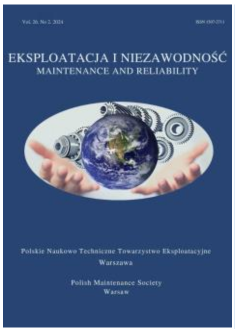

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Statistical Insights: Analyzing Shock Models, Reliability Operations and Testing Exponentiality for NBRUmgf Class of Life Distributions

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Highlights

- Testing exponentiality Classes of life distributions, Reliability operations

Abstract

This study focuses on an innovative life distribution category known as the 'New Better than Renewal used in Moment Generating Function' ($NBRU_{mgf}$) class. It explores the relationships between this particular aging model and established aging categories, and its applicability within a shock model. Moreover, it investigates the consistency of this aging concept through specific reliability operations, which are pivotal tools in reliability engineering. The research involves computing Pitman's asymptotic efficiencies for this testing method and compares them with alternative approaches. Additionally, the study presents an extensive table of percentiles for the test statistic associated with this proposed technique. To underscore the significance of the study's findings, various real-world datasets are employed, demonstrating the efficacy of our test methodology across diverse types of actual data.

Keywords
Classes of real-life distributions, Hypothesis testing, Order stochastic, Reliability theory, Shock model application.

1. Introduction

Classes of life distributions encompass various categories or types of probability distributions utilized to model lifetimes, durations, or intervals between occurrences across diverse fields like actuarial science, survival analysis, and reliability engineering. These distributions aid in comprehending and forecasting the lifespan or duration of entities or events. They establish a unified scientific framework, facilitating collaboration among scientists engaged in aging studies across different disciplines. As a result, statisticians have organized life distributions into distinct classes, delineating aging characteristics. These classes find widespread application in multiple domains, including medicine, engineering, industry, agriculture, among others.

There are several classes of life distributions commonly used to model such phenomena. Some of the key classes include: Increasing Failure Rate (IFR) and its extensions, (IFRA), New

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Better than Used (NBU) and its extensions, (NBUC), (NBUE), (HNBUE), (NBUL), NBRU, NBRUmgf, among many others, for more information, see Klefsjo [1], Ahmed [2], Deshpand et al. [3], Al-Ruzaiza et al. [4], Barlow and Proschan [5], Ahmed [6], Ahsanullah et al. [7], Mahmoud et al. [8], Ghosh and Mitra [9], and EL-Arishy et al.[10] and references therein. These classifications help establish a common ground for discussing and analyzing aging-related phenomena across different scientific domains. Many researchers have explored the connection between age-smooth distributions and distributions with sub-exponential tails, extensively utilized in infinite divisibility and queuing theory. Through multiple comparisons among random variables, diverse age classes have been formulated. In this context, we study a concept of aging derived from the arrangement of the moment generating function. However, before delving into specifics, let's briefly recap some standard notions of stochastic orderings and aging concepts that are under consideration in this paper.

Another significant ordering method extensively utilized in life and reliability testing is as follows:

a) Moment generating function order (denoted by $X \leq_{mgf} Y$), if

$$\int_0^\infty e^{sx} \mathrm{dF(x)} \geq \int_0^\infty e^{sx} \mathrm{dG(x)}\, , s \geq 0, \tag{1}$$

which can be written as:

$$\int_0^\infty e^{sx} \overline{\mathrm{F}}(\mathrm{x})\, \mathrm{dx} \leq \int_0^\infty e^{sx} \overline{\mathrm{G}}(\mathrm{x})\, \mathrm{dx}. \tag{2}$$

The mentioned ordering involves comparing the ratios of survival functions, ensuring that this ratio either increases or remains constant as the variable 'x' increases. This comparison indicates the superiority of one distribution over the other in terms of reliability properties. For further details regarding multiple classes and their testing, please refer to works by Gadallah et al. [11], Mahmoud et al. [12], Abu-Youssef et al. [13], Navarro and Pellerey [14], Bakr [15], Alqifari et. al. [16], Mahmoud El-Morshedy [17], EL-Sagheer et al. [18], and Atallah et al. [19].

Fortunately, the previously mentioned orderings have been utilized in the analysis of lifespan distributions, providing new definitions and descriptions of aging classes. When discussing aging, we refer to the statistical phenomenon where an older system typically exhibits a comparatively shorter remaining lifetime than a younger one.

In this research, we explore a new aging concept known as "new better than renewal used in moment generating function ordering" ($\mathrm{NBRU_{mgf}}$), which originates from moment generating function ordering. Section 2 provides an in-depth definition and explores various relationships associated with this concept. Section 3 delves into crucial reliability properties, closure attributes, convolution, and the establishment of a coherent system within the ($\mathrm{NBRU_{mgf}}$) class. Furthermore, it discusses practical applications of this aging concept within a shock model. Additionally, in section 4, we introduce the development of our testing methodology using the U-statistic and PARE by employing the Mathematica 13.3 program. Through Monte Carlo simulations, we derive critical values for the null distribution. Finally, in section 5, we present several examples illustrating the practical application of the proposed statistical test, demonstrating the relevance of the study's conclusions.

2. Definitions and Preliminaries

Join us on this journey as we navigate the terrain of renewal classes, revealing their role in shaping the reliability landscape. From defining the core concepts to exploring their applications in diverse fields, this article is your compass through the captivating

The renewal survival function is provided by $\overline{W}(t) = \frac{1}{\mu}\int_t^\infty \overline{F}(u)du.$ where X is the lifetime of a device with a finite mean $\mu = \int_0^\infty \overline{F}(u)du$.

