the α and β .

$$\frac{\partial^2 \ell}{\partial \beta^2} = \frac{-D}{\beta^2} - \alpha \sum_{i=1}^{D} (\ln y_i)^2 y_i^{-\beta} - \frac{(n-D)\ \ \alpha\ \ S^{-\beta}(\ln S)^2\ \ e^{-\alpha S^{-\beta}}}{1-e^{-\alpha\ \ S^{-\beta}}} - \frac{(n-D)\ \ \alpha^2\ \ S^{-2\beta}(\ln S)^2\ \ e^{-2\alpha S^{-\beta}}}{(1-e^{-\alpha\ \ S^{-\beta}})^2} - \frac{(n-D)\ \ \alpha^2\ \ S^{-2\beta}(\ln S)^2\ \ e^{-\alpha S^{-\beta}}}{1-e^{-\alpha\ \ S^{-\beta}}}, \tag{10}$$

$$\frac{\partial^2 \ell}{\partial \beta \partial \alpha} = \frac{\partial^2 \ell}{\partial \alpha \partial \beta} = \sum_{i=1}^{D} (\ln y_i) y_i^{-\beta} - \frac{(n-D)\ \ S^{-\beta}(\ln S)\ \ e^{-\alpha S^{-\beta}}}{1-e^{-\alpha\ \ S^{-\beta}}} - \frac{(n-D)\ \ \alpha\ \ S^{-2\beta}(\ln S)\ \ e^{-2\alpha S^{-\beta}}}{(1-e^{-\alpha\ \ S^{-\beta}})^2} - \frac{(n-D)\ \ \alpha\ \ S^{-2\beta}(\ln S)\ \ e^{-\alpha S^{-\beta}}}{1-e^{-\alpha\ \ S^{-\beta}}} \tag{11}$$

$$\frac{\partial^2 \ell}{\partial \alpha^2} = \frac{-D}{\alpha^2} - \frac{(n-D)\ \ S^{-2\beta} e^{-2\alpha S^{-\beta}}}{(1-e^{-\alpha\ \ S^{-\beta}})^2} - \frac{(n-D)\ \ S^{-2\beta} e^{-\alpha S^{-\beta}}}{1-e^{-\alpha\ \ S^{-\beta}}} \tag{12}$$

Now, we formulate the approximate confidence intervals (ACIs) for the parameters α and β. utilizing the asymptotic normal distribution of the MLEs. This involves the utilization of the asymptotic fisher information matrix (FIM). The matrix $I^{-1}(\alpha,\beta)$ is computed by taking the expectation of the negative values of 10–12, and it can be expressed as follows:

$$I^{-1}(\alpha,\beta) = \begin{pmatrix} -\frac{\partial^2 \ell}{\partial \alpha^2} & -\frac{\partial^2 \ell}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \ell}{\partial \beta \partial \alpha} & -\frac{\partial^2 \ell}{\partial \beta^2} \end{pmatrix}^{-1}_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})} = \begin{pmatrix} \widehat{var}(\hat{\alpha}) & cov(\hat{\alpha},\hat{\beta}) \\ cov(\hat{\beta},\hat{\alpha}) & \widehat{var}(\hat{\beta}) \end{pmatrix},$$

The CIs for parameters α and β with a confidence level of $(1-\eta)100\%$ are determined as follows:

$\left(\hat{\alpha} \pm Z_{\eta/2}\sqrt{\widehat{var}(\hat{\alpha})}\right)$ and $\left(\hat{\beta} \pm Z_{\eta/2}\sqrt{\widehat{var}(\hat{\beta})}\right)$, respectively.

The significance of $Z_{\eta/2}$ lies in its representation of the location on the upper tail of the standard normal distribution corresponding to a probability of $\eta/2$.

5. Bayesian Estimation

In this section, our focus is on the Bayesian estimation of the unknown parameters, the survival function, and the hazard rate function of the IWD under the UHCS. We use three balanced loss functions, using BSE, BLINEX, and BGE. Assuming the following gamma priors for α and β:

$$\pi_1(\alpha) \propto \alpha^{d_1-1} e^{-d_2\alpha}, \qquad \alpha > 0,$$
$$\pi_2(\beta) \propto \beta^{d_3-1} e^{-d_4\beta}, \qquad \beta > 0,$$

The joint prior distribution for α and β is

$$\pi(\alpha,\beta) \propto \alpha^{d_1-1}\beta^{d_2-1} e^{-d_1\alpha - d_2\beta}. \tag{13}$$

For Bayesian estimation, We utilize both symmetric and asymmetric balanced loss functions for the Bayesian estimation, as explained below:

5.1. Loss Functions

A loss function, also known as a cost function or objective function, is a mathematical function that measures the penalty or cost associated with the difference between the predicted values of a model and the true values. In the context of Bayesian statistics, loss functions are employed to quantify the discrepancy between predicted values and observed outcomes. In Bayesian statistics, the loss function is crucial for making decisions and assessing the performance of Bayesian estimators. When estimating parameters, Bayesian methods involve specifying a prior distribution and updating it based on observed data to obtain a posterior distribution. The loss function helps in evaluating the quality of the Bayesian point estimates and determining the optimal decision rule under different circumstances. A "balanced" loss function typically refers to a loss function that achieves a balance between competing considerations, such as bias and variance or precision and recall in classification problems.

Commonly used balanced loss functions in Bayesian statistics include:

1. Balanced Squared Error Loss (BSE): This loss function penalizes the squared difference between predicted and true values. It is often used when the goal is to minimize the mean squared error.

2. Balanced LINEX Loss: The BLINEX loss function generalizes the balanced squared error loss by introducing a linear exponential term. It is particularly useful when there is a need to emphasize errors in a specific region of the parameter space.

3. Balanced Generalized Entropy (BGE) Loss: The BGE loss is a family of loss functions that includes squared error and absolute error as special cases. It is versatile and allows for tuning the degree of sensitivity to different levels of errors.

5.2. Symmetric balanced loss functions

5.2.1. The BSE loss function

Jozani et al. [33] presented a generalized balanced loss function, denoted as the

$$L_{\rho,\nu,\phi_0}(\phi,\hat{\phi}) = \nu\rho(\hat{\phi},\hat{\phi}_0) + (1-\nu)\rho(\phi,\hat{\phi}), \tag{14}$$

In this context, ν (with $0 \le \nu \le 1$) serves as a weight parameter, and ρ denotes a user-defined loss function. The target estimator, denoted as $\hat{\phi}_0$, is typically derived using methods such as maximum likelihood or least squares. Specifically, for the Balanced Squared Error (BSE) loss function, we opt for $\rho(\phi,\hat{\phi}) = (\hat{\phi}-\phi)^2$, resulting in the following expression:

$$L(\phi,\hat{\phi}) = \nu(\hat{\phi}_0-\hat{\phi})^2 + (1-\nu)(\hat{\phi}-\phi)^2. \tag{15}$$

The Bayesian estimation for the unknown parameter ϕ is subsequently provided as:

$$\hat{\phi}(y) = \nu\hat{\phi}_0 + (1-\nu)E(\phi|y). \tag{16}$$

In this equation, the parameter $0 \le \nu \le 1$ functions as a weight parameter, and ρ represents a user-defined loss function. The estimator $\hat{\phi}_0$ acts as a general "target" estimator for ϕ, often derived through methods like maximum likelihood, least squares, or unbiasedness. This balanced loss function can be tailored to various loss functions, including absolute value, squared error, LINEX, and general entropy loss functions. By choosing the loss function as $\rho(\phi,\hat{\phi}) = (\hat{\phi}-\phi)^2$, equation (14) simplifies to the BSE loss function in the following manner:

$$L(\phi,\hat{\phi}) = \nu(\hat{\phi}_0-\hat{\phi})^2 + (1-\nu)(\hat{\phi}-\phi)^2, \tag{17}$$

