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Semi-Markov approach for reliability modelling of light utility vehicles

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Highlights

- The 3-state model has been developed for reliability modelling.
- Based on the semi-Markov model, the reliability characteristics were calculated.
- Readiness and suitability indicators were used to assess the operational process.
- The results of the 3-state model have been compared with those of the 9-state model.

Abstract

Vehicles are important elements of military transport systems. Semi-Markov processes, owing to the generic assumption form, are a useful tool for modelling the operation process of numerous technical objects and systems. The suggested approach is an extension of existing stochastic methods employed for a wide spectrum of technical objects; however, research on light utility vehicles complements the subject gap in the scientific literature. This research paper discusses the 3-state semi-Markov model implemented for the purposes of developing reliability analyses. Based on an empirical course of the operation process, the model was validated in terms of determining the conditional probabilities of interstate transitions for an embedded Markov chain, as well as parameters of time distribution functions. The Laplace transform was used to determine the reliability function, the failure probability density function, the failure intensity, and the expected time to failure. The readiness index values were calculated on ergodic probabilities.

Keywords
semi-Markov model, reliability modelling, readiness, maintenance analysis, transportation system.

1. Introduction

The operation of transport means primarily involves three main processes, that is, the implementation of transport tasks together with regular technical activities aimed at verifying correct vehicle functionality; conducting periodic maintenance and servicing; and diagnosing the causes of technical unsuitability and their removal through repair or replacement of spare parts with new ones. The intensity of vehicle operation affects the wear rate of subassemblies and consumables, which directly translates into the frequency of maintenance activities. The duration during which these vehicles remain in these states is strictly correlated with the capacities of a technical subsystem that supports the transport system and the effectiveness of logistics processes associated with the supply chain of consumables and spare parts. In the case of many technical systems, due to the unavailability of fast and flexible material requirements, the unsuitability time of faulty means of transport constitutes a significant factor reducing the values of readiness indices [33].

Technical availability and reliability of vehicles are two of the main determinants of operational effectiveness in modern and advanced transport systems. The appearance of a means of transport failure in the course of the implementation of transport processes generates functional interference with respect to the entire system [10, 23]. For this reason, the operational strategy of many objects and systems assumes periodic preventive maintenance, whose interval and scope depend on both technical and economic factors [18]. The result of such actions is a reduced number of unplanned shutdown periods of machinery and equipment. However, to precisely determine the optimal intervals between subsequent maintenance cycles, it is necessary to develop appropriate reliability models to describe the probability of object failure within a specified period and volume of work [24, 36].

The objective of this publication is to develop an operation process model for light utility vehicles that enables analysing their reliability through determining basic reliability

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characteristics, such as the reliability function, failure probability density function, failure intensity, expected time to failure and readiness indices. A novel contribution to the current state of the source literature is the expansion of existing stochastic modelling methods in terms of their applications within the field of engineering and technical sciences [3, 5, 13]. In addition, the presented research complements the subject matter niche, because none of the previously published papers applied to the reliability of military light utility vehicles based on the semi-Markov model. The research subject matter is wheeled vehicles owned by the Polish Armed Forces. Currently (Russia's aggression against Ukraine), the research topic positions the importance of the content presented taking into account domestic (and international) security issues. Furthermore, it is possible to apply the method discussed in studies within the defence industry.

The research paper has been divided into five chapters. The further part of the article reviews the current state of knowledge in terms of the methods employed and approaches towards modelling the reliability of technical objects and systems. The third chapter contains a description of semi-Markov processes with a method for their implementation in reliability and operational studies. Then, in chapter four, the authors presented the application of the proposed approach with respect to analysing the operation process and reliability of light utility vehicles, based on empirical operational data obtained from an actual military transport system. The results of the studies and analyses presented enable the evaluation of vehicle reliability and readiness, which reflects the technical aspect of the functional effectiveness of transport means. The reliability characteristics developed may be grounds for planning the maintenance and repair potential of a technical system. The article ends with conclusions drawn from the performed computations and indications of future research directions.

2. Literature review

Table 1 contains an overview of the source literature on modelling the reliability of technical objects and systems in the transport, industry, and energy sector.

Statistical methods are some of the basic methods applied when developing reliability models. Selech and Andrzejczak [43] studied the reliability of the cabin door lock reliability in rail vehicles, using the Kaplan-Meier estimator. Next, using an original indicator, they selected a Generalized Gamma distribution as the best fitting of the empirical distribution function. In turn, Wawrzyński et al. [53] used statistical methods to develop a reliability model for aircraft commutators, under the assumption of a serial reliability structure of a tested object.

The results obtained by statistical methods also constitute the foundation for the evaluation of advanced models developed based on the application of fuzzy logic and neural networks. Żyluk et al. [60] developed statistical reliability models for lightweight combat aircraft, with the Weibull model turning out to be the best match. A fuzzy model with a similar accuracy reflected the values of an empirical reliability function. In turn, in [34], the multilayer perceptron (MLP) neural model was slightly better at approximating the light utility vehicle reliability function relative to the exponential and Weibull distributions. Whereas, in the case of fluid filling equipment in the automotive manufacturing industry, Soltanali et al. [45] demonstrated a significant improvement in the accuracy of reliability predictions using the Adaptive Neuro-Fuzzy Inference System model, compared to Weibull and Non-Homogeneous Poisson Process models.

The high accuracy of artificial neural networks in reliability modelling has prompted numerous researchers to employ them, without the need for their verification with other methods. Lolas and Olatunbosun [25] developed models based on MLP networks, used to predict the reliability of motor vehicles. In turn, Du et al. [11] used neural models to allocate reliability to components of industrial machines. On the other hand, Chang [7] suggested a method based on the ordered weighted averaging operator to allocate reliability and applied the developed model to study the liquid crystal display of thin-film transistors. Neural models can be improved by hybridization with other methods. Bai et al. [2] combined an artificial neural network with partial swarm optimization to develop a reliability model that was used to analyse industrial robot systems.