Definition (1): X is new better than renewal used (NBRU) if

$$\overline{\mathrm{W}}_F(t)\overline{F}(x) \geq \overline{\mathrm{W}}_F(x+t);\ t, x > 0. \tag{3}$$

Definition (2): X is renewal new is better than used (RNBU) if

$$\overline{\mathrm{W}}_F(x)\overline{F}(t) \geq \overline{\mathrm{F}}(x+t);\ \ t, x > 0. \tag{4}$$

Definition (3): X is new better than renewal used in expectation (NBRUE) if

$$\mu\, \overline{\mathrm{W}}_F(t) \geq \int_0^\infty \overline{\mathrm{W}}_F(x+t)dx\, ;\ t, x > 0. \tag{5}$$

From the previously stated definitions, a new interpretation of "new better than renewal used" within the context of the moment generating function order can be derived, see Mahmoud El-Morshedy [17], EL-Sagheer et al. [18], and Hassan and Said [20].

Definition (4): The distribution function F is said to be new better than renewal used in moment generating function order (NBRU_{mgf}) if,

$$\overline{W}_F(t)\int_0^\infty e^{sx}\overline{F}(x)\,\mathrm{d}x \ge \int_0^\infty e^{sx}\overline{W}_F(x+t)\,\mathrm{d}x,$$

$$\text{for all } x,t \ge 0, s\ge 0, \tag{6}$$

It's clear that equation (6) is synonymous with,

$$\int_0^\infty \int_t^\infty e^{sx}\overline{F}(x)\overline{F}(y)\,dydx \ge \int_0^\infty \int_t^\infty e^{sx}\overline{F}(x+y)\,dydx$$

Then, we have the following implication:

$$\text{NBRU} \subset \text{NBRU}_{\text{mgf}} \subset \text{NBRUE}$$

3. Preservation results

An effective approach for leveraging reliability class properties in systems analysis involves conducting a RAMS analysis. By employing the attributes inherent in reliability classes, it becomes possible to assess the evolution of a system's failure rate over time and comprehend its impact on availability and maintainability.

The processes of convolution, blending, and constructing cohesive systems within a specific category of life distributions are often highly regarded as essential reliability measures. Studies have shown the closure of NBRU_{mgf} under these operations.

a) Convolution

The convolution property within reliability classes asserts that a system exhibiting a specific reliability class, such as NBRU_{mgf}, will maintain that class when convolved with other systems sharing the same reliability class.

Theorem 1:

The convolution operation maintains closure within the NBRU_{mgf} class of life distributions.

Proof: If F_1 and F_2 belong to the NBRU_{mgf} class, then we obtain the following:

$$\int_0^\infty \int_t^\infty e^{sx}\overline{F}(x+u)\,dudx$$

$$= \int_0^\infty e^{sx}\int_t^\infty \int_0^\infty \overline{F}_1(x+u-z)\ dF_2(z)\,du\,dx$$

$$= \int_0^\infty \int_t^\infty \int_0^\infty e^{sx}\,\overline{F}_1(x+u-z)\ du\,dx\,dF_2(z)$$

$$\le \int_0^\infty \int_0^\infty \int_t^\infty e^{sx}\,\overline{F}_1(x)\overline{F}_1(u-z)\,du\,dx\ dF_2(z)$$

$$= \int_0^\infty e^{sx}\,\overline{F}_1(x)\int_t^\infty \int_0^\infty \overline{F}_1(u-z)\ dF_2(z)\,du\,dx$$

$$= \int_0^\infty e^{sx}\,\overline{F}_1(x)\int_t^\infty \overline{F}(u)\ du\,dx$$

This demonstration illustrates that the NBRU_{mgf} does not maintain closure under the convolution property.

b) Mixture of NWRU_{mgf}:

A system characterized as a mixture comprises diverse components selected randomly based on a specified probability distribution. Reliability classes serve as a means to articulate changes in a system's failure rate over time. An application of the mixture attribute within reliability classes is in the analysis of systems comprising varied components, each with distinct lifetimes and failure rates. The mixing property of reliability classes asserts that a system with a specific reliability class, such as NWRU_{mgf} will retain that classification when formed by any combination of systems sharing the same reliability class.

$$\int_0^\infty \int_t^\infty e^{sx}\overline{F}(x+u)\,dudx$$

$$= \int_0^\infty e^{sx}\int_t^\infty \int_0^\infty e^{sx}\,\overline{F}_\alpha(\alpha + u)\ dG_2(z)\,du\,dx$$

$$= \int_0^\infty \int_0^\infty \int_t^\infty e^{sx}\,\overline{F}_\alpha(\alpha+u)\,du\,dx\ dG(\alpha)$$

Since $\overline{F}_\alpha$ is NWRU_{mgf}, then

$$\int_0^\infty \int_0^\infty \int_t^\infty e^{sx}\,\overline{F}_\alpha(\alpha+u)\,du\,dx\ dG(\alpha)$$

$$\ge \int_0^\infty \int_0^\infty \int_t^\infty e^{sx}\,\overline{F}_\alpha(x)\overline{F}_\alpha(u)\,du\,dx\ dG(z).$$

Using Chebyshev's inequality

$$\int_0^\infty \int_0^\infty \int_t^\infty e^{sx}\,\overline{F}_\alpha(x)\overline{F}_\alpha(u)\,du\,dx\ dG(z)$$

$$= \int_0^\infty \int_0^\infty e^{sx}\,\overline{F}_\alpha(x)\,dG(\alpha)dx\ .\int_t^\infty \int_0^\infty e^{sx}\,\overline{F}_\alpha(u)\,dG(\alpha)du$$

$$= \int_0^\infty e^{sx}\,\overline{F}(x)dx\ .\int_t^\infty \overline{F}(u)\,du = \int_0^\infty \int_t^\infty e^{sx}\overline{F}(x)\overline{F}(u)\,dudx.$$

Then we conclude that the NWRU_{mgf} class is preserved under mixture.

c) Mixing

It is clear that the NBRU_{mgf} class is not preserved under mixing.

d) Building coherent systems:

When every element within the system holds importance and the structural function, indicating the system's performance concerning each element's function, is on the rise, the system is identified as coherent. Engineers in design prioritize the establishment of coherent systems.