The Bayesian estimation for the unknown parameter ϕ is subsequently calculated as:

$$\hat{\phi}(y) = \nu\hat{\phi}_0 + (1-\nu)E(\phi|y). \tag{18}$$

5.3. Asymmetric balanced loss functions

5.3.1.The BLINEX loss function

The BLINEX loss function, incorporating a shape parameter a (where $a \ne 0$), is formulated by defining $\rho(\phi,\hat{\phi}) = e^{a(\hat{\phi}-\phi)} - a(\hat{\phi}-\phi) - 1$, as elucidated by Zellner [56]. Consequently, the Bayesian estimation of ϕ utilizing the BLINEX function is expressed as:

$$\hat{\phi}(y) = \frac{-1}{a}\ln[\nu e^{-a\hat{\phi}_0} + (1-\nu)E(e^{-a\phi}|y)]. \tag{19}$$

5.3.2. The BGE loss function

The BGE loss function, governed by the shape parameter a, is characterized by $\rho(\phi,\hat{\phi}) = (\frac{\hat{\phi}}{\phi})^a - a\ln(\frac{\hat{\phi}}{\phi}) - 1$. Accordingly, the Bayesian estimation of ϕ employing the BGE loss function is articulated as:

$$\hat{\phi}(y) = [\nu(\hat{\phi}_0)^{-a} + (1-\nu)E(\phi^{-a}|y)]^{\frac{-1}{a}}. \tag{20}$$

The adaptability of balanced loss functions is evident, encompassing various special cases, including the maximum likelihood estimate and both symmetric and asymmetric Bayesian estimates. For instance, under the BSE loss function in (18), the Bayesian estimate converges to the maximum likelihood estimate when $\nu = 1$. Conversely, for $\nu = 0$, it transforms into the Bayesian estimate relative to the SE loss function. Similarly, the Bayesian estimator under the BLINEX loss function, as depicted in (19), reduces to the maximum likelihood estimate at $\nu = 1$ and corresponds to the asymmetric LINEX loss function at $\nu = 0$. Likewise, under the BGE loss function in (20), the Bayesian estimator reduces to the maximum likelihood estimate at $\nu = 1$ and corresponds to the GE loss function at $\nu = 0$.

5.4. Lindley's Approximation

Lindley's approach involves employing the Taylor series expansion of the function relevant to the posterior moment. He introduced an asymptotic solution for the ratio of two integrals, commonly encountered in Bayesian estimation (see Lindley [38]).

The ratio of integrals that arises in Bayesian analysis is expressed as:

$$\hat{Q}_B(\alpha,\beta) = E[Q(\alpha,\beta)|y] = \frac{\int_\alpha \int_\beta Q(\alpha,\beta) e^{\ell(y|\alpha,\beta)+\rho(\alpha,\beta)} d\alpha d\beta}{\int_\alpha \int_\beta e^{\ell(y|\alpha,\beta)+\rho(\alpha,\beta)} d\alpha d\beta}. \quad (21)$$

To approximate Lindley's procedure asymptotically, we expand $\rho(\alpha,\beta) = \ln[\pi(\alpha,\beta)]$ and $\ell(\alpha,\beta)$ in (21) using a Taylor series focused on the MLE of (α,β):

$$\hat{Q}_B(\alpha,\beta) = E[Q(\alpha,\beta)|y] = Q(\hat{\alpha},\hat{\beta}) + \frac{1}{2}\sum_{i,j}^{m} [Q_{ij}(\hat{\alpha},\hat{\beta}) + 2Q_i(\hat{\alpha},\hat{\beta})\rho_j(\hat{\alpha},\hat{\beta})]\hat{\varphi}_{ij} + \frac{1}{2}\sum_{i,j,s,k}^{m} \hat{\ell}_{ijs}\hat{Q}_k(\hat{\alpha},\hat{\beta})\hat{\varphi}_{ij}\hat{\varphi}_{sk}, \quad (22)$$

where, $i,j,s,k = 1,2,\ldots,m,$ then,

$$\begin{aligned}\hat{Q}_B(\alpha,\beta) = {} & Q(\hat{\alpha},\hat{\beta}) + \tfrac{1}{2}[(\hat{Q}_{\alpha\alpha} + 2\hat{Q}_\alpha\hat{\rho}_\alpha)\hat{\varphi}_{\alpha\alpha} + (\hat{Q}_{\beta\alpha} + 2\hat{Q}_\beta\hat{\rho}_\alpha)\hat{\varphi}_{\beta\alpha} \\ & + (\hat{Q}_{\alpha\beta} + 2\hat{Q}_\alpha\hat{\rho}_\beta)\hat{\varphi}_{\alpha\beta} + (\hat{Q}_{\beta\beta} + 2\hat{Q}_\beta\hat{\rho}_\beta)\hat{\varphi}_{\beta\beta}] + \tfrac{1}{2}[(\hat{Q}_\alpha\hat{\varphi}_{\alpha\alpha} \\ & + \hat{Q}_\beta\hat{\varphi}_{\alpha\beta})(\hat{\ell}_{\alpha\alpha\alpha}\hat{\varphi}_{\alpha\alpha} + \hat{\ell}_{\alpha\beta\alpha}\hat{\varphi}_{\alpha\beta} + \hat{\ell}_{\beta\alpha\alpha}\hat{\varphi}_{\beta\alpha} + \hat{\ell}_{\beta\beta\alpha}\hat{\varphi}_{\beta\beta}) \\ & + (\hat{Q}_\alpha\hat{\varphi}_{\beta\alpha} + \hat{Q}_\beta\hat{\varphi}_{\beta\beta})(\hat{\ell}_{\beta\alpha\alpha}\hat{\varphi}_{\alpha\alpha} + \hat{\ell}_{\alpha\beta\beta}\hat{\varphi}_{\alpha\beta} + \hat{\ell}_{\beta\alpha\beta}\hat{\varphi}_{\beta\alpha} + \hat{\ell}_{\beta\beta\beta}\hat{\varphi}_{\beta\beta})],\end{aligned} \quad (23)$$

where

$$\hat{\ell}_{ht} = \frac{\partial^{h+t}\ell}{\partial\phi_1^h \partial\phi_2^t}, \quad \rho = \ln\pi(\phi_1,\phi_2), \quad \rho_i = \frac{\partial\rho}{\partial\phi_i}, \quad Q_{\phi_i\phi_j} = \frac{\partial^2 Q}{\partial\phi_i\partial\phi_j} \quad \text{and} \quad Q_{\phi_i} = \frac{\partial Q}{\partial\phi_i}.$$

Where h and t can take the values 0,1,2,3, and $h+t$ equals 3. The indices i and j can be either 1 or 2, corresponding to $\phi_1 = \alpha$ and $\phi_2 = \beta$, respectively. In this context, $\ell(.,.)$ represents the log-likelihood function of the observed data, and $\pi(\phi_1,\phi_2) = \pi(\alpha,\beta)$ denotes the joint prior density function of (α,β). Additionally, φ_{ij} stands for the (i, j)-th element of the inverse of the FIM. Furthermore, $\hat{\alpha}$ and $\hat{\beta}$ are the ML estimators of α and β, respectively.