Macheret et al. [27] applied methods based on probabilistic dependencies and Monte Carlo simulations to study the reliability of military vehicles. A resulting exponential model was satisfactory in describing the time between failures (TBF). In turn, the authors of studies on micro-electro-mechanical system devices [37, 46] proposed the application of probabilistic methods to develop hard and soft failure models. The same failure classification was used by Lyu et al. [26] in relation to multi-state systems. They used probabilistic methods and dispersion to develop a reliability model. Another solution suggested by Miziuła and Navarro [31] is based on the Birnbaum importance measure and was implemented to evaluate the impact of the reliability of individual components on the reliability of the entire system.

The Bayesian approach to estimating the reliability of multi-component systems was presented by Guo and Wilson in [14]. The logistic regression, Weibull and degradation models were applied to describe the reliability of three components of a serial system. Next, using the Bayesian method, the authors developed a combined model that determined the reliability of the entire system.

The models most commonly employed for reliability analyses are Markov and semi-Markov processes. Depending on the complexity of a technical object or system and its operation processes, researchers and engineers develop stochastic models of a diverse number of states. Stawowy et al. [47] presented a 3-state Markov model to analyse power supply systems in transport telematics devices. In reliability studies related to other case studies, researchers constructed more complex models, such as a 4-state bearing model [22], a 5-state model for microelectromechanical systems [54], and a 10-state model for GPS receivers [41]. For a complex port distribution power system, Fang et al. [12] developed several Markov models to describe the reliability of individual subsystems, that is, 8-state models for a solar system, a wind system, and an energy storage system, a 4-state model for a combined cooling, heating and power system, and a 2-state model for a commercial power system.

Semi-Markov models have a significant advantage over Markov models in that they offer a considerably greater scope of possible applications. The applicability of Markov processes is limited due to the need to satisfy the conditions of exponential distributions for the time characteristics of the modelled processes [35]. However, semi-Markov processes are a generalization of Markov processes and allow for any characteristic distributions [9, 49].

Models with a phase space containing 3 operational states dominate the field of application of semi-Markov models. One of the most important 3-state semi-Markov models is the generalized object reliability model developed by Grabski [13], validated with assumed parameter values, without specified references to technical data. In the course of the research, the authors developed original 3-state models applicable to specific objects and systems, e.g., four types of repairable systems [8], single-use systems [9], special vehicles [5], lithium-ion batteries [55] and repairable systems [3]. Mengistu et al. [28] identified a five-state phase space of the semi-Markov model for volunteer cloud systems and Wu et al. [56] for power stations. L. Wang et al. [50] developed 6- and 4-state models to analyse the reliability of repairable systems in alternative environments, which turned out to be more reliable compared to Markov models. More complex models were proposed by Zhang et al. [57], who developed a 7-state model to analyse the reliability of a multilevel modular converter system. Whereas Blasi et al. [4] presented a 9-state semi-Markov model to evaluate the reliability of two machines operating in parallel.

The wide spectrum of applications demonstrated in Table 1 proves the usefulness of Markov and semi-Markov theories in modelling the reliability of technical objects and systems. Diversification of the number of states in the phase space proves the need to adapt the model to the analysed case study, the specificity of the technical object, and the assumptions of the operational strategy in particular. Therefore, the authors of this paper developed a specialized semi-Markov model to analyse the reliability of light utility vehicles. Based on assumptions adopted in military technical systems and the 9-state model [33] developed for a detailed analysis of vehicle readiness. The phase space aggregation was carried out in 3 main operational states.

Table 1. Review of literature on reliability modelling.

Methods	Models	Case study	Paper
Statistical methods	Generalized Gamma distribution	Cabin door lock on rail vehicles	[43]
	Reliability series structure model	Aircraft commutators	[53]
Statistical methods and fuzzy logic	Weibull and fuzzy models	Light combat aircraft	[60]
Statistical methods and neural networks	Exponential, Weibull and MLP models	Light utility vehicles	[34]
	Weibull, Non-Homogeneous Poisson Process and Adaptive Neuro-Fuzzy Inference System models	Fluid filling equipment in automotive manufacturing industry	[45]
Neural networks	Neural models (MLP)	Vehicles	[25]
	Neural model of reliability allocation	Machine tools	[11]
Neural networks and partial swarm optimization	Hybrid model	Industrial robot systems	[2]
Ordered weighted averaging aggregation operator	Reliability allocation model	Thin-film transistor liquid-crystal display	[7]
Probabilistic methods	Reliability models including hard and soft failures	Micro-electro-mechanical systems devices	[46][37]
	Models based on Birnbaum importance measure	2-, 3- and 5-component systems	[31]
Probabilistic methods and Monte Carlo simulation	Exponential model	Military vehicles	[27]
Probabilistic and dispersion methods	Reliability models including hard and soft failures	Multi-state systems	[26]
Bayesian methods	Combined model of component reliability	Multi-component complex system	[14]
Markov processes	3-state model	Power supply systems in transport telematics devices	[47]
	4-state model	Bearings	[22]
	5-state model	Micro-electro-mechanical systems	[54]
	2-, 4- and 8-state model	Port distribution power system	[12]
	10-state model	GPS Receivers	[41]
Semi-Markov processes	3-state model	Technical objects (general)	[13]
		Four types of repairable systems	[8]
		Single-use system	[9]
		Special vehicles	[5]
		Lithium-ion batteries	[55]
		Repairable systems	[3]
	5-state model	Volunteer cloud systems	[28]
		Power station	[56]
	4- and 6-state models	Repairable systems under alternative environments	[50]
	7-state model	Modular multilevel converter system	[57]
	9-state model	Two machines operating in parallel	[4]