For more insights on coherent systems, refer to (Barlow and Proschan [5]).

The theorem below establishes the closure property of the $NBRU_{mgf}$ class under employing the operation to construct a coherent system.

Theorem 2: An $NBRU_{mgf}$ series, composed of n independent components belonging to new better than renewal used in moment generating function order ($NBRU_{mgf}$) class, collectively constitutes an $NBRU_{mgf}$.

Proof: Let $X_1, X_2, \dots, X_n$ be independent $NBRU_{mgf}$ then we have

$$\int_0^\infty \int_t^\infty e^{sy} \frac{p(\min(X_1, \dots, X_n) \geq y+t)}{p(\min(X_1, \dots, X_n) \geq t)} dydt$$

$$= \int_0^\infty \int_t^\infty \prod_{i=1}^n e^{sy} \frac{p(X_i \geq y+t)}{p(X_i \geq t)} dydt$$

$$= \int_0^\infty \int_t^\infty \prod_{i=1}^n e^{sy} \frac{\bar{F}_i(y+t)}{\bar{F}_i(t)} dydt$$

Since F_i is $NBRU_{mgf}$ class of life distribution, then we obtain

$$\int_0^\infty \int_t^\infty \prod_{i=1}^n e^{sy} \frac{\bar{F}_i(y+t)}{\bar{F}_i(t)} dydt \leq \int_0^\infty \int_t^\infty \prod_{i=1}^n e^{sy} \bar{F}_i(y)\, dydt.$$

The proof is now concluded.

e) Applications: Shock model application

The stochastic model known as the homogeneous Poisson shock model delineates system failures resulting from random shocks conforming to the homogeneous Poisson process. This process involves counting independent events occurring at a constant rate within a specified time period. In the shock model, each event induces damage to the system, and system failure ensues when the cumulative damage surpasses a predetermined threshold. Applications of Poisson homogeneous shock models extend to various domains, including modeling phenomena such as insurance claims, health deterioration, machinery breakdowns, or automobile accidents arising from random mechanical failures occurring consistently over time. Consider a device subjected to shocks, where N(t) represents the count of shocks occurring within the time interval $(0, t\,]$. The arrival of the k^{th} shock is denoted by the time T_k. Let $X_k = T_{k+1} - T_k$ denote the time be the time between the k^{th} and $(k+l)^{st}$ shocks. We consider that $X_1, X_2, \dots$ are iid distributed according to F.

Let

$$a_k(t) = p(N(t) = k), \quad k = 1, 2, \dots$$

and define$\bar{P}_k$ as the device's chance of surviving k shocks. Subsequently, the system's survival probability up to time t is

$$\bar{H}(t) = \sum_{k=0}^\infty a_k(t) \bar{P}_k \ .$$

Theorem 3: F is $NBRU_{mgf}$ implies H is $NBRU_{mgf}$.

Proof: Note that $\bar{H}(t)$ can be expressed as

$$\bar{H}(t) = \sum_{k=1}^\infty \bar{F}_k(t)\, p_k$$

Where $p_k = \bar{P}_{k-1} - \bar{P}_k, k = 1, 23, \dots$ and F_k is the distribution function of

T_k, and

$$\int_0^\infty \int_t^\infty e^{sx} \bar{H}(x+u)\, du\, dx$$

$$= \int_0^\infty \int_t^\infty e^{sx} \sum_{n=0}^\infty \bar{P}_n \frac{\lambda^n (x+u)^n}{n!} e^{-\lambda(x+u)}\, du\, dx$$

$$= \int_0^\infty e^{sx} e^{-\lambda x} \int_t^\infty \sum_{n=0}^\infty \bar{P}_n \frac{\lambda^n}{n!} \sum_{m=0}^n \binom{n}{m} \alpha^{n-m} u^m e^{-\lambda u}\, du\, dx$$

$$= \int_0^\infty e^{-x(\lambda - s)} \sum_{n=0}^\infty \sum_{m=0}^n \bar{P}_n \frac{(\lambda x)^{n-m}}{m!\, (n-m)!} \int_t^\infty (\lambda u)^m e^{-\lambda u}\, du\, dx$$

$$= \sum_{n=0}^\infty \sum_{n=m}^\infty \bar{P}_m \frac{e^{-\lambda t}}{\lambda\,(\lambda - s) m!} \frac{\lambda^{n-m}}{(\lambda - s)^{n-m}} \sum_{l=0}^m \frac{m!\,(\lambda t)^l}{l!}$$

$$= \sum_{n=0}^\infty \sum_{n=m}^\infty \sum_{l=0}^m \bar{P}_m \frac{e^{-\lambda t}}{\lambda\,(\lambda - s)} \left(\frac{\lambda}{\lambda\,(\lambda - s)}\right)^{n-m} \frac{(\lambda t)^l}{l!}$$

