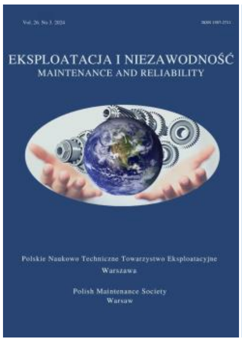

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Comparative Analysis of Stochastic and Uncertain Process Degradation Modeling Based on RQRL

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Highlights

- Small sample sizes cause epistemic uncertainties in reliability estimation.
- Uncertainty theory was utilized to address epistemic uncertainties.
- The Wiener and Liu process degradation models were proposed.
- Sensitivities of degradation models for various sample sizes and measurement times were analyzed based on RQRL.
- Results showed using uncertain process degradation model improved stability of reliability estimation under small-sample conditions.

Abstract

Small sample sizes cause epistemic uncertainties in reliability estimation and even result in potential risks in maintenance strategies. To explore the difference between stochastic- and uncertain-process-based degradation modeling in reliability estimation for small samples, this study proposes a comparative analysis methodology based on the range of quantile reliable lifetime (RQRL). First, considering both unit-to-unit variability and epistemic uncertainty, we proposed the Wiener and Liu process degradation models. Second, based on the RQRL, a comparative analysis method of different degradation models for reliability estimation under various sample sizes and measurement times was proposed. Third, based on a case study, the sensitivities of the Wiener and Liu process degradation models for various sample sizes and measurement times were compared and analyzed based on the RQRL. The results demonstrated that using the uncertain process degradation model improved the uniformity and stability of reliability estimation under small-sample conditions.

Keywords

degradation modeling, epistemic uncertainty, uncertain process, stochastic process, reliability estimation

1. Introduction

Currently, products are expected to be highly reliable to increase system safety and reduce operational costs.[1] However, these highly reliable and long-lifetime products can work for years without failure in regular reliability testing, and obtaining lifetime data is an extremely difficult and expensive task. Therefore, degradation data-based reliability estimation is necessary.[2,3]

The degradation model is a key metric for degradation analysis and reliability estimation. In existing studies, stochastic process models accurately depict the performance degradation caused by the operational environment.[4,5] Three widely used stochastic processes include the Wiener,[6,7] gamma,[8,9] and inverse Gaussian,[10,11] processes. The Wiener process model has attracted significant interest because of its flexible application.[12] Wang et al.[13] investigated the reliability of semiconductor products with recoverability effects. Chen et al.[14] proposed a prognostic framework based on models and data-driven methods. Cai et al.[15] forecasted the remaining useful life using

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current and historical data. Sun et al.[16] analyzed a nonlinear Wiener process by considering the competing causes of failures. Zheng et al.[17] developed a bivariate Wiener model with the initial product status and deterioration speed.

Although stochastic process models have been widely used, Kang et al.[18] reported that probability-theory-based stochastic process models are better adapted for applications wherein sufficient samples are available to fit the probability distributions. However, only a few samples were actually tested owing to limited resources and costs. A small sample size can result in insufficient information to discriminate the product population. This results in epistemic uncertainties in the product reliability estimation, which can result in potential risks in the maintenance strategy. Based on normality, duality, subadditivity, and product axioms, Liu[19,20] proposed the uncertainty theory. Uncertainty theory is a branch of axiomatic mathematics parallel to probability theory and is considered an effective mathematical system for addressing epistemic uncertainties. Based on the uncertainty theory, Kang et al.[21-23] rebuilt the framework of the reliability theory and proposed a new belief reliability theory. Chen et al.[24] proposed a design optimization framework based on belief reliability. Liu et al.[25] introduced a belief reliability growth model for software. Furthermore, certain scholars have studied reliability estimation based on uncertain processes in the uncertainty theory. Li et al.[26,27] and Wu et al.[28] considered different combinations of unit, time, and stress dimensions and proposed three uncertain accelerated degradation testing (ADT) models. Chen et al.[29] integrated multi-source ADT data by constructing evaluation indexes for the datasets. Li et al.[30] quantified the uncertainties in the degradation process of a planetary reducer. Li et al.[31] investigated a multi-state degradation system with epistemic uncertainties. Wang et al.[32] developed an uncertain-process-based reliability model for two-phase deteriorating systems. Chen et al.[33] analyzed the hybrid uncertainties of the dependent competing failure process. Thus, uncertain processes have garnered increasing attention for reliability estimations that consider epistemic uncertainties.

A comparative analysis of stochastic- and uncertain-process-based degradation models is currently among the research highlights, particularly under small-sample conditions (sample size is less than ten[34]). Based on the Liu process in uncertain processes, Li et al.[26] introduced an uncertain ADT model and proposed an assessment index termed "range of quantile reliable lifetime (RQRL) under specific reliability" to compare the differences between Wiener- and Liu-process-based ADT models in unit dimension. However, when considering both product unit-to-unit variability and degradation epistemic uncertainty, there is a lack of comparative analysis between stochastic and uncertain process-based degradation models for reliability estimation under various sample sizes and measurement times. Motivated by Li et al.[26], under small-sample conditions, this study proposed a comparative analysis methodology of stochastic process and uncertain process degradation modeling based on the RQRL under various sample sizes and measurement times. In stochastic processes, the Wiener process has garnered widespread attention owing to its nonmonotonic behavior. Similarly, as an uncertain process for depicting nonmonotonic degradation trajectories, the Liu process is widely used for reliability modeling and estimation. Thus, we selected the Wiener and Liu processes as typical stochastic and uncertain processes for degradation modeling, respectively. The contributions of this study are summarized below. First, considering both unit-to-unit variability and epistemic uncertainty, we proposed the Wiener and uncertain process degradation models. Second, based on the RQRL, we proposed a comparative analysis method for different degradation models for reliability estimation under various sample sizes and measurement times. Finally, using GaAs laser data in a real case study, we compared and analyzed the sensitivities of the Wiener and Liu process degradation models to various sample sizes and measurement times based on the RQRL.

The remainder of this paper is organized as follows. Section 2 proposes the Wiener- and Liu-process-based degradation and reliability models. Section 3 presents statistical methods for the Wiener and Liu process degradation models. Section 4 presents a case study to analyze the sensitivity of the degradation model to various sample sizes and measurement times based on the RQRL. Finally, Section 5 concludes the study.

2. Degradation and reliability modeling

2.1. Degradation modeling

2.1.1. Wiener-process-based degradation modeling

According to Si et al.[35], a Wiener degradation model with a linear trajectory is expressed as

$$Y(t) = \nu_w t + \lambda_w B_w(t), \quad (1)$$

where $Y(t)$ is cumulative degradation at time t, ν_w is drift coefficient representing the degradation rate, λ_w is the diffusion coefficient, and $B_w(t)$ is a standard Brownian motion representing the inherent temporal variability of Wiener process.[35] $B_w(t)$ obeys a normal probability distribution with expected value 0 and variance t, denoted by $B_w(t) \sim N(0, t)$ for each $t \in (0, +\infty)$. The cumulative distribution function (CDF) is expressed as

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{t}\sqrt{2\pi}} exp\left(-\frac{z^2}{2t}\right) dz = \Phi\left(\frac{x}{\sqrt{t}}\right), \quad (2)$$

where $\Phi(\cdot)$ denotes a standard normal probability distribution function.