The following equations represent the Bayesian estimates for different parameters using the BSE, BLINEX, and BGE loss functions, where $\hat{\phi} = [\hat{\alpha},\hat{\beta},\hat{S}(t),\hat{h}(t)]$:

(1) Case of the BSE loss function

If $Q(\hat{\alpha},\hat{\beta}) = \hat{\phi}$, then the Bayesian estimate is given by:

$$\begin{aligned}\hat{\phi}_{BBS} = {} & \nu\hat{\phi}_{ML} + (1-\nu)[\hat{\phi}_{ML} + \tfrac{1}{2}[(\hat{Q}_{\alpha\alpha} + 2\hat{Q}_\alpha\hat{\rho}_\alpha)\hat{\varphi}_{\alpha\alpha} + (\hat{Q}_{\beta\alpha} + 2\hat{Q}_\beta\hat{\rho}_\alpha)\hat{\varphi}_{\beta\alpha} \\ & + (\hat{Q}_{\alpha\beta} + 2\hat{Q}_\alpha\hat{\rho}_\beta)\hat{\varphi}_{\alpha\beta} + (\hat{Q}_{\beta\beta} + 2\hat{Q}_\beta\hat{\rho}_\beta)\hat{\varphi}_{\beta\beta}] + \tfrac{1}{2}[(\hat{Q}_\alpha\hat{\varphi}_{\alpha\alpha} + \hat{Q}_\beta\hat{\varphi}_{\alpha\beta}) \\ & \times (\hat{\ell}_{\alpha\alpha\alpha}\hat{\varphi}_{\alpha\alpha} + \hat{\ell}_{\alpha\beta\alpha}\hat{\varphi}_{\alpha\beta} + \hat{\ell}_{\beta\alpha\alpha}\hat{\varphi}_{\beta\alpha} + \hat{\ell}_{\beta\beta\alpha}\hat{\varphi}_{\beta\beta}) + (\hat{Q}_\alpha\hat{\varphi}_{\beta\alpha} + \hat{Q}_\beta\hat{\varphi}_{\beta\beta}) \\ & \times (\hat{\ell}_{\beta\alpha\alpha}\hat{\varphi}_{\alpha\alpha} + \hat{\ell}_{\alpha\beta\beta}\hat{\varphi}_{\alpha\beta} + \hat{\ell}_{\beta\alpha\beta}\hat{\varphi}_{\beta\alpha} + \hat{\ell}_{\beta\beta\beta}\hat{\varphi}_{\beta\beta})]].\end{aligned} \quad (24)$$

(2) Case of the BLINEX loss function

If $Q(\hat{\alpha},\hat{\beta}) = e^{-a\hat{\phi}}$, then the Bayesian estimate is given by:

$$\begin{aligned}\hat{\phi}_{BBL} = {} & \tfrac{-1}{a}\ln[\nu \quad e^{-a\hat{\phi}_{ML}} + (1-\nu)\{e^{-a\hat{\phi}_{ML}} + \tfrac{1}{2}[(\hat{Q}_{\alpha\alpha} + 2\hat{Q}_\alpha\hat{\rho}_\alpha)\hat{\varphi}_{\alpha\alpha} + (\hat{Q}_{\beta\alpha} + 2\hat{Q}_\beta\hat{\rho}_\alpha) \\ & \times \hat{\varphi}_{\beta\alpha} + (\hat{Q}_{\alpha\beta} + 2\hat{Q}_\alpha\hat{\rho}_\beta)\hat{\varphi}_{\alpha\beta} + (\hat{Q}_{\beta\beta} + 2\hat{Q}_\beta\hat{\rho}_\beta)\hat{\varphi}_{\beta\beta}] + \tfrac{1}{2}[(\hat{Q}_\alpha\hat{\varphi}_{\alpha\alpha} + \hat{Q}_\beta\hat{\varphi}_{\alpha\beta}) \\ & \times (\hat{\ell}_{\alpha\alpha\alpha}\hat{\varphi}_{\alpha\alpha} + \hat{\ell}_{\alpha\beta\alpha}\hat{\varphi}_{\alpha\beta} + \hat{\ell}_{\beta\alpha\alpha}\hat{\varphi}_{\beta\alpha} + \hat{\ell}_{\beta\beta\alpha}\hat{\varphi}_{\beta\beta}) + (\hat{Q}_\alpha\hat{\varphi}_{\beta\alpha} + \hat{Q}_\beta\hat{\varphi}_{\beta\beta})(\hat{\ell}_{\beta\alpha\alpha}\hat{\varphi}_{\alpha\alpha} \\ & + \hat{\ell}_{\alpha\beta\beta}\hat{\varphi}_{\alpha\beta} + \hat{\ell}_{\beta\alpha\beta}\hat{\varphi}_{\beta\alpha} + \hat{\ell}_{\beta\beta\beta}\hat{\varphi}_{\beta\beta})]\}].\end{aligned} \quad (25)$$

(3) Case of the BGE loss function

If $Q(\hat{\alpha},\hat{\beta}) = [\hat{\phi}]^{-a}$, then the Bayesian estimate is given by:

$$\begin{aligned}\hat{\phi}_{BBG} = {} & [\nu \quad [\hat{\phi}_{ML}]^{-a} + (1-\nu)\{[\hat{\phi}_{ML}]^{-a} + \tfrac{1}{2}[(\hat{Q}_{\alpha\alpha} + 2\hat{Q}_\alpha\hat{\rho}_\alpha)\hat{\varphi}_{\alpha\alpha} \\ & + (\hat{Q}_{\beta\alpha} + 2\hat{Q}_\beta\hat{\rho}_\alpha)\hat{\varphi}_{\beta\alpha} + (\hat{Q}_{\alpha\beta} + 2\hat{Q}_\alpha\hat{\rho}_\beta)\hat{\varphi}_{\alpha\beta} + (\hat{Q}_{\beta\beta} + 2\hat{Q}_\beta\hat{\rho}_\beta)\hat{\varphi}_{\beta\beta}] \\ & + \tfrac{1}{2}[(\hat{Q}_\alpha\hat{\varphi}_{\alpha\alpha} + \hat{Q}_\beta\hat{\varphi}_{\alpha\beta})(\hat{\ell}_{\alpha\alpha\alpha}\hat{\varphi}_{\alpha\alpha} + \hat{\ell}_{\alpha\beta\alpha}\hat{\varphi}_{\alpha\beta} + \hat{\ell}_{\beta\alpha\alpha}\hat{\varphi}_{\beta\alpha} + \hat{\ell}_{\beta\beta\alpha}\hat{\varphi}_{\beta\beta}) \\ & + (\hat{Q}_\alpha\hat{\varphi}_{\beta\alpha} + \hat{Q}_\beta\hat{\varphi}_{\beta\beta})(\hat{\ell}_{\beta\alpha\alpha}\hat{\varphi}_{\alpha\alpha} + \hat{\ell}_{\alpha\beta\beta}\hat{\varphi}_{\alpha\beta} \\ & + \hat{\ell}_{\beta\alpha\beta}\hat{\varphi}_{\beta\alpha} + \hat{\ell}_{\beta\beta\beta}\hat{\varphi}_{\beta\beta})]\}]^{\frac{-1}{a}}.\end{aligned} \quad (26)$$

6. Medical Data

Bladder cancer is a prevalent malignancy affecting millions of individuals worldwide, with varying degrees of severity and outcomes. Understanding the remission times of bladder cancer patients is crucial for evaluating treatment efficacy, disease progression, and patient prognosis. Remission time refers to the period during which patients remain free of cancer after receiving treatment, and its study provides valuable insights into the effectiveness of therapeutic interventions.