Let i=n-m

$$= \sum_{i=0}^\infty \sum_{m=0}^\infty \sum_{l=0}^m \bar{P}_{i+m} \frac{e^{-\lambda t}}{\lambda\,(\lambda - s)} \left(\frac{\lambda}{\lambda\,(\lambda - s)}\right)^i \frac{(\lambda t)^l}{l!}$$

$$= \sum_{i=0}^\infty \sum_{l=0}^\infty \sum_{m=l}^\infty \bar{P}_{i+m} \frac{e^{-\lambda t}}{\lambda\,(\lambda - s)} \left(\frac{\lambda - s}{\lambda}\right)^{-i} \frac{(\lambda t)^l}{l!}$$

Since F is $NBRU_{mgf}$

$$\leq \sum_{i=0}^\infty \sum_{l=0}^\infty \sum_{m=l}^\infty \bar{P}_i \bar{P}_m \left(\frac{\lambda - s}{\lambda}\right)^{-i} \frac{e^{-\lambda t}}{\lambda\,(\lambda - s)} \frac{(\lambda t)^l}{l!}$$

$$= \sum_{i=0}^\infty \sum_{m=0}^\infty \sum_{l=0}^m \bar{P}_i \bar{P}_m \left(\frac{\lambda - s}{\lambda}\right)^{-i} \frac{e^{-\lambda t}}{\lambda\,(\lambda - s)} \frac{(\lambda t)^l}{l!}$$

$$= \sum_{i=0}^\infty \sum_{m=0}^\infty \sum_{l=0}^\infty \bar{P}_i \bar{P}_m \frac{(\lambda t)^l}{l!} \frac{e^{-\lambda t}}{\lambda\,(\lambda - s)} \left(\frac{\lambda}{\lambda - s}\right)^{n-m}$$

$$= \int_0^\infty e^{sx} \sum_{i=0}^{\infty} \bar{P}_i \frac{(\lambda x)^{n-m}}{(n-m)!} e^{-\lambda x} dx \int_t^\infty \sum_{m=0}^{\infty} \bar{P}_m \frac{(\lambda u)^m}{m!} e^{-\lambda u} du$$

$$= \int_0^\infty e^{sx} \sum_{i=0}^{\infty} \bar{P}_i \frac{(\lambda x)^{i}}{i!} e^{-\lambda x} dx \int_t^\infty \sum_{m=0}^{\infty} \bar{P}_m \frac{(\lambda u)^m}{m!} e^{-\lambda u} du$$

$$= \int_0^\infty e^{sx} \bar{H}(x) dx \int_t^\infty \bar{H}(u) du$$

➔

$$\int_0^\infty \int_t^\infty e^{sx} \bar{H}(x) \bar{H}(u)\, du\, dx \geq \int_0^\infty \int_t^\infty e^{sx} \bar{H}(x+u)\, du\, dx$$

The proof is now concluded.

4. Non-parametric hypothesis testing

Many researchers use non-parametric tests to assess data exponentiality against many classes of life distributions using various techniques, for example, Abu-Youssef and El-Toony [20], Mahmoud et al, [21], Etman et al, [22], Ghosh and Mitra [23], Navarro [24], and Belzunce [25], among others. Now, we develop an exponential departure measure towards the NBRU_{mgf} class.

4.1 Testing exponentiality

In order to construct our testing exponentiality, we will integrate both sides of Eq. (6) as defined in definition (4) concerning t across the interval [0, ∞),

$$\int_0^\infty \int_0^\infty e^{sx} \bar{\text{W}}_F(x+t)\, \text{d}x\, dt \leq \int_0^\infty \bar{\text{W}}_F(t) dt \int_0^\infty e^{sx} \bar{F}(x)\, \text{d}x$$

Following several computations, we obtain

$$\frac{1}{s^3}(\varphi(s)-1) - \frac{1}{s^2}\mu \leq \frac{\mu_2}{2s}\varphi(s), \qquad \varphi(s) = E(e^{sx}).$$

The test that follows is predicated on a sample $X_1, X_2, \dots, X_n$ from a population with distribution F, we test

$H_0 : F$ is exponential distribution versus $H_1 : F$ is $NBRU_{mgf}$.

Using $\Delta(s)$ as a deviation measure from H_0 yields,

$$\Delta(s) = \left(\frac{\mu_2}{2s} - \frac{1}{s^3}\right)\varphi(s) + \frac{1}{s^2}\mu + \frac{1}{s^3}, \tag{7}$$

Take note that whereas $\Delta(s) > 0$ under H_1 and 0 under H_0, we use the following to guarantee the scale invariance of the test

$$\widehat{\Delta}(s) = \frac{\Delta(s)}{\overline{X}^2}. \tag{8}$$

As stated in Eq. (8), the empirical estimate of $\widehat{\Delta}(s)$ is

$$\widehat{\Delta}_n(s) = \frac{1}{n^2 s^3\, \overline{X}^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\left(\frac{s^2}{2} X_i^2 - 1\right) e^{s X_j} + s X_i + 1\right], \tag{9}$$

Let,

$$\omega(X_1, X_2) = \left(\frac{s^2}{2} X_1^2 - 1\right) e^{s X_2} + s X_1 + 1, \tag{10}$$

and define the symmetric kernel as

$$\varphi(X_1, X_2) = \frac{1}{2!} \sum_R \omega(X_1, X_2).$$

Where the total includes all of X_i and X_j arrangements. This demonstrates that the U_n -statistic provided by $\widehat{\Delta}_n(s)$ is identical to

$$U_n = \frac{1}{\binom{n}{2}} \sum_R \varphi(X_1, X_2).$$

This theorem encapsulates $\widehat{\Delta}_n(s)$ asymptotic normality.