Individuality reveals different degradation rates because of the impact of product techniques. To demonstrate this unit-to-unit variability, specific parameters can be rendered randomly.[36] According to probability theory, to depict unit-to-unit degradation rate in Equation (1), ν_w can be assumed to follow a normal distribution[36], which is expressed as

$$\nu_w \sim N(\mu_{wv}, \sigma_{wv}^2), \quad (3)$$

where $N(\cdot)$ denotes a normal probability distribution, and μ_{wv} and σ_{wv} are the expected value and standard deviation, respectively.

Equations (1) and (3) are denoted as M_1. Within the framework of the probability theory, Model M_1 considers both unit-to-unit and temporal variabilities. Based on model M_1, the degradation values obey a normal distribution with expected value $\mu_{wv}t$ and variance $\sigma_{wv}^2 t^2 + \lambda_w^2 t$, that is, $Y(t) \sim N(\mu_{wv}t, \sigma_{wv}^2 t^2 + \lambda_w^2 t)$. The probability density function (PDF) is

$$f_{(w)Y}(y) = \frac{1}{\sqrt{2\pi(\sigma_{wv}^2 t^2 + \lambda_w^2 t)}} exp\left(-\frac{(y-\mu_{wv}t)^2}{2(\sigma_{wv}^2 t^2 + \lambda_w^2 t)}\right). \quad (4)$$

2.1.2. Liu-process-based degradation modeling

According to uncertainty theory, when discriminating the population of products, a small sample size can provide insufficient information and result in epistemic uncertainties in reliability estimation. Chen et al.[37] used a diffusion term with respect to time to describe epistemic uncertainties owing to a small sample size. Thus, a linear degradation model based on the Liu process is expressed as

$$Y(t) = \nu_u t + \lambda_u C_u(t), \quad (5)$$

where ν_u is the drift coefficient representing degradation rate, λ_u is the diffusion coefficient, $\lambda_u C_u(t)$ is the diffusion term with respect to time representing epistemic uncertainties, and $C_u(t)$ is an uncertain Liu process. $C_u(t)$ obeys a normal uncertainty distribution with expected value 0 and variance t^2, denoted by $C_u(t) \sim N_u(0, t)$ for each $t \in (0, +\infty)$.[20,38] The uncertainty distribution function is

$$\Phi_{(u)t}(x) = \left(1 + exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad (6)$$

where $\Phi_{(u)}(\cdot)$ denotes an uncertainty distribution function.

Note that there are three differences between the Wiener process $B_w(t)$, and Liu process $C_u(t)$.

(i) $B_w(t)$ and $C_u(t)$ are different mathematical systems. $B_w(t)$ belongs to the probability-theory-based stochastic process, where almost all sample paths are continuous. In contrast, $C_u(t)$ belongs to the uncertainty-theory-based uncertain process, where almost all sample paths are Lipschitz continuous.[20]

(ii) The distribution functions of $B_w(t)$ and $C_u(t)$ are different. $B_w(t)$ follows the normal probability distribution as in Equation (2). However, $C_u(t)$ follows a normal uncertainty distribution, as in Equation (6).

(iii) The rates of change for the increments in $B_w(t)$ and $C_u(t)$ are different. For $B_w(t)$ and for each $\Delta t > 0$, the change rate for increments $(B_w(t + \Delta t) - B_w(t))/\Delta t$ is a random variable with expected value 0 and variance $1/\Delta t$, that is,

$$\frac{B_w(t+\Delta t) - B_w(t)}{\Delta t} \sim N\left(0, \frac{1}{\Delta t}\right). \quad (7)$$

In Equation (7), the distribution of change rate for increments relates to Δt. When Δt approaches zero, $(B_w(t + \Delta t) - B_w(t))/\Delta t \sim N(0, \infty)$. For $C_u(t)$ and for each $\Delta t > 0$, the change rate for increments $(C_u(t + \Delta t) - C_u(t))/\Delta t$ is an uncertain variable with expected 0 and variance 1, that is,

$$\frac{C_u(t+\Delta t) - C_u(t)}{\Delta t} \sim N_u(0,1). \quad (8)$$

In Equation (8), the distribution of change rate for increments and the Δt are independent, which is more suitable for describing a uniform motion process.

Based on the Liu process, to depict unit-to-unit degradation rate in Equation (5), ν_u can be assumed to follow a normal uncertainty distribution under small sample conditions[28], which is expressed as

$$\nu_u \sim N_u(\mu_{uv}, \sigma_{uv}), \tag{9}$$

where $N_u(\cdot)$ denotes normal uncertainty distribution, and μ_{uv} and σ_{uv} are the expected value and standard deviation, respectively.

Equations (5) and (9) are denoted as M_2. Within the framework of the uncertainty theory, Model M_2 considers both unit-to-unit variability and epistemic uncertainty. Based on model M_2, the degradation values follow a normal uncertainty distribution with expected value $\mu_{uv}t$ and variance $(\sigma_{uv}+\lambda_u)^2 t^2$, that is, $Y(t) \sim N_u(\mu_{uv}t, (\sigma_{uv}+\lambda_u)t)$. The uncertainty distribution function is expressed as:

$$\Phi_{(u)t}(y) = \left(1 + exp\left(\frac{\pi(\mu_{uv}t - y)}{\sqrt{3}t(\sigma_{uv}+\lambda_u)}\right)\right)^{-1}. \tag{10}$$

The two degradation models based on the Wiener and Liu processes are presented in Table 1. As evident, the primary difference between the models M_1 and M_2 is the uncertainty degradation modeling. In model M_1, the unit-to-unit and temporal variabilities are represented by a normal probability distribution, whereas in model M_2, the unit-to-unit variability and epistemic uncertainty are represented by a normal uncertainty distribution.

Table 1. Wiener- and Liu-process-based degradation models.

Notation	Model
M_1	$Y(t) = \nu_w t + \lambda_w B_w(t)$, $B_w(t) \sim N(0, t)$, $\nu_w \sim N(\mu_{wv}, \sigma_{wv}^2)$
M_2	$Y(t) = \nu_u t + \lambda_u C_u(t)$, $C_u(t) \sim N_u(0, t)$, $\nu_u \sim N_u(\mu_{uv}, \sigma_{uv})$

2.2. Reliability modeling

The moment when the performance degradation reaches a pre-given threshold D for the first time is generally termed the lifetime T_D or first hitting time (FHT).[26]

$$T_D = inf\{t_D \geq 0 | Y(t) = D\}. \tag{11}$$

2.2.1. Wiener-process-based reliability modeling

It has been proven that the FHT of the Wiener process obeys an inverse Gaussian distribution.[2] Using Equation (1), when only temporal variability is considered, the PDF of FHT is

$$f_T(t) = \frac{D}{\sqrt{2\pi\lambda_w^2 t^3}} exp\left(-\frac{(D-\nu_w t)^2}{2\lambda_w^2 t}\right), \tag{12}$$

and the CDF of FHT is

$$F_T(t) = \Phi\left(\frac{\nu_w t - D}{\lambda_w\sqrt{t}}\right) + exp\left(\frac{2\nu_w D}{\lambda_w^2}\right)\Phi\left(\frac{-D-\nu_w t}{\lambda_w\sqrt{t}}\right). \tag{13}$$