This application is drawn from Lee and Wang [37] and focuses on the remission times (in months) of 128 patients afflicted with bladder cancer. Recent analyses of this dataset have been conducted in several papers, including Okasha et al. [46] Klakattawi [36] and Hamdeni and Gasmi [27]. This data is given in Appendix B.

We applied the Kolmogorov-Smirnov (K-S) test to the data to determine if it follows an IWD. The null hypothesis of the test is that the data comes from an IWD. The test statistic was 0.07149 and the p-value was 0.5069. Since the p-value is greater than the significance level of 0.05, we failed to reject the null hypothesis. This means that the data is consistent with an IWD. In other words, the IWD is a good model for the remission times of bladder cancer patients. This is because the IWD fits the data more closely than other distributions. We also created a plot of the empirical CDF of the data and the CDF of the IWD as shown in Figure (5). The plot shows that the two CDFs are very close, which further supports the conclusion that the IWD is a good model for the remission times of bladder cancer patients.

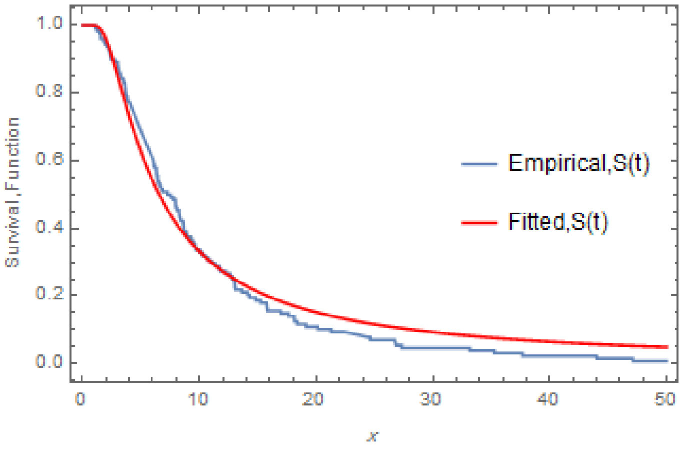

Figure 5. Plots of fitted functions of the IWD.

Now, we consider the case when the data are censored. We generate six artificially UHCS data sets from the uncensored data set as:

1. $T_1 = 17.0, T_2 = 25.0, k = 90, d = 100$. In this instance, $D = 110, S = T_1 = 17.0$.

2. $T_1 = 17.30, T_2 = 25.0, k = 90, d = 103$. In this instance, $D = 103, S = y_{d:n} = 13.50$.

3. $T_1 = 17.8, T_2 = 25.0, k = 90, d = 105$. In this instance, $D = 120, S = T_2 = 25.0$.

4. $T_1 = 9.0, T_2 = 17.0, k = 95, d = 108$. In this instance, $D = 108, S = y_{d:n} = 15.0$.

5. $T_1 = 9.0, T_2 = 13.90, k = 98, d = 108$. In this instance, $D = 106, S = T_2 = 13.90$.

6. $T_1 = 9.0, T_2 = 9.50, k = 100, d = 108$. In this instance, $D = 100, S = y_{k:n} = 13.0$.

In the context of Lindley's approximation, we employed non-informative gamma priors for both α and β. In this scenario, the hyperparameters are set to 0 ($d_1 = d_2 = d_3 = d_4 = 0$). We then investigated the impact of various loss functions, namely BSE loss, BLINEX loss, and BGE loss, while considering different values of the shape parameter "a" for the BLINEX and BGE loss functions. Additionally, we explored varying values of "ν " for the parameters $\alpha = 2.76$, $\beta = 1.28$, $S(t = 0.8)$, and $h(t = 0.8)$. The comprehensive results are outlined in Table 1 in Appendix B..

7. Simulation Study

This section focuses on the comparison of estimation performance between classical statistics and Bayesian statistics for the IWD distribution. It involves employing both the method of ML and the Lindley method within the framework of UHCS, considering three types of balanced loss functions: BSE, BLINEX, and BGE loss functions.

A simulation study is conducted to visually elucidate all the outcomes expounded in the preceding sections. The estimators derived from both the MLE and Bayesian estimation methods are procured following the subsequent steps:

• A sample is generated using the IWD model with the parameter values $\alpha = 2.76$ and $\beta = 1.28$.

• The MLEs for α and β are determined by numerically solving the two nonlinear (8) and (9).

• Bayesian estimates for α and β are computed using the Lindley method with a dataset comprising 10,000 observations. These estimations are conducted under three distinct balanced loss functions: BSE loss function (18), BLINEX loss function (14), and BGE loss function (20) with hyper-parameter $d_1 = d_3 = 0.2$ and $d_2 = d_4 = 0.4$.

• The above steps are iterated 1,000 times to evaluate the performance of these estimates.

• The simulation is executed with varying values for k, d, T_1, and T_2, alongside the parameters α and β. We examine different combinations of (n, k, d) including (30, 15, 20), (50, 20, 35), (50, 25, 30), (60, 30, 38), (50, 35, 45), and (50, 40, 43). The pre-determined termination times (T_1, T_2) are also selected as (1, 1.3), (0.5, 1.4), (0.9, 1.5), (0.6, 1.7), (0.5, 1.2), and (0.4, 0.9).

• Additionally, the RMSE of the estimators is calculated using the subsequent formula:

$$\mathrm{RMSE}(\hat{\psi}) = \sqrt{\sum_{i=1}^{1000} \frac{(\hat{\psi}_i - \psi)^2}{1000}}. \tag{27}$$

All of these findings are displayed in Tables 3–8 in Appendix C.

8. Results Analysis and Discussion

These simulations serve as invaluable tools for gauging the methods' performance under a spectrum of scenarios, thereby affording profound insights into their relative effectiveness. Furthermore, these conclusions are substantiated by the analysis of Tables 3–8, which unequivocally demonstrate that the performance of Bayesian estimates for α, β, S(t) and h(t) surpasses that of MLEs in terms of RMSEs, and also we notice that when the sample size increases RMSEs decreases and when the ν increases RMSEs increases.

We observe the following trends from the results:

1. At $a = 0.3$, Bayes estimates under BGE outperform Bayes estimates under BLINEX and BSE, demonstrating the minimum RMSEs.

2. Bayes estimates under BSE surpass Bayes estimates under MLE, showcasing the minimum RMSEs.

3. Bayes estimates under BLINEX at $a = 0.3$ and $a = 5$ outshine Bayes estimates under BSE, exhibiting the minimum RMSEs.

4. Bayes estimates under BSE are superior to Bayes estimates under BLINEX and BGE at $a = -5$, indicating the minimum RMSEs.

5. At $a = 5$, Bayes estimates under BLINEX excel over Bayes estimates under BGE and BSE, demonstrating the minimum RMSEs.

9. Conclusions

This investigation has been dedicated to the meticulous exploration of parameter estimation within the context of the IWD. Additionally, our inquiry has encompassed the estimation of the survival and hazard rate functions, particularly when the data is subject to observation under the UHCS mechanism. In this comprehensive endeavor, both the ML and Bayesian estimators have been harnessed for the purpose of deriving the pertinent parameters that define this lifetime distribution. To this end, we initially derive the MLEs as foundational components.

Further advancing into the Bayesian realm, we delve into the derivation of Bayesian estimators through the application of the Lindley method, employing diverse loss functions such as BSE, BLINEX, and BGE. The outcomes of these estimators are then presented utilizing an actual dataset, adding practicality to our findings. Additionally, to comprehensively assess and compare the efficacy of the proposed methods, we have conducted an extensive simulation study. This study encompasses various sample sizes (d, k) and encompasses distinct scenarios (I, II, III, IV, V, VI).