3. Methods

Operation processes are a complex composition of deterministic and random processes, whereas random components are interpreted as stochastic processes $X(t)$ reflecting changes in the operational states of the technical object studied in discrete or continuous time. At any time t, an object is only in one of the states identified within the phase space $S = X(t)$. This assumption requires a precise identification of all possible operational states in which vehicles may remain during the operation process [33]. Stochastic processes that satisfy the Markov property are important in terms of applicability. According to Markov theory, the conditional probabilities of reaching future states $X(t_{n+1})$ result solely from the current state $X(t_n)$ [51]. In mathematical notation, the property presented [38, 42, 44] is consistent with the dependence (1):

$$P\begin{Bmatrix} X(t_n) = x_n | X(t_{n-1}) = x_{n-1}, \\ X(t_{n-2}) = x_{n-2}, \dots, X(t_0) = x_0 \end{Bmatrix} = P\{X(t_n) = x_n | X(t_{n-1}) = x_{n-1}\} \quad (1)$$

The source literature is dominated by the division of Markov processes based on state space and time, which distinguishes the following process types, i.e.:

1) discrete in states and discrete over time,
2) discrete in states and continuous over time,
3) continuous in states and discrete over time,
4) continuous in states and continuous over time,

Models based on discrete-state processes developed for both the discrete [30] and continuous times [42]–[45].

3.1. Semi-Markov processes

Semi-Markov processes are a generalization of Markov processes in terms of time characteristic distributions. Markov models assume exponential distributions of transition times between individual states within the phase space, which significantly narrows down their applicability when modelling reliability. Furthermore, their use without verifying the adopted assumptions may lead to significant errors in the obtained results [33, 48]. Semi-Markov models are a solution to this problem. They allow any distribution of time characteristics [13]. The values of sojourn times in states are calculated as time intervals from the moment an object entered the S_i state until it transitioned to the next S_j state. On the basis of the realization set of these variables, an approximation of the distributions is conducted based on the nonlinear least squares method.

The basic description of the semi-Markov process is the $\boldsymbol{Q}(t)$ renewal kernel matrix, consisting of products of the conditional probability of transition from the S_i state to the S_j state and distribution functions of the condition duration distribution of the S_i state before transition to the S_j state, according to the equation [19, 29]:

$$\boldsymbol{Q}(t) = \begin{bmatrix} 0 & Q_{12}(t) & \cdots & Q_{1(k-1)}(t) & Q_{1k}(t) \\ Q_{21}(t) & 0 & \cdots & Q_{2(k-1)}(t) & Q_{2k}(t) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ Q_{(k-1)1}(t) & Q_{(k-1)2}(t) & \cdots & 0 & Q_{(k-1)k}(t) \\ Q_{k1}(t) & Q_{k2}(t) & \cdots & Q_{k(k-1)}(t) & 0 \end{bmatrix} \quad (2)$$

whereas:

$$Q_{ij}(t) = p_{ij}F_{ij}(t), \quad (3)$$

where p_{ij} means the probability of transition from the S_i state to the S_j state, and $F_{ij}(t)$ is the distribution function of time spent in the S_i state befor transitioning to the S_j state.

An embedded Markov chain is constructed for a semi-Markov process over continuous time. It describes changes in process states, not taking into account the times of residence in individual states. The possibility of a transition from the S_i state to the S_j state is assumed for an embedded Markov chain, provided that $i \neq j$. The matrix of conditional probabilities of interstate transitions $\boldsymbol{P}$ may have non-zero elements, except only for the main diagonal, which can be written using the formula:

$$\boldsymbol{P} = \begin{bmatrix} 0 & p_{12} & \cdots & p_{1(k-1)} & p_{1k} \\ p_{21} & 0 & \cdots & p_{2(k-1)} & p_{2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{(k-1)1} & p_{(k-1)2} & \cdots & 0 & p_{(k-1)k} \\ p_{k1} & p_{k2} & \cdots & p_{k(k-1)} & 0 \end{bmatrix}, \quad (4)$$

under the assumption of meeting the condition of the stochastic matrix [21, 32]:

$$\sum_{j=1}^{k} p_{ij} = 1. \quad (5)$$

Constructing an embedded Markov chain based on an empirical process waveform implies the need to acquire numerical data on interstate transitions. For this purpose, it is justified to construct a population matrix of interstate transitions $\boldsymbol{N}$, according to (6):

$$\boldsymbol{N} = \begin{bmatrix} 0 & n_{12} & \cdots & n_{1(k-1)} & n_{1k} \\ n_{21} & 0 & \cdots & n_{2(k-1)} & n_{2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n_{(k-1)1} & n_{(k-1)2} & \cdots & 0 & n_{(k-1)k} \\ n_{k1} & n_{k2} & \cdots & n_{k(k-1)} & 0 \end{bmatrix}. \quad (6)$$

The elements n_{ij} of the empirical matrix $\boldsymbol{N}$ count the transitions in one step between all combinations of states S_i and S_j of the empirical embedded Markov chain. The process of transitions between states should be recorded for so long that all possible transitions can be observed. Based on the empirical data obtained contained in the matrix $\boldsymbol{N}$, the unknown elements p_{ij} of the transition matrix $\boldsymbol{P}$ are estimated, according to the formula (7):

$$\hat{p}_{ij} = \frac{n_{ij}}{\sum_{j=1}^{k} n_{ij}}, \quad (7)$$

where $\hat{p}_{ij}$ is the maximum likelihood estimator of the unknown value of p_{ij}. This estimator is consistent and unbiased, and the standard error of this estimator decreases rapidly as the number of transitions n_{ij} increases. The standard error $SE(p_{ij})$ of the estimation of the transition probability p_{ij} is given by the formula (8) [6, 15]:

$$SE(p_{ij}) = \sqrt{\frac{\hat{p}_{ij}(1-\hat{p}_{ij})}{\sum_{j=1}^{k} n_{ij}}}. \quad (8)$$

Estimating the probabilities of transitions p_{ij} with an acceptable error may require long-term observations for each state S_i.