Considering both unit-to-unit and temporal variabilities, Si et al.[35] derived the PDF and CDF of the FHT for the Wiener process. Thus, based on model M_1, the PDF of the FHT is expressed as

$$f_T(t) = \frac{D}{\sqrt{2\pi t^3(\sigma_{wv}^2 t + \lambda_w^2)}} exp\left(-\frac{(D-\mu_{wv}t)^2}{2t(\sigma_{wv}^2 t + \lambda_w^2)}\right), \tag{14}$$

and the reliability model is expressed as

$$R_T(t) = 1 - F_T(t) = \Phi\left(\frac{D-\mu_{wv}t}{\sqrt{\sigma_{wv}^2 t^2 + \lambda_w^2 t}}\right) - exp\left(\frac{2\mu_{wv}D}{\lambda_w^2} + \frac{2\sigma_{wv}^2 D^2}{\lambda_w^4}\right)\Phi\left(-\frac{2\sigma_{wv}^2 Dt + \lambda_w^2(D+\mu_{wv}t)}{\lambda_w^2\sqrt{\sigma_{wv}^2 t^2 + \lambda_w^2 t}}\right). \tag{15}$$

2.2.2. Liu-process-based reliability modeling

According to the definition of the FHT for an uncertain process[39], the uncertainty distribution of the FHT for model M_2 is expressed as

$$\mathrm{H}_u(t) = \mathrm{M}\{t_D \leq t\} = \mathrm{M}\{sup\ \ Y_u(t) \geq D\}, \tag{16}$$

where $\mathrm{H}_u(t)$ is the uncertainty distribution of FHT, and $\mathrm{M}\{\cdot\}$ is the uncertain measure.

Referring to Liu[40] and Wu et al.[28], we can conclude that $Y(t)$ in model M_2 is an uncertain process with independent degradation increments. Applying the extreme value theorem[39] of the uncertain process, the analytical expression of the FHT for model M_2 is expressed as

$$\mathrm{H}_u(t) = \left(1 + exp\left(\frac{\pi(D-\mu_{uv}t)}{\sqrt{3}(\sigma_{uv}+\lambda_u)t}\right)\right)^{-1}. \tag{17}$$

Based on the uncertainty theory, Kang et al.[22] defined two new reliability metrics, that is, the belief reliability $R_u(t)$ and the belief reliable lifespan $BL_u(\alpha)$. Within the required operating status, $R_u(t)$ is the belief degree for a product performing a specific function at time t.[22] Based on model M_2, the reliability model is expressed as

$$R_u(t) = 1 - \mathrm{H}_u(t) = \left(1 + exp\left(\frac{\pi(\mu_{uv}t - D)}{\sqrt{3}(\sigma_{uv}+\lambda_u)t}\right)\right)^{-1}. \tag{18}$$

$BL_u(\alpha)$ is defined as the supremum lifespan when $R_u(t)$ is higher than the belief degree α ($\alpha \in [0, 1]$)[22], which is expressed as

$$BL_u(\alpha) = sup\{t | R_u(t) \geq \alpha\}. \tag{19}$$

2.3. RQRL

The assessment index "RQRL" is short for "range of quantile reliable lifetime" under specific reliability.[26] RQRL refers to the distance between the lower and upper limits of reliable lifespan; thus, it stands for the uniformity of reliability estimation. A higher RQRL indicates a less uniform reliability estimation. Based on the RQRL, we proposed a comparative analysis method for different degradation models for reliability estimation under various sample sizes and measurement times.

First, we analyzed the difference between the Wiener and Liu process degradation models in terms of reliability estimation for various sample sizes. We assume that the total sample size is p, from which we randomly selected n samples. From this, there are C_p^n combinations of sample sizes, and we can obtain C_p^n reliability curves. Using Equation (20), we can calculate the lower and upper limits of the reliability estimation.

$$\begin{cases} R_{(n)}^{ll}(t) = \overset{C_p^n}{\underset{g=1}{\wedge}} R_{(n)}^{g}(t) \\ R_{(n)}^{ul}(t) = \overset{C_p^n}{\underset{g=1}{\vee}} R_{(n)}^{g}(t) \end{cases}, \tag{20}$$

where $R_{(n)}^{ll}(t)$, $R_{(n)}^{ul}(t)$, and $R_{(n)}^{g}(t)$ denote the lower limit, upper limit, and g-th reliability estimation results, respectively, for n sample conditions.

Furthermore, we utilized an assessment index $RQRL_{(n)}(R)$ to denote the difference between the upper limit of reliable lifespan $t_{(n)}^{ul}(R)$ and the lower limit of reliable lifespan $t_{(n)}^{ll}(R)$ with reliability of R under n sample conditions, which is expressed as

$$RQRL_{(n)}(R) = t_{(n)}^{ul}(R) - t_{(n)}^{ll}(R). \tag{21}$$

Thereafter, we analyzed the differences between the Wiener and Liu process degradation models for reliability estimation under different measurement times. We assume that the total measurement time is q, from which we randomly select m measurements. From this, there are C_q^m combinations of measurement times, and we can obtain C_q^m reliability curves. Using Equation (22), we can calculate the lower and upper limits of the reliability estimation for different measurement times.

$$\begin{cases} R_{(m)}^{ll}(t) = \overset{C_q^m}{\underset{g=1}{\wedge}} R_{(m)}^{g}(t) \\ R_{(m)}^{ul}(t) = \overset{C_q^m}{\underset{g=1}{\vee}} R_{(m)}^{g}(t) \end{cases}, \tag{22}$$

where $R_{(m)}^{ll}(t)$, $R_{(m)}^{ul}(t)$, and $R_{(m)}^{g}(t)$ are the lower limit, upper limit, and g-th reliability estimation results, respectively, at measurement time m.

Similarly, we used an assessment index $RQRL_{(m)}(R)$ to quantify the distance between the upper and lower limits of the reliable lifespan with a reliability of R under measurement time m, that is,

$$RQRL_{(m)}(R) = t_{(m)}^{ul}(R) - t_{(m)}^{ll}(R). \tag{23}$$

In a real case study, based on the RQRL of models M_1 and M_2, we compared and analyzed the two degradation models for reliability estimation under various sample sizes and measurement times.

3. Statistical method

In the testing, assuming that the sample size is n, we performed m equally periodic degradation observations. The interval between measurements at the j-th measurement of i-th sample is denoted as Δt_{ij}, that is, $\Delta t_{ij} = t_{i,j} - t_{i,j-1}$, where $j = 1, 2, \ldots, m$, $i = 1, 2, \ldots, n$, and the degradation increment is $\Delta y_{ij} = Y_{i,j} - Y_{i,j-1}$.

Because the Wiener-process-based model M_1 and the Liu-process-based model M_2 belong to different mathematical systems, we adopted the probability-theory-based maximum likelihood estimation (MLE) and the uncertainty-theory-based principle of least squares (PLS)[41] to estimate the unknown parameters in Section 3.1 and 3.2, respectively.

3.1. Statistical method for model M_1

For model M_1, we estimated the unknown parameter vector $\Theta_{M_1} = [\mu_{wv}, \sigma_{wv}, \lambda_w]$ by applying a probability-theory-based MLE. The assessment procedure for charity can be divided into two stages.