Lindley's approximation method, also known as the Lindley equation or Lindley's formula, is a statistical technique used for Bayesian analysis. While it has its merits, it also has some limitations. Here are a few:

• Approximation Accuracy: Lindley's approximation is, by nature, an approximation method. Its accuracy may be compromised in certain scenarios, especially when dealing with complex models or datasets with unique characteristics.

• Sample Size Sensitivity: The method's performance can be sensitive to sample size. In situations with small sample sizes, the approximation may not provide accurate results, and alternative Bayesian methods may be more appropriate.

• Dependence on Priors: Lindley's method, like many Bayesian approaches, relies on prior distributions. The results obtained can be sensitive to the choice of priors, and the subjectivity in selecting them may pose a challenge in certain applications.

• Complex Models: Lindley's approximation might face limitations when applied to intricate statistical models. Its simplicity might not adequately capture the nuances of complex relationships within the data.

• Computational Demands: For large datasets or models, Lindley's method may pose computational challenges. Its efficiency could be a limiting factor when dealing with extensive and computationally demanding analyses.

Prospects for Future Research:

• Refinement of Approximation Techniques: Future research could focus on refining Lindley's approximation method or developing alternative approximations that address its limitations. This may involve exploring more accurate approximations for specific scenarios or extending its applicability to a broader range of models.

• Robustness Studies: Investigating the robustness of Lindley's method across different types of data and model structures could be valuable. This research could identify conditions under which the method performs well and areas where improvements are needed.

• Integration with Other Methods: Combining Lindley's approximation with other Bayesian methods or statistical techniques may enhance its overall performance. Research could explore hybrid approaches that leverage the strengths of various methods to provide more robust and accurate results.

• Handling Complex Models: Addressing the limitations associated with complex models is crucial. Future research may focus on developing Bayesian methods that can effectively handle intricate statistical models without sacrificing accuracy or computational efficiency.

• User-Friendly Implementation: Simplifying the implementation of Lindley's method or creating user-friendly tools could broaden its accessibility. This would facilitate its adoption by researchers and practitioners who may not have extensive expertise in Bayesian statistics.

• Real-world Applications: Further research could concentrate on applying Lindley's method to a diverse set of real-world problems to assess its performance in practical settings. Understanding its strengths and limitations in various applications can guide researchers in choosing appropriate statistical tools.

• By addressing these aspects, future research can contribute to refining Lindley's approximation method and expanding its applicability in Bayesian analysis.

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Appendix A

Abbreviations used in this manuscript

Abbreviation	Meaning
UHCS	Unified Hybrid Censoring Scheme
IWD	Inverse Weibull Distribution
PDF	Probability Density Function
CDF	Cumulative Distribution Function
MLEs	Maximum Likelihood Estimators
BSE	Balanced Squared Error Loss Function
BLINEX	Balanced Linear Exponential Loss Function
BGE	Balanced General Entropy Loss Function
ML	Maximum Likelihood
FIM	Fisher Information Matrix
RMSEs	Root Mean Squared Errors
K-S	Kolmogorov-Smirnov

Appendix B

The remission times, measured in months, are provided below 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Appendix C

Table 1. Estimating α, β, $S(t)$, and $h(t)$ using MLE and Lindley method for real data.

Cases	Parameters	MLE	v	BSE	BLINEX			BGE		
					a = -5	a = 0.3	a =5	a = -5	a = 0.3	a = 5
Case I	β	2.415	0.0	2.408	2.505	2.401	2.316	2.447	2.395	2.353
			0.3	2.410	2.482	2.401	2.341	2.438	2.401	2.371
			0.9	2.415	2.426	2.414	2.403	2.418	2.413	2.409
	α	0.712	0.0	0.710	0.714	0.709	0.705	0.715	0.708	0.702
			0.3	0.710	0.714	0.710	0.707	0.714	0.709	0.705
			0.9	0.712	0.712	0.712	0.711	0.712	0.711	0.711
	S(t)	0.911	0.0	0.909	0.910	0.909	0.908	0.910	0.909	0.908
			0.3	0.910	0.910	0.910	0.909	0.910	0.910	0.909
			0.9	0.911	0.911	0.911	0.911	0.911	0.911	0.911
	h(t)	0.461	0.0	0.465	0.479	0.465	0.451	0.488	0.457	0.433
			0.3	0.464	0.474	0.463	0.454	0.480	0.458	0.440
			0.9	0.461	0.463	0.461	0.460	0.464	0.461	0.458
Case II	β	2.413	0.0	2.405	2.503	2.398	2.313	2.444	2.392	2.350
			0.3	2.407	2.480	2.398	2.338	2.435	2.398	2.368
			0.9	2.412	2.424	2.411	2.400	2.416	2.411	2.406
	α	0.697	0.0	0.695	0.700	0.694	0.690	0.700	0.693	0.687
			0.3	0.695	0.699	0.695	0.692	0.699	0.694	0.690
			0.9	0.697	0.697	0.697	0.696	0.697	0.697	0.696
	S(t)	0.915	0.0	0.912	0.913	0.912	0.911	0.913	0.912	0.911
			0.3	0.913	0.914	0.913	0.912	0.914	0.913	0.912
			0.9	0.914	0.914	0.914	0.914	0.914	0.914	0.914
	h(t)	0.444	0.0	0.448	0.462	0.448	0.434	0.471	0.440	0.416
			0.3	0.447	0.457	0.447	0.437	0.463	0.441	0.423
			0.9	0.445	0.446	0.444	0.443	0.447	0.444	0.441
Case III	β	2.422	0.0	2.415	2.511	2.408	2.323	2.454	2.402	2.360
			0.3	2.417	2.489	2.408	2.348	2.444	2.408	2.378
			0.9	2.421	2.433	2.421	2.410	2.425	2.420	2.415
	α	0.735	0.0	0.733	0.737	0.733	0.728	0.738	0.731	0.726
			0.3	0.733	0.737	0.733	0.730	0.737	0.732	0.728
			0.9	0.735	0.735	0.735	0.734	0.735	0.734	0.734
	S(t)	0.906	0.0	0.904	0.905	0.904	0.903	0.905	0.904	0.903
			0.3	0.905	0.906	0.905	0.904	0.905	0.905	0.904
			0.9	0.906	0.906	0.906	0.906	0.906	0.906	0.906
	h(t)	0.487	0.0	0.491	0.506	0.490	0.476	0.514	0.483	0.459
			0.3	0.490	0.500	0.489	0.479	0.506	0.484	0.466
			0.9	0.487	0.489	0.487	0.486	0.490	0.486	0.483
Case IV	β	2.415	0.0	2.407	2.504	2.400	2.316	2.446	2.394	2.353
			0.3	2.410	2.482	2.400	2.341	2.437	2.401	2.370
			0.9	2.414	2.426	2.413	2.402	2.418	2.413	2.408
	α	0.710	0.0	0.708	0.712	0.707	0.703	0.713	0.706	0.700
			0.3	0.708	0.712	0.708	0.705	0.712	0.707	0.703
			0.9	0.710	0.710	0.710	0.709	0.710	0.709	0.709
	S(t)	0.912	0.0	0.910	0.910	0.909	0.909	0.910	0.909	0.908
			0.3	0.910	0.911	0.910	0.910	0.911	0.910	0.909
			0.9	0.911	0.912	0.911	0.911	0.912	0.911	0.911
	h(t)	0.459	0.0	0.463	0.477	0.462	0.449	0.486	0.455	0.431
			0.3	0.462	0.472	0.461	0.452	0.478	0.456	0.438
			0.9	0.459	0.461	0.459	0.458	0.462	0.458	0.455