3.2. Reliability modelling

The reliability function $R(t)$ determines the probability of an event, in which a technical object operated under assumed conditions remains continuously in a state of technical suitability from time 0 to time t [1, 16, 17, 34]. The mathematical reliability function description is presented by the dependence (9):

$$R(t) = P(T \geq t) \text{for } t \geq 0, \quad (9)$$

where T is the failure time of the technical object.

For an n-state semi-Markov model, if at time $t = 0$ an object is in state S_i belonging to a subset of suitability states A', its first time reaching state S_j belonging to a subset of unsuitability states A means losing suitability and the appearance of failure [13]. The probability of reaching a subset of states A up to time t corresponds to the value of the unreliability function $F_i(t)$.

$$F_i(t) = \Phi_{iA}(t) = P(\Theta_A \leq t | X(t=0) = i), \tag{10}$$

where Θ_A is a random variable that determines the time when the object reaches a subset of states A.

Using the formula (11):

$$R_i(t) = 1 - F_i(t), \tag{11}$$

the reliability function $R(t)$ is determined as (12):

$$R_i(t) = 1 - \Phi_{iA}(t) = 1 - P(\Theta_A \leq t | X(t=0) = i). \tag{12}$$

The cumulative distribution function $\Phi_{iA}(t)$ is calculated using the equation (13):

$$\Phi_{iA}(t) = \sum_{j \in A} Q_{ij}(t) + \sum_{k \in A} \int_0^t \Phi_{kA}(t-x) dQ_{ik}(x), \tag{13}$$

which, after a Laplace – Stieltjes transform, takes the form:

$$\tilde{\phi}_{iA}(s) = \sum_{j \in A} \tilde{q}_{ij}(s) + \sum_{k \in A} \tilde{q}_{ik}(s) \tilde{\phi}_{kA}(s), \tag{14}$$

where:

$$\tilde{\phi}_{iA}(s) = \int_0^\infty e^{-st} d\Phi_{iA}(t), \tag{15}$$

$$\tilde{q}_{ij}(s) = \int_0^\infty e^{-st} dQ_{ij}(t). \tag{16}$$

Equation (13) can be written in matrix form as (17):

$$\left(\boldsymbol{I} - \tilde{\boldsymbol{q}}_{A'}(s)\right) \tilde{\boldsymbol{\phi}}_{A'}(s) = \tilde{\boldsymbol{b}}(s), \tag{17}$$

where $\boldsymbol{I}$ is an identity matrix, $\tilde{\boldsymbol{q}}_{A'}$ is a square submatrix of the transform matrix $\tilde{\boldsymbol{q}}(s)$, while matrices $\tilde{\boldsymbol{\phi}}_{A'}(s)$ and $\tilde{\boldsymbol{b}}(s)$ are single-column matrices of relevant transforms, according to dependencies (18) and (19):

$$\tilde{\Phi}_{A'}(s) = \left[\tilde{\phi}_{iA}(s) : i \in A'\right]^{\mathrm{T}}, \tag{18}$$

$$\tilde{b}(s) = \left[\sum_{j \in A} \tilde{q}_{ij}(s) : i \in A'\right]^{\mathrm{T}}. \tag{19}$$

3.3 Instantaneous probabilities of states

Instantaneous probabilities that an object remains in the S_j states can be used to determine readiness indices at a given time t. Knowing the matrix $\boldsymbol{P} = p_j(t)$ of the $S_i \rightarrow S_j$ transition probabilities for the semi-Markov process and the initial distribution vector of the state probability $p_j(0)$, the matrix of instantaneous probabilities $p_j(t)$ is calculated as a matrix product according to formula (20):

$$\mathrm{p}_j(t) = \mathrm{p}_j(0) \cdot \mathrm{p}(t), \tag{20}$$

The probability matrix $\boldsymbol{p_j}(\mathrm{t})$ can be calculated by solving the matrix equation (21):

$$\tilde{\mathrm{p}}(s) = \frac{1}{s}\left(\mathrm{I} - \tilde{\mathrm{q}}(s)\right)^{-1}\left(\mathrm{I} - \tilde{\mathrm{h}}(s)\right), \tag{21}$$

whereas the matrix elements are calculated according to the dependencies [52] (22-30):

$$\tilde{p}_{ij}(s) = \int_0^\infty e^{-st} dP_{ij}(t), \tag{22}$$

$$\tilde{q}_{ij}(s) = \int_0^\infty e^{-st} dQ_{ij}(t), \tag{23}$$

$$q_{ij}(t) = \frac{dQ_{ij}(t)}{dt} = p_{ij} \frac{dF_{ij}(t)}{dt} = p_{ij} f_{ij}(t), \tag{24}$$

$$\tilde{h}_{ij}(s) = \int_0^\infty e^{-st} dH_{ij}(t), \tag{25}$$

$$h_{ij}(t) = \delta_{ij} \sum_{j=1}^n q_{ij}(t) = \delta_{ij} g_i(t), \tag{26}$$

$$g_i(t) = \frac{dG_i(t)}{dt}, \tag{27}$$

$$G_i(t) = \sum_{k \in S} Q_{ik}(t), \tag{28}$$

$$\tilde{g}_i(s) = \int_0^\infty e^{-st} dG_i(t), \tag{29}$$

where δ_{ij} are elements of the identity matrix (30):