Theorem: As $n \to \infty$, $\sqrt{n}\left(\widehat{\Delta}_n(s) - \Delta(s)\right)$ is asymptotically normal with mean 0 and variance σ^2 given as in Eq. (11). Under H_0 , the variance σ^2 reduces to Eq. (12)

Proof: Using Eq. (10) then, let

$$\beta_1(X_1) = E(\omega(X_1, X_2)|X_1) = \frac{1}{s^2(s-1)} + \frac{1}{s^2} X_1 - \frac{X_1^2}{2s(s-1)}$$

And,

$$\beta_2(X_2) = E(\omega(X_1, X_2)|X_2) = \frac{(s^2-1)}{s^3} e^{s X_2} + \frac{1+s}{s^3}$$

Considering,

$$\beta(X) = \beta_1(X_1) + \beta_2(X_2)$$
$$= \frac{(s^2-1)}{s^3} e^{s X} - \frac{X^2}{2s(s-1)} + \frac{1}{s^2} X + \frac{(s^2+s-1)}{s^3(s-1)}.$$

Then the variance is,

$$\sigma^2 = Var\left[\frac{(s^2-1)}{s^3} e^{s X} - \frac{X^2}{2s(s-1)} + \frac{1}{s^2} X + \frac{(s^2+s-1)}{s^3(s-1)}\right]. \tag{11}$$

Under H_0 it is easy to prove that $\mu_0 = E\big(\beta(X)\big) = 0$, and the variance σ^2 reduces to

$$\sigma^2 = \frac{5-s}{(s-1)^3(2s-1)}, s \neq 1, \frac{1}{2}. \tag{12}$$

The proof is now concluded.

4.2 The Pitman Asymptotic Efficiencies (PAE's) of $\delta(s)$

The Pitman asymptotic efficiencies (PAEs) for the Makeham, Weibull, and Linear failure rate families (LFR) are calculated in this section.

$$PAE(\Delta(S)) = \frac{1}{\sigma_0}\left|\frac{d}{d\theta}\Delta(s)\right|_{\theta\to\theta_0},$$

where,

$$\frac{d}{d\theta}\Delta(s) = \frac{1}{s^3}\left[\frac{s^2}{2}\left(\int_0^\infty e^{s\,x}\,dF_\theta(x)\right).\left(\int_0^\infty x^2\,d\grave{F}_\theta(x)\right) + \left(\frac{s^2}{2}\int_0^\infty x^2\,dF_\theta(x) - 1\right)\int_0^\infty e^{s\,x}\,d\grave{F}_\theta(x) + s\int_0^\infty x\,d\grave{F}_\theta(x)\right].$$

In this case we obtain,

$$PAE(\Delta(s), LFR) = \frac{1}{\sigma_0}\left|\frac{2}{(s-1)^2}\right|.$$

$$PAE(\Delta(s), Makeham) = \frac{1}{\sigma_0}\left|\frac{s^2+6s-4}{4s^2(s-1)(s-2)}\right|.$$

$$PAE(\Delta(s), Weibull) = \frac{1}{\sigma_0}\left|\frac{s+(1+s)log(1-s)}{s^2(s-1)}\right|.$$

$$PAE(\Delta(s), LFR) = \frac{1}{\sigma_0}\left|\frac{-s(2+s)+2(s^2-1)\log(1-s)}{2s^3(s-1)}\right|.$$

Table 1. Includes the asymptotic efficiencies of $\Delta(s)$ at different values of s.

Distribution	$S = 0.01$	$S = 0.04$	$S = 0.05$	$S = 0.1$	$S = 0.2$	$S = 0.3$
LFR	0.89079	0.8791185	0.874957	0.851835	0.7905694	0.697368
Makeham	2182.49	126.4316	78.8050	17.0984	3.0305160	0.841514
Wiebull	0.665112	0.647437	0.641354	0.609357	0.5357867	0.443891
Gamma	0.295069	0.285630	0.282408	0.265679	0.2285611	0.184648

Table 1 shows that the PAE's of Δ(s) decrease as s increases, and that the Makeham distribution has higher PAE's than the LFR, Weibull, and Gamma distributions.

Contrasting this values with others that could be relevant to this issue. Here, tests $\widehat{\delta}(s)$ which represented by Hassan and Said [26], at $s = 0.04$, 0.2, is our choice.

Table 2. The asymptotic relative efficiencies at $s = 0.04$, 0.2.

Distribution	$(\Delta(s), \hat{\delta}(s))_{s=0.04}$	$(\Delta(s), \hat{\delta}(s))_{s=0.2}$
LFR	0.105897	0.174398
Makeham	43.5343	2.049196
Wiebull	0.293054	1.868284

As evident from Table 2, the proposed test demonstrates relatively high efficiency compared to some nonparametric hypothesis tests for certain other classes. This certainly provides a positive impression regarding the proposed test for this class.

4.3 Monte Carlo Null Distribution Critical Points

In this section, we calculate the lower and upper percentiles of $\widehat{\Delta}_n(s)$ given in Eq. (9) based on 10000 simulated samples of size $n = 5(50)5$, as in Tables 3 and 4.

Table 3. Critical values of statistic $\widehat{\Delta}_n(s)$ at $s = 0.01$.