Stage 1: The degradation increments of i-th sample obey $\Delta y_{ij} \sim N(\nu_{wi}\Delta t, \sigma_{wi}^2 \Delta t)$. The PDF is

$$f_{(w)}(\Delta y_{ij}) = \frac{1}{\sqrt{2\pi\sigma_{wi}^2 \Delta t}} exp\left(-\frac{(\Delta y_{ij} - \nu_{wi}\Delta t)^2}{2\sigma_{wi}^2 \Delta t}\right), \tag{24}$$

and logarithm likelihood function is

$$l_{(w)i} = -\frac{m}{2} ln(2\pi\Delta t) - \frac{m}{2} ln\, \sigma_{wi}^2 - \sum_{j=1}^{m} \frac{(\Delta y_{ij} - \nu_{wi}\Delta t)^2}{2\sigma_{wi}^2 \Delta t}. \tag{25}$$

We set $\partial l_{(w)i}/\partial \nu_{wi} = 0$ and $\partial l_{(w)i}/\partial \sigma_{wi}^2 = 0$ to obtain the MLEs of ν_{wi} and σ_{wi}^2 as

$$\begin{cases} \hat{\nu}_{wi} = \frac{1}{m}\sum_{j=1}^{m} \frac{\Delta y_{ij}}{\Delta t} \\ \hat{\sigma}_{wi}^2 = \frac{1}{m}\sum_{j=1}^{m} \frac{(\Delta y_{ij} - \hat{\nu}_{wi}\Delta t)^2}{\Delta t} \end{cases}. \tag{26}$$

Stage 2: In model M_1, $v_{wi} \sim N(\mu_{wv}, \sigma_{wv}^2)$ and $\sigma_{wi}^2 \Delta t = \sigma_{wv}^2 \Delta t^2 + \lambda_{wi}^2 \Delta t$. Thus, we obtain the MLEs of μ_{wv}, σ_{wv}^2, and λ_w^2 as

$$\begin{cases} \hat{\mu}_{wv} = \frac{1}{n}\sum_{i=1}^{n} \hat{v}_{wi} \\ \hat{\sigma}_{wv}^2 = \frac{1}{n}\sum_{i=1}^{n} (\hat{v}_{wi} - \hat{\mu}_{wv})^2 \\ \hat{\lambda}_w^2 = \left(\frac{1}{n}\sum_{i=1}^{n} \sqrt{\left|\hat{\sigma}_{wi}^2 - \hat{\sigma}_{wv}^2 \Delta t\right|}\right)^2 \end{cases}. \tag{27}$$

3.2. Statistical method for model M_2

For model M_2, the unknown parameters $\Theta_{M_2} = [\mu_{uv}, \sigma_{uv}, \lambda_u]$ are estimated by applying an uncertainty-theory-based PLS[41]. The estimation procedure is divided into two stages.

Stage 1: Belief degrees calculation

In probability theory, the unknown parameters are estimated by constructing the PDF of the degradation increments. However, the density function no longer exists in the uncertainty theory.[42] Instead, only an uncertainty distribution function can be used. The belief degree is adopted in the uncertainty theory to represent the strength of an event.[19] Thus, before estimating the parameters, we should calculate the belief degrees of the degradation increments.

Following the Liu process, the degradation increments at the j-th measurement $\Delta \boldsymbol{y}_{ij} = (\Delta y_{1j}, \Delta y_{2j}, \ldots, \Delta y_{nj})$ follow a normal uncertainty distribution, denoted as $\Delta y_{ij} \sim N_u(\mu_{uv}\Delta t, (\sigma_{uv} + \lambda_u)\Delta t)$. The uncertainty distribution function is as follows:

$$\Phi_{(u)t}(\Delta y_{ij}) = \left(1 + exp\left(\frac{\pi(\mu_{uv}\Delta t - \Delta y_{ij})}{\sqrt{3}\Delta t(\sigma_{uv} + \lambda_u)}\right)\right)^{-1}, \tag{28}$$

where each degradation increment Δy_{ij} has a belief degree α_{ij}. The elements $\Delta \boldsymbol{y}_{ij} = (\Delta y_{1j}, \Delta y_{2j}, \ldots, \Delta y_{nj})$ are sorted in ascending order. The belief degrees of the degradation increments are calculated using the approximate mean rank function[26] as follows:

$$\alpha_{ij} = \frac{i - 0.3}{n_j + 0.4}, i = 1,2, \ldots n_j. \tag{29}$$

Stage 2: Parameters estimation

According to PLS, the objective function is the sum of the squares of the distance between the calculated belief degrees and the uncertainty distribution of the degradation increments.[41] The unknown parameters are estimated by minimizing the objective function as follows:

$$\hat{\Theta}_{M_2} = min \sum_{i=1}^{n_j} \sum_{j=1}^{m} \left(\Phi_{(u)}(\Delta y_{ij} | \Theta) - \alpha_{ij}\right)^2. \tag{30}$$

4. Case study

4.1. Degradation and uncertainty analysis

Meeker and Escobar[43] recorded the percentage growth in the operating current of GaAs lasers in a real engineering scenario. In this study, these data were used for degradation modeling and uncertainty analysis. The total sample size was set at $n = 15$. The termination test time was 4000 h and the interval between degradation measurements was set to $\Delta t = 250$ h. The failure threshold was set to $D = 10$, that is, when the operating current increased by 10%, the GaAs laser was out of service. The degradation paths of the operating currents of the GaAs lasers are presented in Fig. 1; the degradation path of each sample was approximately linear.

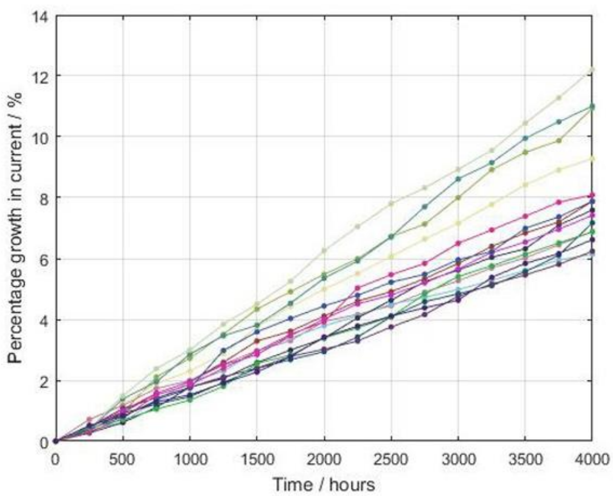

Fig. 1. Percentage growth in operating current of GaAs lasers.

The MLE and PLS methods were used to assess unknown parameters, and the results are summarized in Table 2.

Table 2. Parameter estimations of models M_1 and M_2.

Model	Estimation method	Parameter estimations		
		μ_v	σ_v	λ
M_1	MLE	$0.0020\ (\mu_{wv})$	$4.6740 \times 10^{-4}\ (\sigma_{wv})$	$0.0173\ (\lambda_w)$
M_2	PLS	$0.0019\ (\mu_{uv})$	$1.3316 \times 10^{-4}\ (\sigma_{uv})$	$7.5946 \times 10^{-4}\ (\lambda_u)$

As shown in Table 2, the estimated value of μ_v in model M_1 was almost equal to the estimation in M_2, which indicated that the degradation rates of M_1 and M_2 were nearly identical. However, the estimated values of σ_v and λ in model M_1 were different than those in M_2. This is because the uncertainty modeling and statistical methods for M_1 and M_2 were different.