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}. \tag{30}$$

3.4. Ergodic probabilities of states

Ergodic probabilities values of an embedded Markov chain π_j are calculated by solving the matrix equation (31) [21]:

$$(\boldsymbol{P}^T - \boldsymbol{I}) \cdot \boldsymbol{\Pi} = 0, \tag{31}$$

assuming that the standardization condition is met, according to the formula (32):

$$\sum_{j=1}^n \pi_j = 1. \tag{32}$$

The random variable T_i determines the sojourn time in state S_i before the transition to another state. In turn, the variable T_{ij} determines the sojourn time in the S_i state before the direct transition to the S_j state. If at time $t = 0$, the object is in the state $S_i \in A'$, then the sum of the times T_{ij} until the transition to the state $S_k \in A$ is equal to the value of Θ_A, as described by the formula (33):

$$\Theta_A = \sum_{j:S_j \in A'}^k T_{ij}, \; S_k \in A. \tag{33}$$

If an embedded Markov chain exhibits ergodicity and there are expected values $E(T_i)$ of state sojourn times, the values of ergodic probabilities p_j for a semi-Markov process are determined using the dependence (34-35):

$$p_j = \frac{\pi_j E(T_j)}{\sum_{i=1}^k \pi_i E(T_i)}, \tag{34}$$

$$E(T_j) = \sum_{i=1}^k p_{ij} E(T_{ij}), \tag{35}$$

where π_i is the ergodic probability of an embedded Markov chain for the S_i state, and $E(T_{ij})$ is the expected time for the direct transition from the S_i state to the S_j state.

4. Results and discussions

4.1. 3-state Semi-Markov model of operation process

The operation process of light utility vehicles functioning within military transport systems is executed within a multi-state phase space. The identification of a phase-space state set should take into account the objectives of a developed stochastic model.

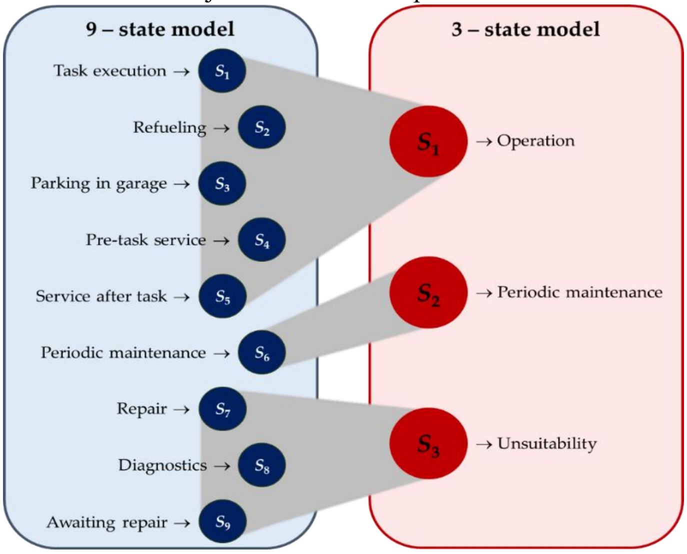


Figure 1. Aggregation of states of the 9-state model [33] modified to the 3-state model

For the purposes of a thorough analysis and evaluation of the operation process in terms of functional readiness indices, technical readiness, and technical suitability, the authors of this publication developed the 9-state semi-Markov model [33]. The calculation of instantaneous probabilities and probabilities of first-time reaching a given state subset by a technical object within a multi-state model requires having significant computing power suitable for complex mathematical operations. A solution to this problem is the aggregation of process phase space states.

The semi-Markov 3-state model was developed, which is a modification of the 9-state model proposed in [33] for reliability analyses. Reducing the number of operational states enabled us to achieve a 3-state phase space containing: S_1 – Operation, S_2 – Periodic maintenance, S_3 – Unsuitability. The state aggregation diagram is shown in Fig. 1.

In the 3-state operation model, the S_1 state defines technical readiness of a vehicle, which is implementing a task or awaiting a transport task as part of the analysed transport system. Short-term activities associated with daily vehicle maintenance and refuelling are frequently assumed. The S_2 state refers to periodic maintenance associated with the checking of the correct functioning of essential vehicle mechanisms, the replacement of specified parts and operating liquids according to the vehicle manual, and maintenance activities. The S_3 state means the unsuitability of a technical object and the need to conduct diagnostic activities in order to identify the causes behind the failure and to repair or replace damaged parts, mechanisms, subassemblies or assemblies. It also includes the time to wait for the availability of qualified personnel as well as materials and technical resources of the system to carry out diagnostic and repair activities. A directed graph of interstate transitions shown in Fig. 2 has been developed for such an identified phase space.

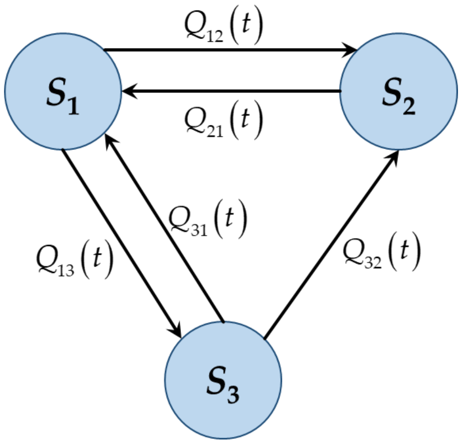


Figure 2. Transition graph of the 3-state semi-Markov model.

According to the assumptions of the operational strategy adopted within the analysed transport system, reaching state S_3 is possible only from state S_1. Periodic maintenance activities may cover only a vehicle in a state of technical suitability. If a vehicle is damaged during operation and is qualified to perform periodic maintenance due to completing a standard interval between subsequent maintenance cycles, it is first brought to a state of technical suitability through repair activities, followed by periodic maintenance. In the 3-state model, this principle has been implemented as an inability of a transition from state S_2 to state S_3.