Table 2. *Cont.*

Cases	Parameters	MLE	v	BSE	BLINEX			BGE		
					a = -5	a = 0.3	a =5	a = -5	a = 0.3	a = 5
Case V	β	2.41473	0.0	2.407	2.504	2.40	2.315	2.446	2.394	2.352
			0.3	2.409	2.481	2.400	2.340	2.437	2.400	2.370
			0.9	2.414	2.425	2.413	2.402	2.418	2.412	2.408
	α	0.707	0.0	0.705	0.710	0.705	0.701	0.711	0.704	0.697
			0.3	0.706	0.709	0.706	0.703	0.710	0.705	0.700
			0.9	0.707	0.708	0.707	0.707	0.708	0.707	0.706
	S(t)	0.912	0.0	0.910	0.911	0.910	0.909	0.911	0.910	0.909
			0.3	0.911	0.911	0.911	0.910	0.911	0.910	0.910
			0.9	0.912	0.912	0.912	0.912	0.912	0.912	0.912
	h(t)	0.456	0.0	0.461	0.474	0.460	0.446	0.483	0.452	0.428
			0.3	0.459	0.469	0.459	0.449	0.476	0.453	0.435
			0.9	0.457	0.458	0.457	0.455	0.459	0.456	0.453
Case VI	β	2.412	0.0	2.404	2.502	2.397	2.312	2.444	2.391	2.349
			0.3	2.406	2.479	2.397	2.337	2.434	2.397	2.367
			0.9	2.411	2.423	2.410	2.399	2.415	2.410	2.405
	α	0.688	0.0	0.685	0.690	0.685	0.680	0.691	0.683	0.677
			0.3	0.686	0.689	0.686	0.683	0.690	0.685	0.680
			0.9	0.687	0.688	0.687	0.687	0.688	0.687	0.687
	S(t)	0.917	0.0	0.915	0.915	0.914	0.914	0.915	0.914	0.913
			0.3	0.915	0.916	0.915	0.915	0.916	0.915	0.914
			0.9	0.916	0.917	0.916	0.916	0.917	0.916	0.916
	h(t)	0.433	0.0	0.438	0.451	0.437	0.424	0.460	0.429	0.405
			0.3	0.437	0.446	0.436	0.427	0.453	0.431	0.413
			0.9	0.434	0.435	0.434	0.433	0.436	0.433	0.430

Table 3. Average estimates and RMSEs in bold for α, β, S, and h for Case I.

Parameters	MLE	v	BSE	BLINEX			BGE		
				a = -5	a = 0.3	a =5	a = -5	a = 0.3	a = 5
β	1.876	0.0	1.869	1.964	1.964	1.779	1.917	1.853	1.804
	0.651		**0.645**	**0.740**	**0.637**	**0.554**	**0.692**	**0.692**	**0.581**
		0.3	1.871	1.941	1.866	1.803	1.905	1.860	1.824
			0.647	**0.719**	**0.641**	**0.577**	**0.680**	**0.636**	**0.600**
		0.6	1.873	1.916	1.870	1.831	1.893	1.867	1.845
			0.648	**0.692**	**0.645**	**0.605**	**0.667**	**0.642**	**0.621**
		0.9	1.875	1.887	1.874	1.864	1.880	1.873	1.868
			0.650	**0.662**	**0.649**	**0.638**	**0.655**	**0.648**	**0.643**
α	3.244	0.0	3.052	3.503	3.015	2.916	3.234	3.007	2.917
	1.050		**0.813**	**1.277**	**0.781**	**0.834**	**0.977**	**0.786**	**0.763**
		0.3	3.110	3.456	3.081	2.970	3.237	3.075	2.994
			0.883	**1.242**	**0.853**	**0.852**	**1.000**	**0.857**	**0.818**
		0.6	3.167	3.393	3.149	3.043	3.240	3.146	3.085
			0.954	**1.195**	**0.932**	**0.886**	**1.047**	**0.936**	**0.894**
		0.9	3.224	3.297	3.219	3.167	3.243	3.219	3.199
			1.026	**1.111**	**1.019**	**0.970**	**1.044**	**1.020**	**1.003**
S(t)	0.969	0.0	0.695	0.704	0.695	0.688	0.705	0.693	0.684
	0.031		**0.284**	**0.275**	**0.286**	**0.293**	**0.275**	**0.288**	**0.296**
		0.3	0.702	0.708	0.701	0.696	0.709	0.700	0.693
			0.279	**0.273**	**0.279**	**0.284**	**0.272**	**0.282**	**0.289**
		0.6	0.708	0.712	0.708	0.705	0.712	0.707	0.702
			0.273	**0.270**	**0.273**	**0.277**	**0.270**	**0.275**	**0.279**
		0.9	0.714	0.715	0.714	0.713	0.715	0.714	0.713
			0.268	**0.268**	**0.268**	**0.270**	**0.268**	**0.268**	**0.270**
h(t)	0.151	0.0	0.663	0.689	0.661	0.629	0.693	0.649	0.603
	0.100		**0.583**	**0.611**	**0.582**	**0.548**	**0.614**	**0.570**	**0.524**
		0.3	0.654	0.674	0.653	0.630	0.678	0.644	0.611
			0.332	**0.356**	**0.330**	**0.303**	**0.360**	**0.320**	**0.284**
		0.6	0.646	0.658	0.645	0.632	0.661	0.640	0.620

			0.567	**0.580**	**0.566**	**0.553**	**0.583**	**0.562**	**0.542**
		0.9	0.637	0.641	0.637	0.634	0.641	0.636	0.631
			0.559	**0.563**	**0.559**	**0.555**	**0.563**	**0.558**	**0.553**

Table 4. Average estimates and RMSEs in bold for α, β, S, and h for Case II.

				BLINEX			BGE		
Parameters	MLE	v	BSE	a = -5	a = 0.3	a =5	a = -5	a = 0.3	a = 5
β	1.602	0.0	1.600	1.670	1.670	1.532	1.639	1.587	1.546
	0.375		**0.372**	**0.441**	**0.368**	**0.308**	**0.409**	**0.360**	**0.322**
		0.3	1.601	1.652	1.597	1.550	1.6291	1.592	1.561
			0.374	**0.424**	**0.370**	**0.325**	**0.400**	**0.364**	**0.337**
		0.6	1.601	1.632	1.599	1.571	1.617	1.596	1.578
			0.374	**0.404**	**0.372**	**0.344**	**0.388**	**0.368**	**0.352**
		0.9	1.602	1.610	1.601	1.594	1.606	1.601	1.596
			0.374	**0.383**	**0.374**	**0.366**	**0.378**	**0.372**	**0.372**
α	2.997	0.0	2.846	3.240	2.812	2.687	3.014	2.800	2.705
	0.667		**0.543**	**0.846**	**0.523**	**0.545**	**0.648**	**0.526**	**0.517**
		0.3	2.891	3.194	2.866	2.740	3.009	2.857	2.775
			0.577	**0.815**	**0.559**	**0.457**	**0.653**	**0.561**	**0.538**
		0.6	2.936	3.133	2.921	2.812	3.004	2.916	2.857
			0.614	**0.772**	**0.602**	**0.562**	**0.659**	**0.602**	**0.575**
		0.9	2.982	3.042	2.978	2.930	2.999	2.976	2.958
			0.653	**0.705**	**0.649**	**0.616**	**0.664**	**0.664**	**0.638**
S(t)	0.763	0.0	0.743	0.750	0.742	0.737	0.750	0.740	0.734
	0.221		**0.240**	**0.232**	**0.240**	**0.246**	**0.232**	**0.242**	**0.248**
		0.3	0.749	0.754	0.748	0.744	0.754	0.747	0.742
			0.234	**0.228**	**0.234**	**0.238**	**0.228**	**0.236**	**0.240**
		0.6	0.755	0.758	0.755	0.752	0.758	0.754	0.751
			0.228	**0.225**	**0.228**	**0.232**	**0.225**	**0.230**	**0.232**
		0.9	0.761	0.762	0.761	0.760	0.762	0.761	0.760
			0.223	**0.221**	**0.223**	**0.223**	**0.221**	**0.223**	**0.223**
h(t)	0.510	0.0	0.536	0.554	0.535	0.513	0.561	0.524	0.485
	0.433		**0.458**	**0.477**	**0.457**	**0.433**	**0.483**	**0.446**	**0.407**
		0.3	0.528	0.542	0.527	0.512	0.548	0.520	0.491
			0.450	**0.465**	**0.449**	**0.433**	**0.459**	**0.442**	**0.414**
		0.6	0.520	0.529	0.520	0.511	0.533	0.515	0.499
			0.443	**0.452**	**0.442**	**0.433**	**0.456**	**0.439**	**0.421**
		0.9	0.512	0.515	0.512	0.510	0.516	0.511	0.507
			0.435	**0.438**	**0.435**	**0.433**	**0.439**	**0.434**	**0.431**