N	0.01	0.05	0.1	0.90	0.95	0.99
5	-0.375824	-0.030835	0.068469	0.551491	0.650733	0.893734
10	-1.129950	-0.308111	-0.075279	0.445400	0.515587	0.660711
15	-1.374750	-0.415991	-0.155317	0.395854	0.448736	0.549449
20	-1.367950	-0.457097	-0.196224	0.362855	0.412356	0.508138
25	-1.382270	-0.499651	-0.232127	0.341043	0.383692	0.465291
30	-1.272850	-0.504211	-0.249057	0.323280	0.362719	0.443570
35	-1.356820	-0.510378	-0.237370	0.311543	0.349686	0.415424
40	-1.442650	-0.523181	-0.249273	0.300121	0.333519	0.395044
45	-1.289410	-0.499169	-0.269296	0.290136	0.322209	0.384358
50	-1.142050	-0.466088	-0.256345	0.281686	0.315020	0.371146

Table 4. Critical values of statistic $\widehat{\Delta}_n(s)$ at $s = 0.05$

n	0.01	0.05	0.1	0.90	0.95	0.99
5	-0.407104	-0.060054	0.0502313	0.502602	0.597123	0.818009
10	-1.112520	-0.328200	-0.103944	0.406543	0.473638	0.591903
15	-1.257930	-0.427818	-0.182174	0.365368	0.415446	0.509395

n	0.01	0.05	0.1	0.90	0.95	0.99
20	-1.339420	-0.467233	-0.194242	0.336037	0.374662	0.462578
25	-1.168680	-0.464453	-0.220472	0.314295	0.354454	0.418497
30	-1.156760	-0.444027	-0.221505	0.301222	0.335915	0.397601
35	-1.222870	-0.509019	-0.248571	0.291506	0.324503	0.384477
40	-1.112590	-0.484394	-0.244659	0.278267	0.309309	0.363083
45	-1.072240	-0.427151	-0.226628	0.268368	0.298926	0.355678
50	-1.002410	-0.433660	-0.239529	0.260921	0.293162	0.349944

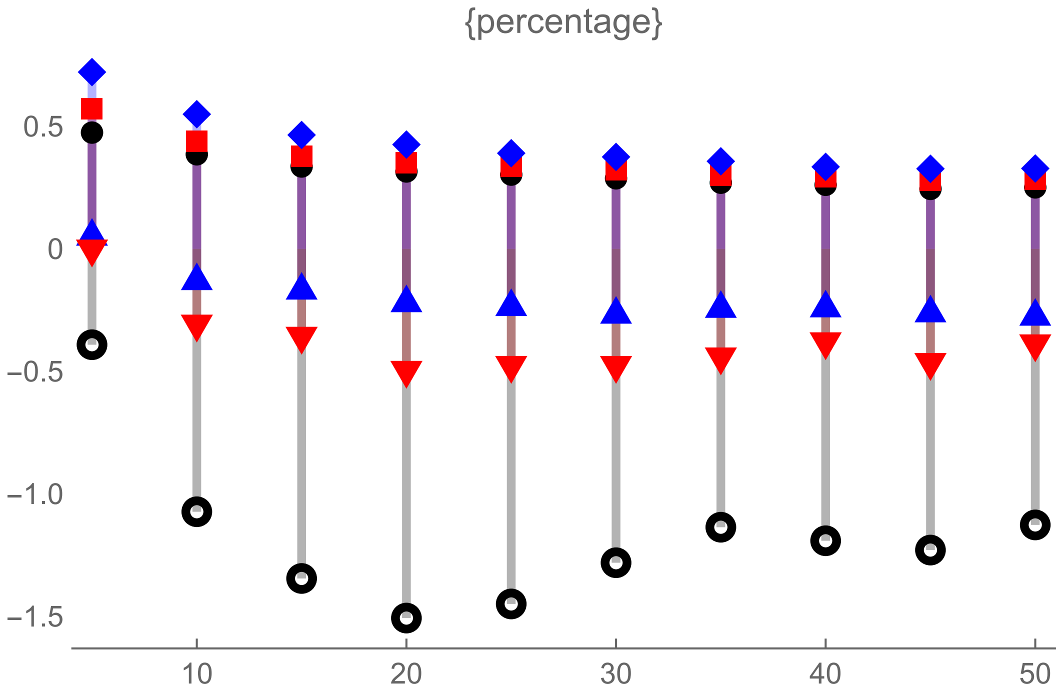

Fig.1. at s = 0.01.