Moreover, we analyzed the degradation trends and envelopments of the sample paths. For model M_1, we used the Monte Carlo simulation method to compute the degradation increments and generate sample degradation paths. For model M_2, we used an uncertain simulation method[28] to generate sample degradation paths. The uncertainty distribution function represents the relationship between the degradation increment and belief degree. According to the uncertainty distribution of the degradation increments, the inverse uncertainty distribution function can be derived.[38] Thus, assuming that the belief degree is calculated, the inverse uncertainty distribution function can be utilized to compute the degradation increments. Based on model M_2, the uncertainty distribution of the degradation increments is shown in Equation (28). Therefore, the inverse uncertainty distribution function is derived as

$$\Phi_{(u)\Delta t}^{-1}(\alpha_{ij}) = \mu_{uv}\Delta t + \frac{\sqrt{3}\Delta t(\sigma_{uv}+\lambda_u)}{\pi} ln\left(\frac{\alpha_{ij}}{1-\alpha_{ij}}\right) = \Delta y_{ij}, \alpha_{ij} \in (0,1), \tag{31}$$

where $\Phi_{(u)}^{-1}(\cdot)$ denotes an inverse uncertainty distribution function.

The uncertain simulation method is illustrated as follows:

Step 1: Initialize the parameter $j = 0$;

Step 2: Let $j = j + 1$, and arbitrarily sample the belief degree α_{ij} from (0, 1);

Step 3: Using Equation (31), compute the degradation increments Δy_{ij};

Step 4: Repeat Step2–Step3, until $j = m$.

Following the algorithm above, we obtained the cumulative degradation at measurement time t_m for i-th product as follows:

$$Y_i(t_m) = \sum_{j=1}^{m} \Delta y_i(t_j) = \sum_{j=1}^{m} \Delta y_{ij}. \tag{32}$$

Using Monte Carlo and uncertain simulation methods, we obtained the degradation trends for M_1 and M_2. The results are presented in Fig. 2. The degradation tendencies fit well with the actual degradation data. The results demonstrated that the parameter estimations were reasonable and that models M_1 and M_2 were both suitable for modeling the degradation trends.

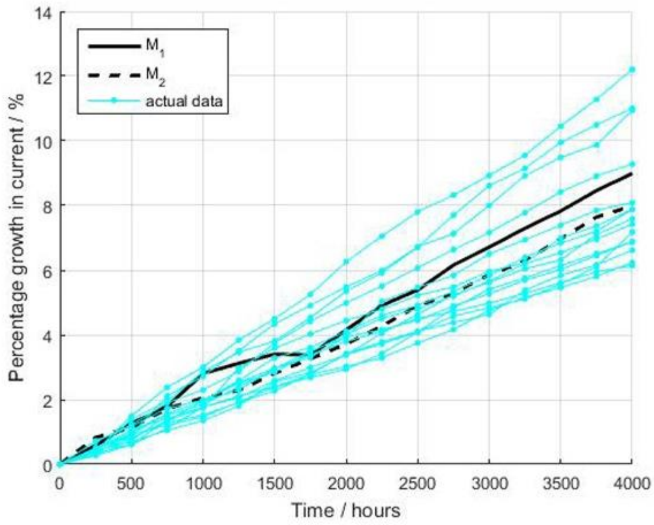

Fig. 2. Degradation trends based on models M_1 and M_2.

In the degradation envelopment analysis, we simulated 1000 sample degradation paths and obtained the lower and upper boundaries based on models M_1 and M_2. The results are presented in Fig. 3. As evident, the lower and upper boundaries enveloped almost all the degradation data, which demonstrated that the models M_1 and M_2 were both suitable for modeling the degradation data.

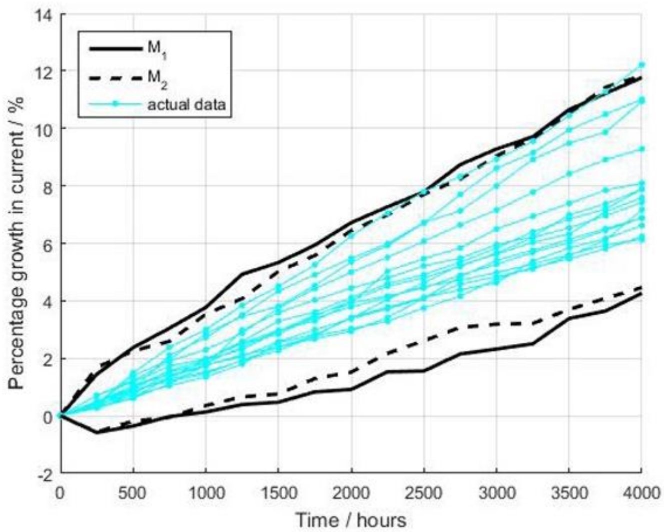

Fig. 3 Degradation envelopments based on models M_1 and M_2.

4.2. Reliability estimation and sensitiveness analysis

To explore the difference between the Wiener and Liu process degradation modeling in reliability estimation, we estimated the reliability of GaAs lasers based on the models M_1 and M_2 and analyzed the model sensitivity to various sample sizes and measurement times based on RQRL.

First, we maintained the measurement times and analyzed the model sensitivity to various sample sizes. The total sample size and measurement time were 15 and 16, respectively. The measurement times were set equal to the total measurement times, that is $m = 16$. Refer to the optimal design of ADT[29], and to fully represent the trend of reliability estimation curves under different sample sizes, we set five different sample sizes and set to have equal intervals; that is, the sample sizes were set as n = 3(3)15. There were C_{15}^{n} combinations of samples when the sample size was n; therefore, we obtained C_{15}^{n} reliability curves. Using Equations (15), (18), and (20), we obtained the lower and upper limits of the reliability estimation for various sample sizes based on the models M_1 and M_2. The results are shown in Fig. 4.