Table 5. Average estimates and RMSEs in bold for α, β, S, and h for Case III.

				BLINEX			BGE		
Parameters	MLE	v	BSE	a = -5	a = 0.3	a =5	a = -5	a = 0.3	a = 5
β	1.555	0.0	1.553	1.618	1.618	1.489	1.590	1.541	1.501
	0.334		**0.334**	**0.396**	**0.330**	**0.275**	**0.368**	**0.324**	**0.288**
		0.3	1.554	1.601	1.550	1.507	1.580	1.545	1.516
			0.334	**0.380**	**0.331**	**0.289**	**0.359**	**0.327**	**0.308**
		0.6	1.554	1.582	1.552	1.526	1.569	1.549	1.532
			0.334	**0.361**	**0.333**	**0.308**	**0.349**	**0.330**	**0.314**
		0.9	1.554	1.562	1.554	1.547	1.558	1.553	1.549
			0.334	**0.342**	**0.334**	**0.327**	**0.339**	**0.334**	**0.330**
α	2.988	0.0	2.849	3.223	2.817	2.689	3.007	2.805	2.711
	0.634		**0.515**	**0.808**	**0.495**	**0.517**	**0.617**	**0.498**	**0.488**
		0.3	2.890	3.178	2.867	2.740	3.001	2.858	2.778
			0.548	**0.777**	**0.531**	**0.518**	**0.622**	**0.531**	**0.508**
		0.6	2.932	3.118	2.918	2.811	2.996	2.913	2.857
			0.583	**0.736**	**0.571**	**0.532**	**0.626**	**0.571**	**0.545**
		0.9	2.974	3.031	2.970	2.925	2.990	2.969	2.951
			0.621	**0.670**	**0.617**	**0.584**	**0.631**	**0.617**	**0.605**
S(t)	0.776	0.0	0.757	0.763	0.756	0.751	0.764	0.755	0.749
	0.204		**0.221**	**0.214**	**0.223**	**0.228**	**0.214**	**0.223**	**0.230**

Parameters	MLE	v	BSE	BLINEX a = -5	BLINEX a = 0.3	BLINEX a =5	BGE a = -5	BGE a = 0.3	BGE a = 5
		0.3	0.762	0.767	0.762	0.758	0.767	0.761	0.756
			0.216	**0.212**	**0.216**	**0.221**	**0.212**	**0.219**	**0.223**
		0.6	0.768	0.771	0.768	0.766	0.771	0.767	0.764
			0.212	**0.209**	**0.212**	**0.214**	**0.207**	**0.212**	**0.216**
		0.9	0.774	0.775	0.774	0.773	0.775	0.774	0.773
			0.207	**0.204**	**0.207**	**0.207**	**0.204**	**0.207**	**0.207**
h(t)	0.484	0.0	0.508	0.525	0.507	0.487	0.532	0.497	0.459
	0.403		**0.426**	**0.444**	**0.425**	**0.404**	**0.451**	**0.415**	**0.378**
		0.3	0.501	0.514	0.500	0.486	0.520	0.493	0.466
			0.419	**0.433**	**0.418**	**0.403**	**0.438**	**0.412**	**0.384**
		0.6	0.493	0.501	0.493	0.485	0.506	0.489	0.473
			0.412	**0.420**	**0.412**	**0.403**	**0.424**	**0.408**	**0.392**
		0.9	0.486	0.488	0.486	0.484	0.490	0.485	0.481
			0.406	**0.407**	**0.406**	**0.403**	**0.167**	**0.404**	**0.400**

Table 6. Average estimates and RMSEs in bold for α, β, S, and h for Case IV.

Parameters	MLE	v	BSE	BLINEX a = -5	BLINEX a = 0.3	BLINEX a =5	BGE a = -5	BGE a = 0.3	BGE a = 5
β	2.182	0.0	2.166	2.293	2.293	2.054	2.224	2.147	2.090
	0.955		**0.939**	**1.069**	**0.928**	**0.824**	**0.997**	**0.920**	**0.862**
		0.3	2.171	2.266	2.164	2.084	2.212	2.157	2.115
			0.944	**1.042**	**0.937**	**0.854**	**0.985**	**0.930**	**0.887**
		0.6	2.176	2.234	2.172	2.120	2.199	2.168	2.142
			0.949	**1.009**	**0.944**	**0.891**	**0.973**	**0.941**	**0.915**
		0.9	2.180	2.196	2.179	2.164	2.186	2.178	2.171
			0.953	**0.970**	**0.952**	**0.937**	**0.959**	**0.951**	**0.944**
α	2.513	0.0	2.401	2.717	2.376	2.254	2.547	2.361	2.274
	0.658		**0.630**	**0.684**	**0.634**	**0.722**	**0.617**	**0.644**	**0.696**
		0.3	2.435	2.676	2.416	2.302	2.537	2.405	2.331
			0.634	**0.679**	**0.635**	**0.698**	**0.629**	**0.642**	**0.676**
		0.6	2.469	2.623	2.457	2.365	2.527	2.451	2.399
			0.642	**0.674**	**0.641**	**0.671**	**0.641**	**0.645**	**0.659**
		0.9	2.502	2.548	2.499	2.462	2.517	2.497	2.481
			0.654	**0.664**	**0.653**	**0.651**	**0.654**	**0.654**	**0.654**
S(t)	0.551	0.0	0.539	0.552	0.538	0.527	0.558	0.533	0.516
	0.431		**0.442**	**0.431**	**0.442**	**0.423**	**0.423**	**0.448**	**0.464**
		0.3	0.542	0.552	0.542	0.534	0.556	0.538	0.525
			0.439	**0.430**	**0.439**	**0.200**	**0.181**	**0.443**	**0.456**
		0.6	0.546	0.551	0.546	0.541	0.554	0.544	0.536
			0.435	**0.430**	**0.435**	**0.441**	**0.427**	**0.438**	**0.446**
		0.9	0.550	0.558	0.550	0.549	0.552	0.549	0.547
			0.432	**0.431**	**0.432**	**0.433**	**0.430**	**0.432**	**0.434**
h(t)	0.933	0.0	0.945	0.994	0.942	0.892	0.988	0.929	0.881
	0.860		**0.871**	**0.920**	**0.867**	**0.816**	**0.913**	**0.854**	**0.806**
		0.3	0.942	0.978	0.939	0.903	0.973	0.931	0.895
			0.867	**0.904**	**0.865**	**0.828**	**0.898**	**0.856**	**0.820**
		0.6	0.938	0.960	0.937	0.915	0.957	0.932	0.910
			0.864	**0.887**	**0.863**	**0.840**	**0.883**	**0.857**	**0.836**
		0.9	0.935	0.940	0.934	0.929	0.940	0.933	0.927
			0.861	**0.867**	**0.860**	**0.854**	**0.866**	**0.859**	**0.853**

Table 7. Average estimates and RMSEs in bold for α, β, S, and h for Case V.