A mathematical description of the 3-state semi-Markov model is a renewal kernel matrix $\boldsymbol{Q}(t)$, the elements of which are the products of conditional probabilities of the embedded Markov chain and distribution functions of conditions times of residence in individual states, as represented by formula (36):

$$\boldsymbol{Q}(t) = \begin{bmatrix} 0 & Q_{12}(t) & Q_{13}(t) \\ Q_{21}(t) & 0 & 0 \\ Q_{31}(t) & Q_{32}(t) & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & p_{12}F_{12}(t) & p_{13}F_{13}(t) \\ p_{21}F_{21}(t) & 0 & 0 \\ p_{31}F_{31}(t) & p_{32}F_{32}(t) & 0 \end{bmatrix}. \quad (36)$$

An alternative model definition is the matrix $\mathbf{q}(t)$, the elements of which are products of conditional probabilities of an embedded Markov chain and densities of conditional probabilities of state residence times, according to the dependence (37):

$$\boldsymbol{q}(\mathrm{t}) = \begin{bmatrix} 0 & q_{12}(t) & q_{13}(t) \\ q_{21}(t) & 0 & 0 \\ q_{31}(t) & q_{32}(t) & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & p_{12}f_{12}(t) & p_{13}f_{13}(t) \\ p_{21}f_{21}(t) & 0 & 0 \\ p_{31}f_{31}(t) & p_{32}f_{32}(t) & 0 \end{bmatrix}. \quad (37)$$

4.2. Estimation of model parameters

The 3-state semi-Markov model was validated on the basis of the empirical waveform of the operation process of a sample of 19 Honker 2000 vehicles. These vehicles are part of a military unit transport system and are intended for transporting people and cargo weighing up to 1000 kg. A collective empirical database has been developed based on operating documents that cover a 3-year study period. The graphical visualization of the database can be found in Fig. 3, where each month is marked with a relevant colour, referring to the vehicles staying in specified operational states. The periods in which a vehicle remained in an unsuitable state were expressed in days. The total number of interstate transitions for the entire sample was 416. This was used as a base to estimate interstate transition probabilities for an embedded Markov chain presented by the formula (38):

$$\boldsymbol{P} = [p_{ij}] = \begin{bmatrix} 0 & 0.51 & 0.49 \\ 1 & 0 & 0 \\ 0.9 & 0.1 & 0 \end{bmatrix}, \quad (38)$$

whereas standard estimation errors amounted to:

$$\boldsymbol{SE} = \begin{bmatrix} 0 & 0.0352 & 0.0352 \\ 0 & 0 & 0 \\ 0.0300 & 0.0300 & 0 \end{bmatrix}. \quad (39)$$

$SE(p_{ij})$ values did not exceed 0.04, which can be adopted as a satisfactory and acceptable level in engineering applications [33, 39, 40].

The next model validation stage involves matching distributions to process time characteristics and estimating the parameters of these distributions. Matlab software was used for this purpose. The results are presented in Fig. 4 and Table 2. The four characteristics T_{12}, T_{13}, T_{31} and T_{32} were matched with an exponential distribution with parameter λ (λ_{12}, λ_{13}, λ_{31} and λ_{32}, respectively). Whereas the characteristic T_{21} was described by a gamma distribution with the shape k_{21} and the scale θ_2 parameters. The R^2 coefficient of determination was used as a measure of the quality of the match between the empirical and theoretical distribution functions. For characteristics T_{12}, T_{13}, and T_{21}, coefficient R^2 adopted values above 0.96, while for characteristics T_{31} and T_{32}, it ranged from 0.85 to 0.86.