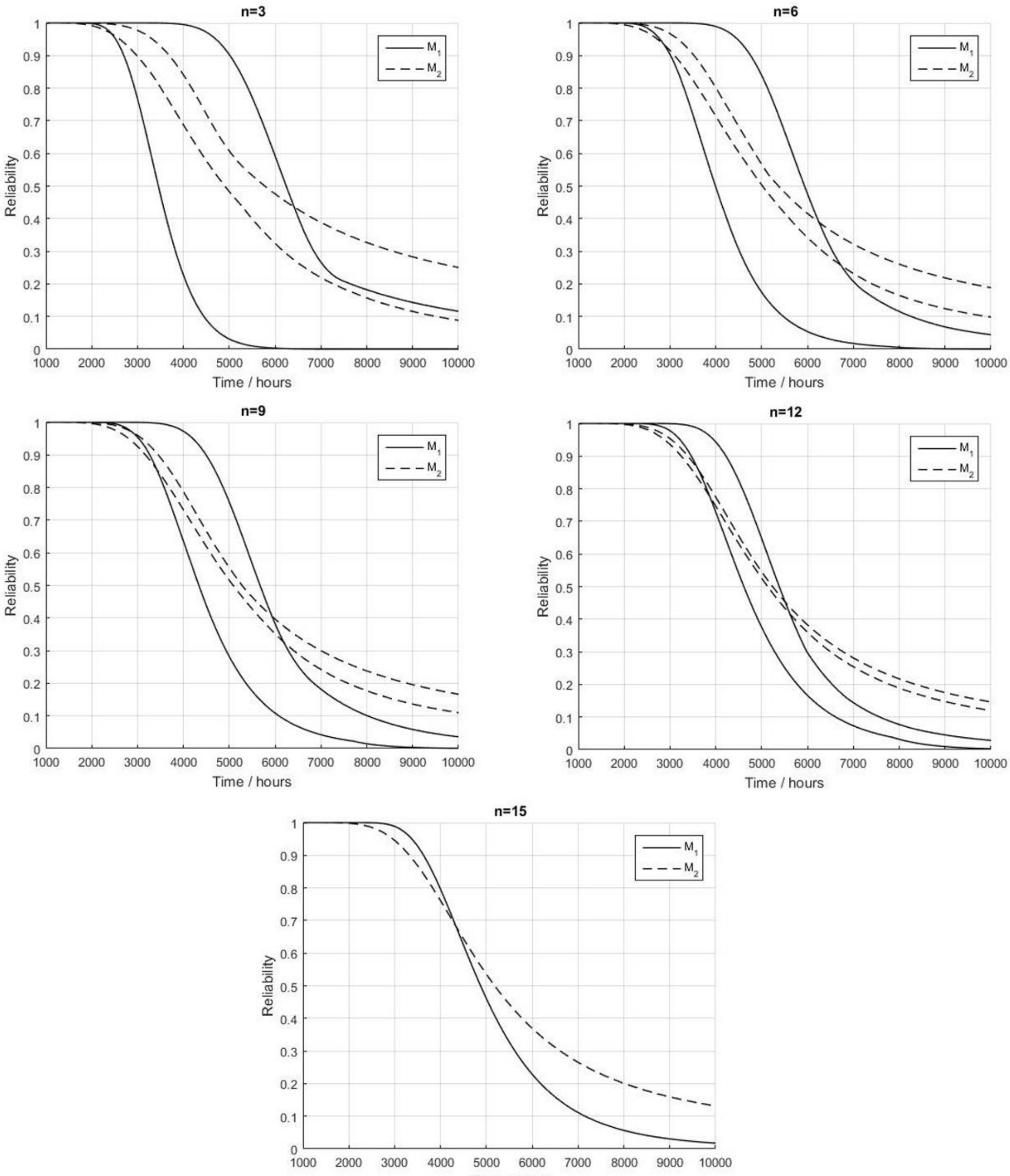

Fig. 4. Lower and upper limits of reliability estimation under n = 3(3)15 and m = 16.

We utilized the assessment index "$RQRL_{(n)}(R)$" in Equation (21) to quantify the sensitiveness of models M_1 and M_2 to various sample sizes. In practical engineering, engineers typically focus on high reliability, which is above 0.8.[28] In Fig. 4, $RQRL_{(n)}(R) = 0$ when $n = 15$. Using Equation (21), we calculated $RQRL_{(n)}(R)$ with $R = [0.8, 0.99]$ for n = 3(3)12. The results are shown in Fig. 5.

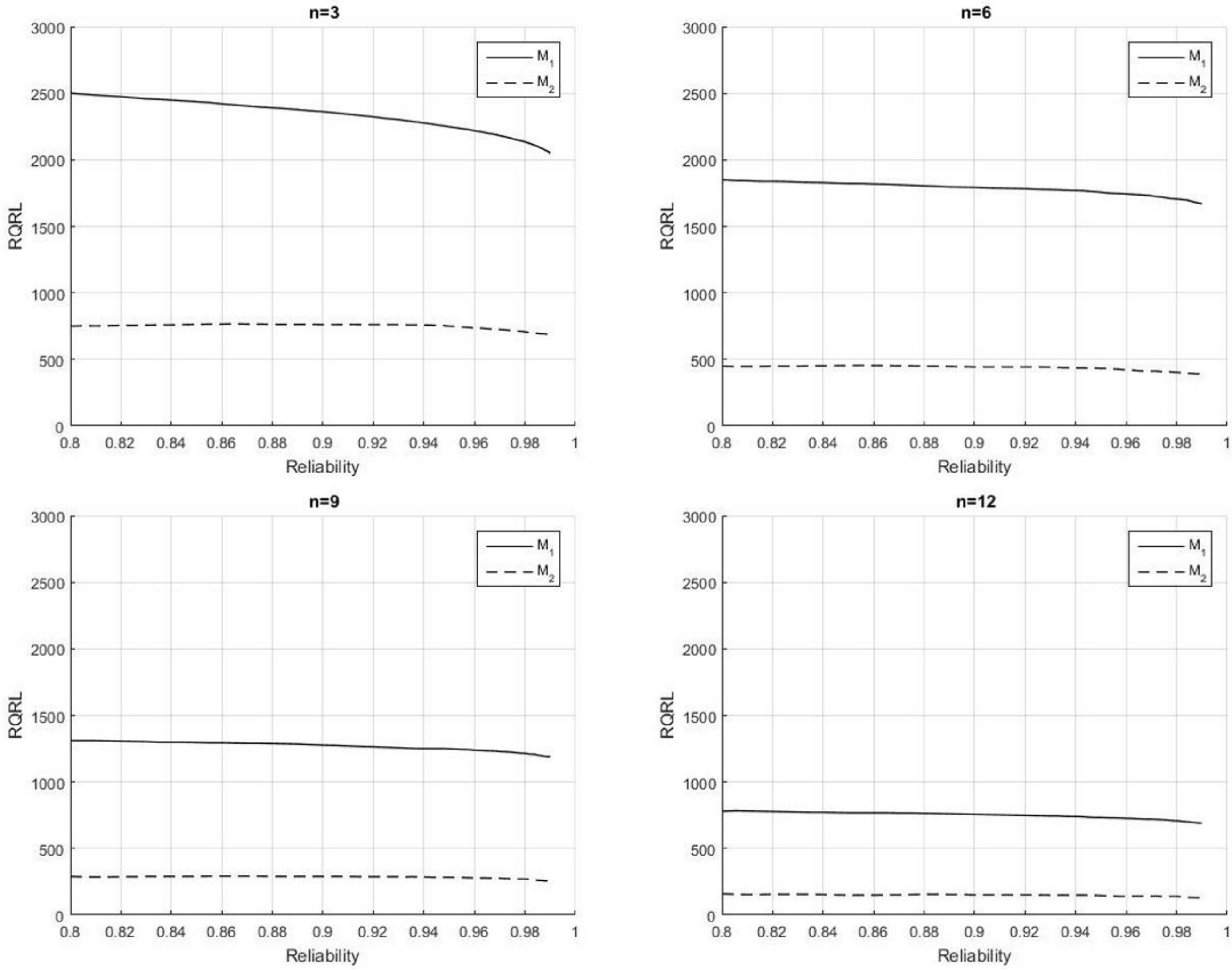

Fig. 5. $RQRL_{(n)}$ under n = 3(3)12 and m = 16.