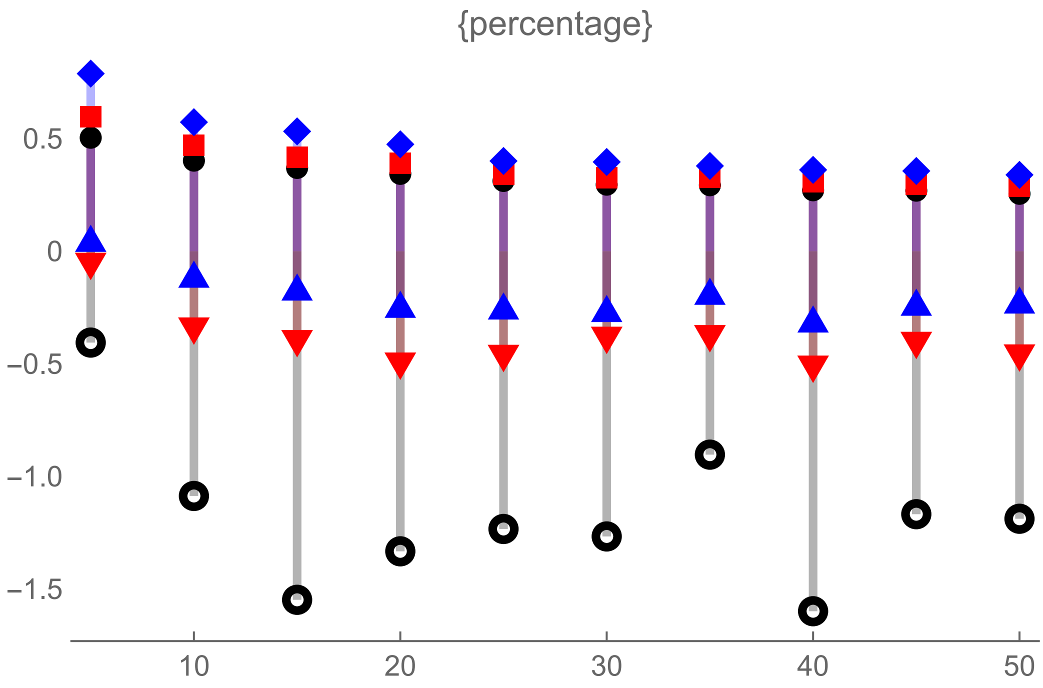

Fig.2. at s = 0.05.

Upon examining Tables 2 and 3, as well as Figures 1 and 2, it's noticeable that the behavior of critical values tends to approach a normal distribution as the sample size increases.

1. Applications

In order to showcase the relevance of the study's conclusions, we utilize distinct real-world datasets at 95% confidence level. The following results illustrate the effectiveness of our test across various types of real data.

Data-set #1. Take into consideration the data set in Murthy et al. [27], which represents the time takes for thirty repairable components to fail (see Figure 3). We obtain $\widehat{\Delta}_n(s) = 0.46555$ at $s = 0.1$ and $\widehat{\Delta}_n(s) = 1.06107$ at $s = 0.05$ in this instance. These results fall in the reject region of H_0. Then, we can reject the exponential property of this data, at $\alpha = 0.05$.

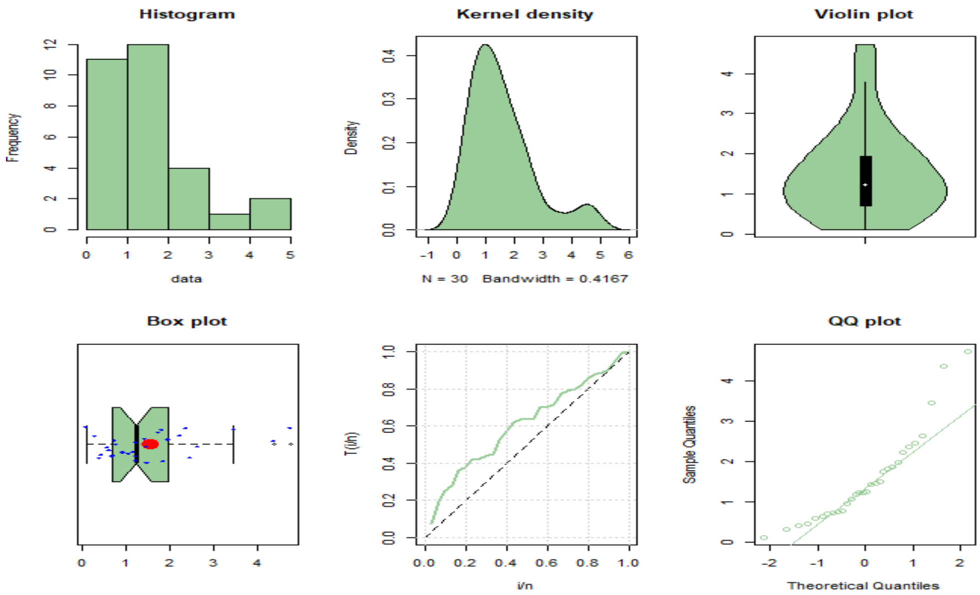

Fig. 3.

Data-set #2. Consider the data set given in Almetwally et al.[28], this data set shows a 36-day period of COVID-19 data that belongs to Canada, from April 10 to May 15, 2020, (see Figure 4). The drought death rate was the basis for this data. It is easily to show that $\widehat{\Delta}_n(s) = 1.3572$ at $s = 0.1$ and $\widehat{\Delta}_n(s) = 0.8076$ at $s = 0.05$, which are greater than the critical value of Tables 2, 3. Then, we can reject the exponential property of this data, and conclude that, the data have $NBRU_{mgf}$ property.