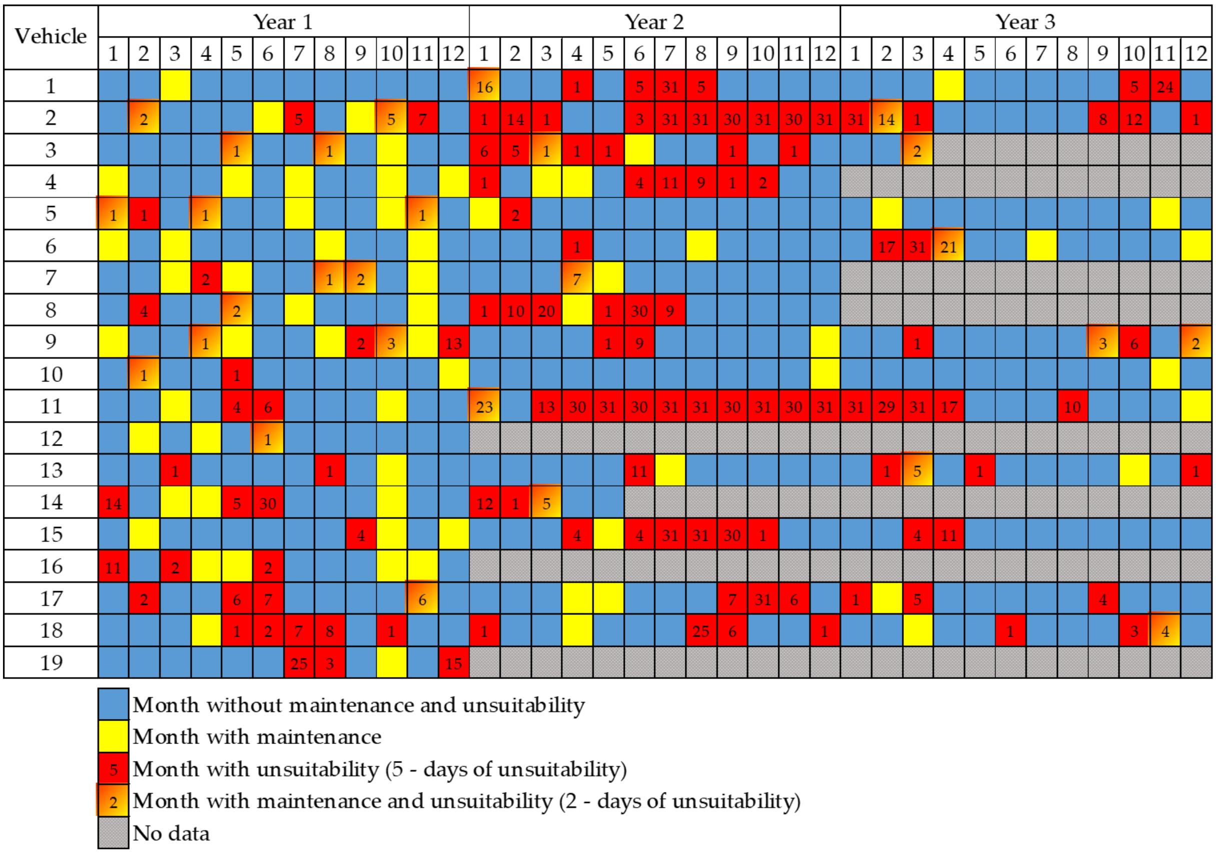


Figure 3. Graphical visualization of database.

Table 2. Estimation of distribution parameters.

Variable	Distribution	Equation	Parameters	R^2
T_{12}	Exponential	$F_{12}(t) = 1 - e^{-\lambda_{12}t}$	$\lambda_{12} = 1.33\times10^{-5}$	0.9909
T_{13}	Exponential	$F_{13}(t) = 1 - e^{-\lambda_{13}t}$	$\lambda_{13} = 1.07\times10^{-5}$	0.9961
T_{21}	Gamma	$F_{21}(t) = \frac{1}{\Gamma(k_{21})}\gamma\left(k_{21}, \frac{t}{\theta_{21}}\right)$	$k_{21} = 4$ $\theta_{21} = 65.85$	0.9630
T_{31}	Exponential	$F_{31}(t) = 1 - e^{-\lambda_{31}t}$	$\lambda_{31} = 4.67\times10^{-5}$	0.8560
T_{32}	Exponential	$F_{32}(t) = 1 - e^{-\lambda_{32}t}$	$\lambda_{32} = 1.46\times10^{-4}$	0.8662

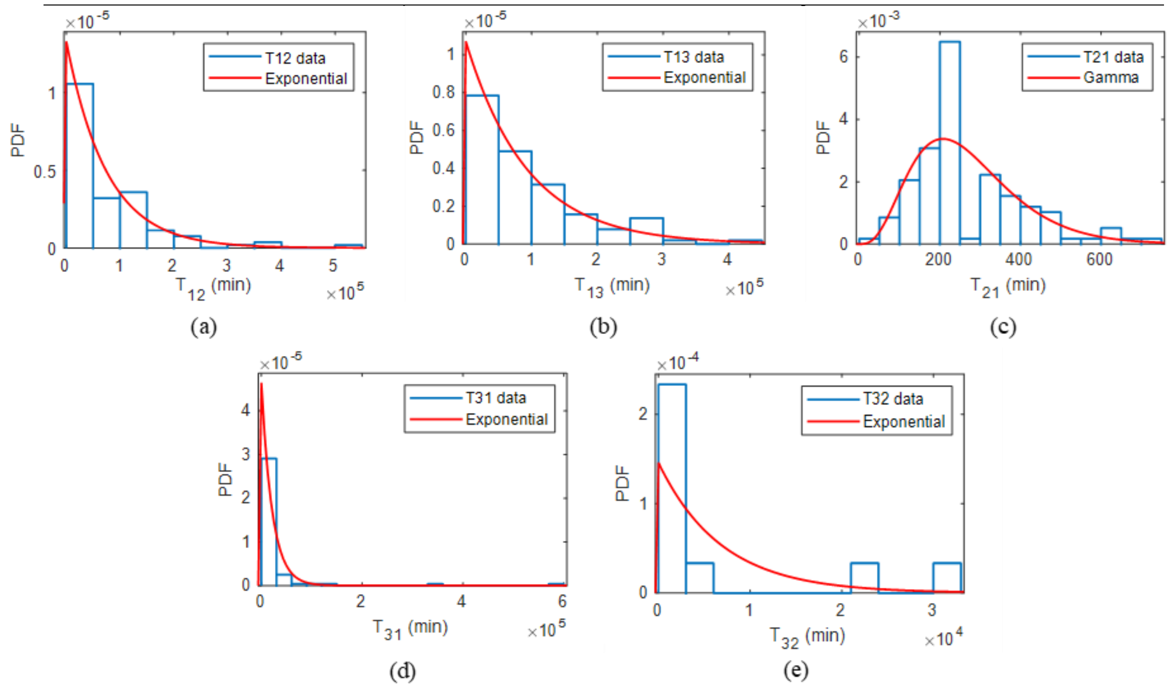


Figure 4. PDFs of the time of sojourn T_{ij} in state S_i before transitioning to state S_j: (a) – T_{12}, (b) – T_{13}, (c) – T_{21}, (d) – T_{31}, (e) – T_{32}.