				BLINEX			BGE		
Parameters	MLE	v	BSE	a = -5	a = 0.3	a =5	a = -5	a = 0.3	a = 5
β	1.630	0.0	1.627	1.699	1.699	1.557	1.667	1.614	1.572
	0.412		**0.409**	**0.480**	**0.404**	**0.342**	**0.447**	**0.397**	**0.359**
		0.3	1.628	1.681	1.624	1.576	1.656	1.619	1.588
			0.411	**0.462**	**0.407**	**0.360**	**0.437**	**0.402**	**0.374**
		0.6	1.629	1.660	1.627	1.597	1.645	1.623	1.605
			0.411	**0.442**	**0.408**	**0.380**	**0.426**	**0.406**	**0.389**
		0.9	1.630	1.638	1.629	1.621	1.634	1.628	1.623
			0.412	**0.420**	**0.411**	**0.403**	**0.415**	**0.411**	**0.406**
α	2.920	0.0	2.775	3.154	2.743	2.621	2.938	2.731	2.639
	0.676		**0.564**	**0.835**	**0.457**	**0.584**	**0.653**	**0.553**	**0.554**
		0.3	2.818	3.109	2.795	2.672	2.933	2.786	2.706
			0.594	**0.807**	**0.579**	**0.582**	**0.660**	**0.582**	**0.568**
		0.6	2.862	3.050	2.847	2.743	2.927	2.842	2.785
			0.628	**0.770**	**0.617**	**0.855**	**0.667**	**0.618**	**0.597**
		0.9	2.905	2.963	2.901	2.856	2.922	2.900	2.882
			0.664	**0.637**	**0.660**	**0.630**	**0.673**	**0.660**	**0.651**
S(t)	0.747	0.0	0.727	0.734	0.726	0.720	0.735	0.724	0.717
	0.236		**0.254**	**0.246**	**0.254**	**0.260**	**0.244**	**0.256**	**0.264**
		0.3	0.733	0.738	0.732	0.728	0.739	0.731	0.725
			0.248	**0.242**	**0.248**	**0.252**	**0.242**	**0.250**	**0.256**
		0.6	0.739	0.742	0.738	0.736	0.742	0.737	0.734
			0.242	**0.240**	**0.242**	**0.246**	**0.238**	**0.244**	**0.248**
		0.9	0.745	0.745	0.744	0.744	0.745	0.744	0.743
			0.238	**0.236**	**0.238**	**0.238**	**0.236**	**0.238**	**0.238**
h(t)	0.538	0.0	0.563	0.583	0.562	0.538	0.589	0.551	0.512
	0.460		**0.484**	**0.504**	**0.482**	**0.458**	**0.509**	**0.472**	**0.433**
		0.3	0.556	0.570	0.555	0.538	0.576	0.547	0.519
			0.477	**0.292**	**0.476**	**0.459**	**0.523**	**0.472**	**0.440**
		0.6	0.548	0.557	0.547	0.538	0.561	0.543	0.526
			0.470	**0.479**	**0.469**	**0.459**	**0.482**	**0.465**	**0.448**
		0.9	0.541	0.543	0.540	0.538	0.544	0.539	0.535
			0.462	**0.465**	**0.462**	**0.460**	**0.466**	**0.461**	**0.457**

Table 8. Average estimates and RMSEs in bold for α, β, S, and h for Case VI.

				BLINEX			BGE		
Parameters	MLE	v	BSE	a = -5	a = 0.3	a =5	a = -5	a = 0.3	a = 5
β	1.560	0.0	1.558	1.623	1.623	1.493	1.595	1.545	1.505
	0.336		**0.334**	**0.397**	**0.330**	**0.273**	**0.368**	**0.322**	**0.288**
		0.3	1.558	1.607	1.555	1.511	1.585	1.550	1.521
			0.334	**0.380**	**0.331**	**0.289**	**0.359**	**0.327**	**0.301**
		0.6	1.559	1.588	1.557	1.531	1.575	1.554	1.537
			0.334	**0.363**	**0.333**	**0.308**	**0.349**	**0.330**	**0.314**
		0.9	1.560	1.567	1.559	1.552	1.564	1.559	1.554
			0.336	**0.343**	**0.336**	**0.328**	**0.339**	**0.334**	**0.330**
α	2.947	0.0	2.801	3.184	2.768	2.644	2.965	2.757	2.663
	0.653		**0.542**	**0.819**	**0.524**	**0.555**	**0.634**	**0.529**	**0.527**
		0.3	2.845	3.138	2.821	2.696	2.960	2.812	2.731
			0.527	**0.789**	**0.556**	**0.553**	**0.639**	**0.558**	**0.542**
		0.6	2.888	3.079	2.874	2.767	2.954	2.868	2.811
			0.572	**0.750**	**0.594**	**0.562**	**0.645**	**0.594**	**0.572**
		0.9	2.932	2.990	2.928	2.882	2.948	2.927	2.909

Parameters	MLE	v	BSE	BLINEX			BGE		
				a = -5	a = 0.3	a =5	a = -5	a = 0.3	a = 5
			0.641	**0.689**	**0.637**	**0.607**	**0.651**	**0.637**	**0.627**
S(t)	0.769	0.0	0.749	0.756	0.748	0.743	0.756	0.746	0.740
	0.212		**0.232**	**0.223**	**0.232**	**0.238**	**0.223**	**0.234**	**0.240**
		0.3	0.755	0.760	0.754	0.750	0.760	0.753	0.748
			0.225	**0.221**	**0.225**	**0.230**	**0.221**	**0.228**	**0.232**
		0.6	0.761	0.764	0.760	0.758	0.764	0.760	0.756
			0.221	**0.216**	**0.221**	**0.223**	**0.216**	**0.221**	**0.225**
		0.9	0.767	0.767	0.767	0.766	0.767	0.766	0.765
			0.214	**0.214**	**0.214**	**0.216**	**0.214**	**0.214**	**0.216**
h(t)	0.493	0.0	0.519	0.536	0.517	0.497	0.543	0.507	0.469
	0.413		**0.437**	**0.454**	**0.435**	**0.414**	**0.461**	**0.425**	**0.388**
		0.3	0.511	0.524	0.510	0.496	0.530	0.503	0.475
			0.430	**0.443**	**0.428**	**0.413**	**0.449**	**0.421**	**0.394**
		0.6	0.504	0.512	0.503	0.495	0.516	0.499	0.482
			0.423	**0.431**	**0.421**	**0.413**	**0.434**	**0.418**	**0.402**
		0.9	0.496	0.498	0.496	0.494	0.499	0.495	0.491
			0.415	**0.418**	**0.415**	**0.413**	**0.419**	**0.414**	**0.411**