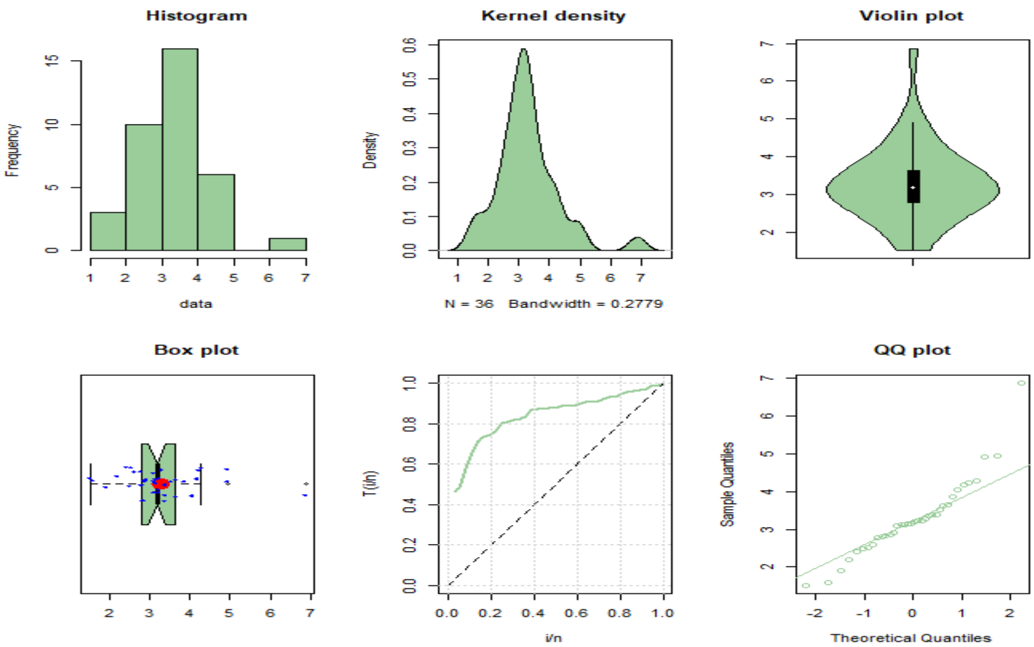

Fig. 4.

Data-set #3. Examining the data provided in Kochar [29], goldfish exposed to varying doses of methyl mercury were used in an experiment at Florida State University to determine the impact of methyl mercury poisoning on fish life spans. The prescribed times to death each day at a single dosage level (see Figure 5). It is evident that the test statistic values $\widehat{\Delta}_n(s) = 7153.72$ at $s = 0.1$ and $\widehat{\Delta}_n(s) = 285.07$ at $s = 0.05$ exceed the tabulated critical value found in Tables 2, 3. This indicates that the data collection possesses the $NBRU_{mgf}$ characteristic.

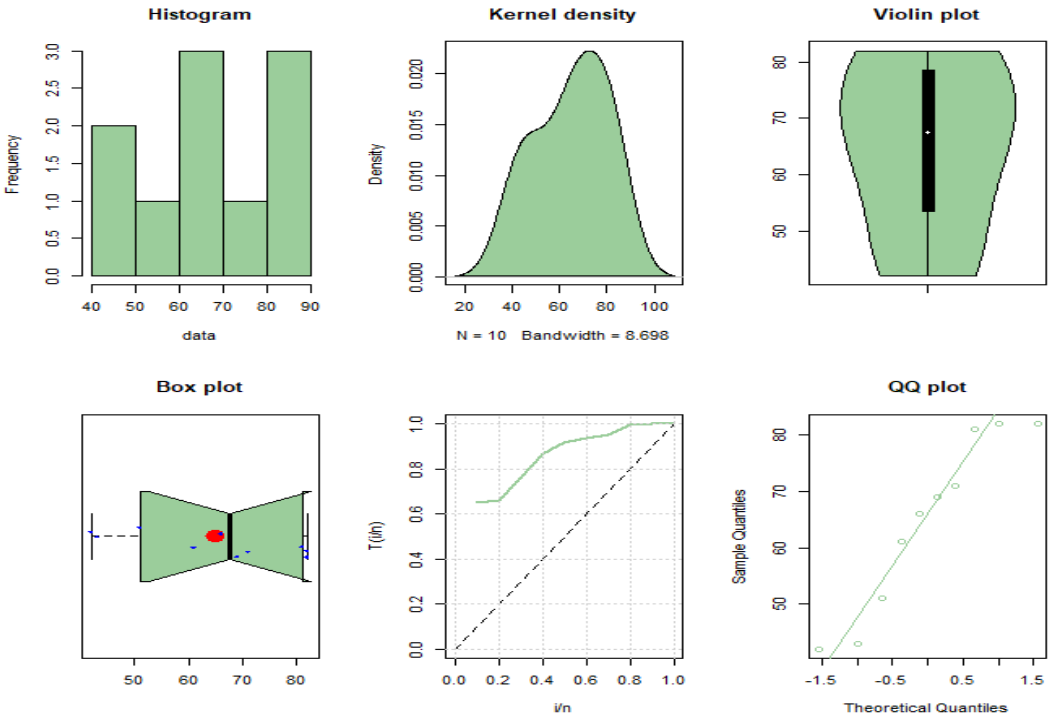

Fig. 5.

5. Conclusions

In unraveling the intricate tapestry of aging theories, we embarked on a captivating exploration of an avant-garde concept known as "New Better than Renewal Used in Moment Generating Function Ordering" (NBRUmgf), born from the depths of moment generating function ordering. Our journey led us through unraveling the intricate threads that weave various relationships within this innovative paradigm. Practical applications of this aging notion in a shock model are outlined, and we also covered certain closure attributes, convolution, mixing, and the creation of a coherent system within the ((NBRUmgf) class. Our testing methodology, born from the essence of the U-statistic, emerged as a robust instrument in the symphony of statistical analysis. With the powerful Mathematica 13.3 program as our conductor, we orchestrated Monte Carlo simulations to extract the elusive critical values, unlocking the secrets hidden within the null distribution. Furthermore, we explored the Makeham, Weibull, and linear failure rate (LFR) Pitman asymptotic efficiency. The significance of our research extends beyond theoretical realms to practical applications, with the suggested test poised to evaluate the effectiveness of treatment approaches across diverse domains, including but not limited to engineering and medical research. As exemplified in the applications section, our findings provide a versatile tool that can be harnessed to gauge the impact and efficacy of various methodologies, making it a valuable asset in the toolkit of researchers and practitioners across different fields. Additionally, the support of the Mathematica 13.3 program streamlines the implementation process, making it accessible for practitioners in both academia and industry.

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