Figures 4 and 5 indicated the following results. First, with increasing test time, the reliability decreased from 1, whereas $RQRL_{(n)}$ gradually increased. Under equal sample conditions, a higher $RQRL_{(n)}$ indicated that the reliability estimation was less uniform. Thus, the reliability estimation became less uniform with an increase in the test time. Second, the reliability curves based on models M_1 and M_2 had cross points (e.g., in Fig. 4 "n = 15," the cross point was 4350 h). Before the crossing point, the reliability estimation based on model M_2 was conservative; however, after the crossing point, the results were the opposite. Third, with an increasing sample size, the $RQRL_{(n)}$ based on models M_1 and M_2 decreased. This indicates that obtaining more information about the samples greatly reduced the degradation epistemic uncertainties and accordingly improved the uniformity of the reliability estimation. Finally, for the same sample size, the $RQRL_{(n)}$ based on the model M_2 was much lower than that based on M_1, indicating that this model offered a more uniform reliability estimation result. This demonstrates that the sensitivity of the uncertain process degradation model to sample sizes (particularly for small sample sizes) was much lower than that of the stochastic process degradation model. Therefore, an uncertain process degradation model can provide a more stable reliability estimation result.

Next, we maintained the sample size and analyzed the sensitivity of the degradation models to the measurement times. We set the sample size equal to that of the total sample size; that is, n = 15. Similar to the setting of n in Fig. 4, to fully represent the trend of the reliability estimation curves under different measurement times, we set five different measurement times at equal intervals, that is, the measurement times were set as m = 4(3)16. Thus, in this case, we obtained the C_{16}^{m} reliability curves. Using Equations (15), (18), and (22), we obtained the lower and upper limits of the reliability estimation for different combinations of measurement times. The results are presented in Fig. 6.

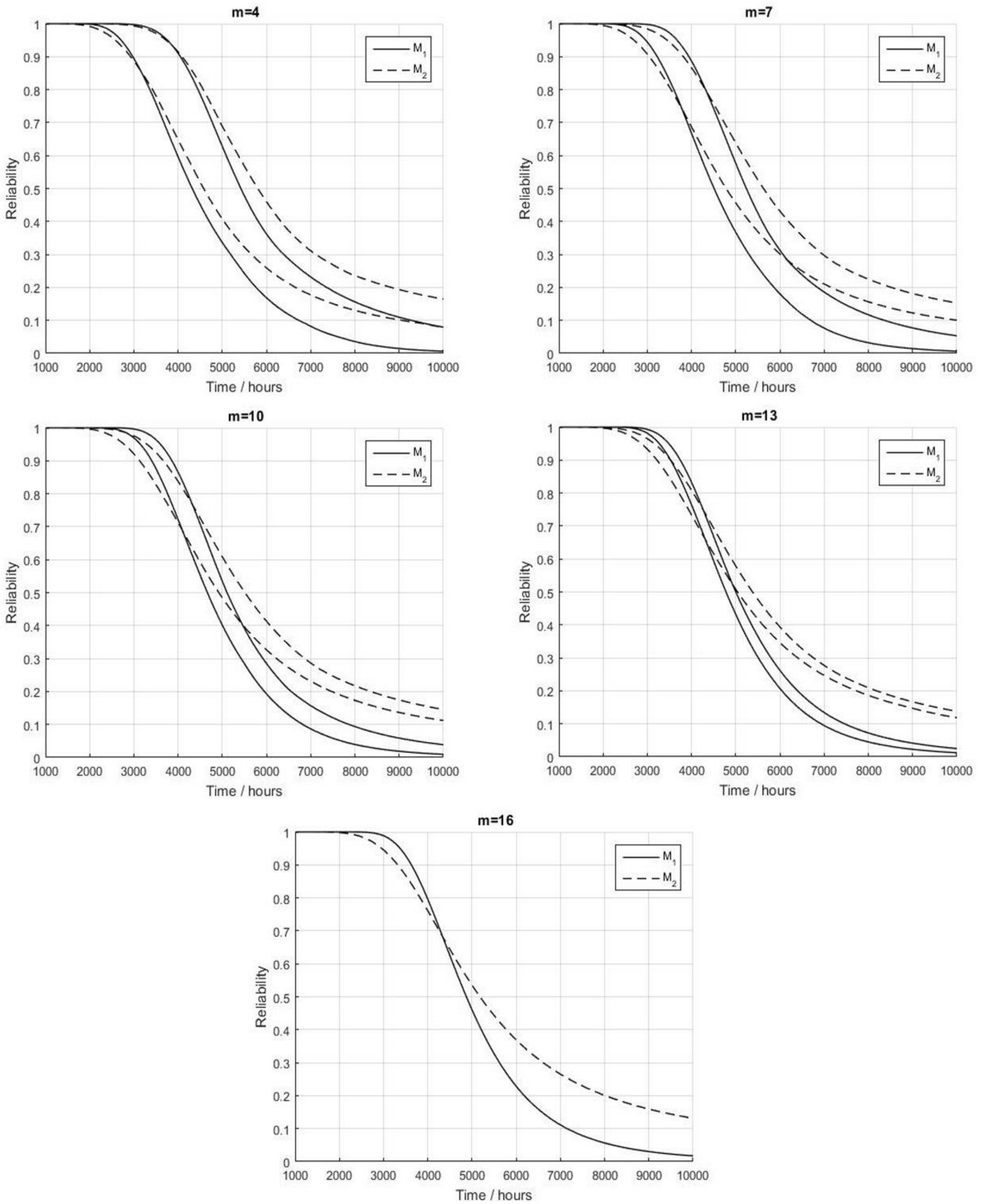

Fig. 6. Lower and upper limits of reliability estimation under n = 15 and m = 4(3)16.

Similarly, we utilized the assessment index "$RQRL_{(m)}(R)$," as in Equation (23), to quantify the sensitiveness of models M_1 and M_2 to various measurement times. In Fig. 6, $RQRL_{(m)}(R)$ = 0 when m = 16. Following Equation (23), we calculated $RQRL_{(m)}(R)$ using R = [0.8, 0.99] for m = 4(3)13. The results are shown in Fig. 7.

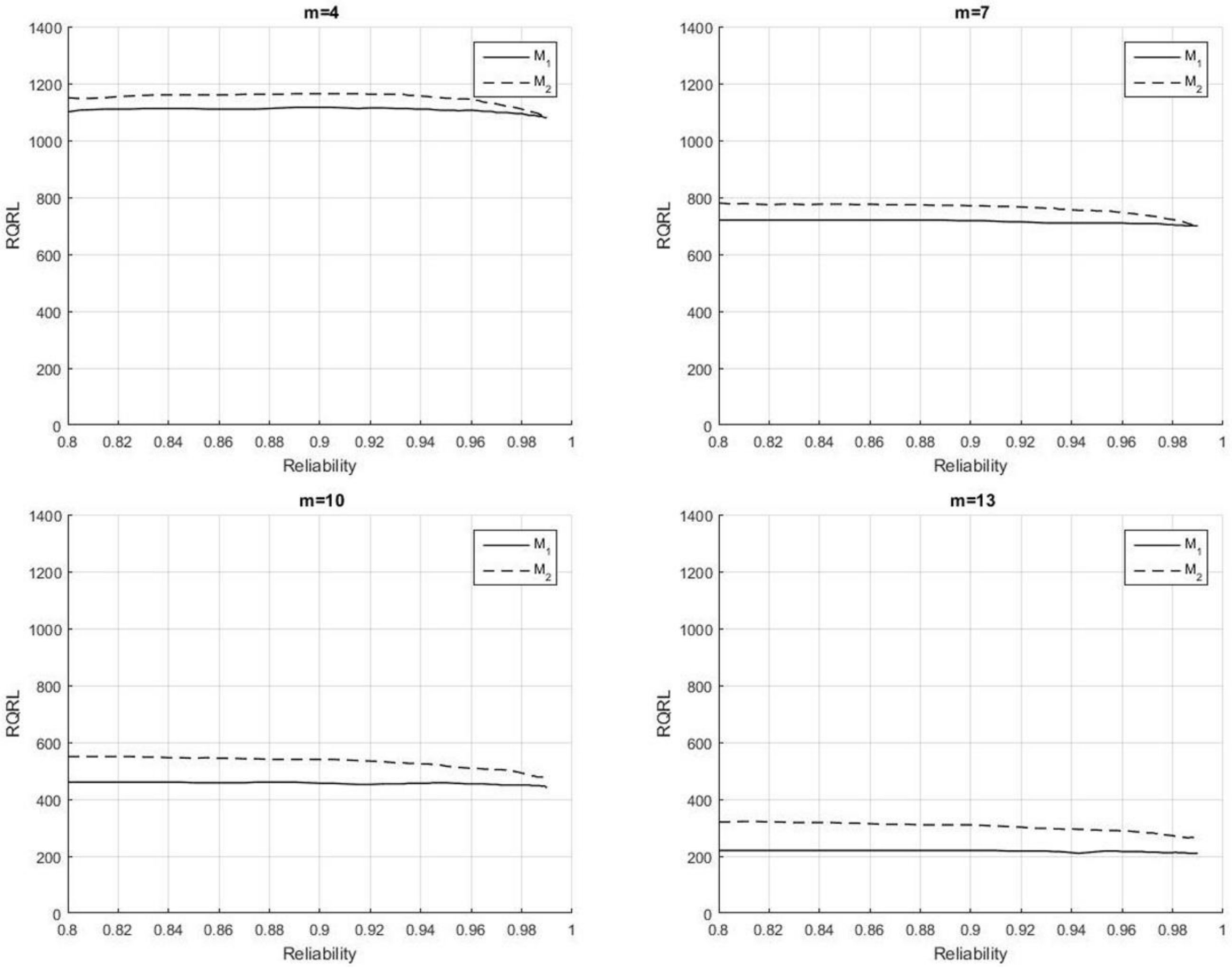

Fig. 7. $RQRL_{(m)}$ under n = 15 and m = 4(3)13.

Based on Fig. 6 and 7, we obtained the following observations. First, with increasing test time, $RQRL_{(m)}$ gradually increased, and a higher $RQRL_{(m)}$ indicated that the reliability estimation was less uniform. Second, $RQRL_{(m)}$ based on the models M_1 and M_2 both decreased with increasing measurement times, demonstrating that more degradation observations significantly reduced epistemic uncertainties and improved the uniformity of reliability estimation. Finally, under the same measurement times, the $RQRL_{(m)}$ based on model M_2 was slightly higher than that based on M_1, which indicated that the sensitivity of the uncertain process degradation model to measurement times was slightly higher than that of the stochastic process degradation model. Therefore, the stochastic process degradation model can provide a more stable reliability estimation result.

5. Conclusion

In this study, considering both the unit-to-unit variability and epistemic uncertainty, Wiener-process-based (M_1) and Liu-process-based (M_2) degradation models were proposed. Based on the RQRL, we compared and analyzed the sensitivity of models M_1 and M_2 to various sample sizes and measurement times. The major conclusions are drawn as follows:

1) The stochastic- and uncertain-process-based degradation models both fit well with the actual sample paths; therefore, both models were suitable for modeling the degradation data.

2) With an increasing sample size and measurement time, the RQRL gradually decreased. This demonstrates that the larger product sample sizes and more measurement times reduced the degradation epistemic uncertainties and improved the uniformity and stability for reliability estimation.

3) Under the same sample sizes, particularly under the same small sample sizes (i.e., n = 3, n = 6, n = 9 in Fig. 5), $RQRL_{(n)}$ based on model M_2 was considerably lower than that based on M_1, demonstrating that the uncertain process degradation model provided a much more uniform reliability estimation result than

the stochastic process degradation model. However, for the same measurement times (i.e., $m = 4(3)13$ in Fig. 7), the $RQRL_{(m)}$ based on model M_2 was slightly higher than that based on M_1, demonstrating that the stochastic process degradation model offered a more uniform reliability estimation result. Thus, under small-sample conditions, the uncertain-process-based degradation model was preferable for decreasing epistemic uncertainties and increasing the stability of the reliability estimation.

The following topics are worthwhile for further research:

1) This study analyzed the differences between stochastic process and uncertain process degradation modeling in reliability estimation based on the RQRL. In the future, we will compare the two uncertainty degradation models through further theoretical analysis and engineering case studies.

2) This study utilized the probability-theory-based MLE and uncertainty-theory-based PLS to estimate unknown parameters. We plan to explore new statistical methods that can be used for comparative verification with existing algorithms.

3) The Bayesian method is typically used to describe uncertainties based on subject probability. We can compare and analyze the differences between the Bayesian method and uncertainty theory in quantifying epistemic uncertainties.

4) To update the belief degree of the uncertain variable with the current state, we will attempt to combine the uncertain process with the study of conditional reliability, such that the product belief reliability under dynamic conditions can be estimated.

Acronyms

CDF	cumulative distribution function
FHT	first hitting time
MLE	maximum likelihood estimation
PDF	probability density function
PLS	principle of least squares
RQRL	range of quantile reliable lifetime

Notation

$\Phi(\cdot)$	standard normal probability distribution function
$\Phi_{(u)}(\cdot)$	uncertainty distribution function
$\Phi_{(u)}^{-1}(\cdot)$	inverse uncertainty distribution function
$N(\cdot)$	normal probability distribution
$N_u(\cdot)$	normal uncertainty distribution
$B_w(\cdot)$	standard Brownian motion
$C_u(\cdot)$	Liu process
v	drift coefficient
λ	diffusion coefficient
μ_v	expected value of v
σ_v	standard deviation of v
t	testing time
Δt	interval between testing observations
$Y_i(t_j)$	degradation of i-th sample at observation time t_j
Δy	degradation increments
D	failure threshold
T_D	lifetime of products
n	number of samples
m	number of testing observations
$\boldsymbol{\Theta}$	set of degradation model parameters
α	belief degree
$R(t)$	reliability at time t
$R^{ll}(t)$	lower limit of reliability estimation at time t
$R^{ul}(t)$	upper limit of reliability estimation at time t
$t^{ll}(R)$	lower limit of reliable lifespan at reliability R
$t^{ul}(R)$	upper limit of reliable lifespan at reliability R
$RQRL(R)$	difference between lower and upper limit of reliable lifespan at reliability R

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