4.3. Reliability assessment of light utility vehicles

Based on empirical data and the estimated parameters of the 3-state semi-Markov model, calculations were performed according to the dependencies presented in Section 3 to assess the reliability characteristics of light utility vehicles. The reliability function $R(t)$ was determined using Equation (17), assuming that a vehicle at time $t = 0$ is in one of the technical suitability states, namely, S_1 or S_2. In the case of such initial conditions, the form of the square matrix $\tilde{\mathbf{q}}_{A'}(s)$ is shown by the formula (40). While single-column matrices $\tilde{\boldsymbol{\phi}}_{A'}(s)$ and $\tilde{\boldsymbol{b}}(s)$ are consistent with formulas (41) and (42).

$$\tilde{\mathbf{q}}_{A'}(s) = \begin{bmatrix} 0 & \tilde{q}_{12}(s) \\ \tilde{q}_{21}(s) & 0 \end{bmatrix}, \quad (40)$$

$$\tilde{\boldsymbol{\Phi}}_{A'}(s) = \begin{bmatrix} \tilde{\phi}_{13}(s) \\ \tilde{\phi}_{23}(s) \end{bmatrix}, \quad (41)$$

$$\tilde{\mathbf{b}}(s) = \begin{bmatrix} \tilde{q}_{13}(s) \\ 0 \end{bmatrix}. \quad (42)$$

Substituting the aforementioned dependencies into formula (17) provided the equation:

$$\begin{bmatrix} 1 & -\tilde{q}_{12}(s) \\ -\tilde{q}_{21}(s) & 1 \end{bmatrix} \cdot \begin{bmatrix} \tilde{\phi}_{13}(s) \\ \tilde{\phi}_{23}(s) \end{bmatrix} = \begin{bmatrix} \tilde{q}_{13}(s) \\ 0 \end{bmatrix}. \quad (43)$$

Using Mathematica software, the authors obtained the solutions to Equation (43), which are the values of the variables $\tilde{\phi}_{13}(s)$ and $\tilde{\phi}_{23}(s)$ presented using the formula (44):

$$\begin{cases} \tilde{\phi}_{13}(s) = \frac{\tilde{q}_{13}(s)}{1-\tilde{q}_{12}(s)\tilde{q}_{21}(s)} \\ \tilde{\phi}_{23}(s) = \frac{\tilde{q}_{13}(s)\tilde{q}_{21}(s)}{1-\tilde{q}_{12}(s)\tilde{q}_{21}(s)} \end{cases}. \quad (44)$$

Solutions to Equation (42) are Laplace transforms of the density function of the probability of the first transition from states S_1 and S_2, respectively, to the state of technical unsuitability S_3. The probability of a transition from any given time t to state S_3 depends on the initial state of the process. Therefore, the form of the reliability function also depends on the initial state. The authors adopted the designation of the reliability function $R_1(t)$ for an object that stayed at time $t = 0$ in state S_1 and $R_2(t)$ for the initial state S_2. The Laplace transforms $\tilde{R}_1(s)$ and $\tilde{R}_2(s)$ for the reliability functions $R_1(t)$ and $R_2(t)$, respectively, have been determined using the dependence (45):

$$\begin{cases} \tilde{R}_1(s) = \frac{1-\tilde{\phi}_{13}(s)}{s} = \frac{1-\tilde{q}_{12}(s)\tilde{q}_{21}(s)-\tilde{q}_{13}(s)}{s(1-\tilde{q}_{12}(s)\tilde{q}_{21}(s))} \\ \tilde{R}_2(s) = \frac{1-\tilde{\phi}_{23}(s)}{s} = \frac{1-(\tilde{q}_{12}(s)+\tilde{q}_{13}(s))\tilde{q}_{21}(s)}{s(1-\tilde{q}_{12}(s)\tilde{q}_{21}(s))}, \end{cases} \quad (45)$$

while transforms $\tilde{q}_{ij}(s)$ are expressed through formulas (46):

$$\begin{cases} \tilde{q}_{12}(s) = p_{12}\left(\frac{\lambda_{12}}{s+\lambda_{12}}\right) \\ \tilde{q}_{13}(s) = p_{13}\left(\frac{\lambda_{13}}{s+\lambda_{13}}\right) \\ \tilde{q}_{21}(s) = p_{21}\left(\frac{1}{1+\theta_{21}s}\right)^{k_{21}} \\ \tilde{q}_{31}(s) = p_{31}\left(\frac{\lambda_{31}}{s+\lambda_{31}}\right) \\ \tilde{q}_{32}(s) = p_{32}\left(\frac{\lambda_{32}}{s+\lambda_{32}}\right) \end{cases}. \quad (46)$$

In turn, $q_{ij}(t)$ values for the analysed case study have been determined as dependencies (47):

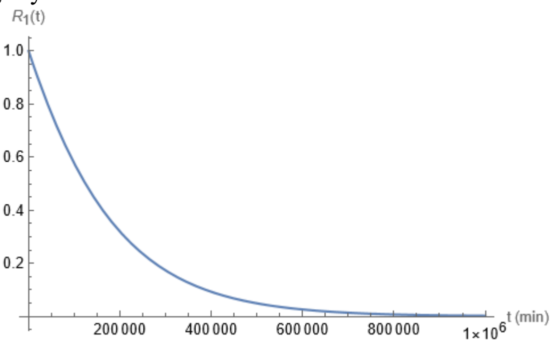
$$\begin{cases} q_{12}(t) = p_{12}\lambda_{12}e^{-\lambda_{12}t} \\ q_{13}(t) = p_{13}\lambda_{13}e^{-\lambda_{12}t} \\ q_{21}(t) = p_{21}\frac{1}{\Gamma(k_{21})\theta_{21}{}^{k_{21}}}t^{k_{21}-1}e^{-\frac{t}{\theta_{21}}} \\ q_{31}(t) = p_{31}\lambda_{31}e^{-\lambda_{31}t} \\ q_{32}(t) = p_{32}\lambda_{32}e^{-\lambda_{32}t} \end{cases}. \quad (47)$$

After substituting estimated values of the time characteristic distribution parameters for the semi-Markov process, the authors determined the formulas of the reliability functions $R_1(t)$ and $R_2(t)$ in the time domain t, using the inverse Laplace transforms of the functions $\tilde{R}_1(s)$ and $\tilde{R}_2(s)$:

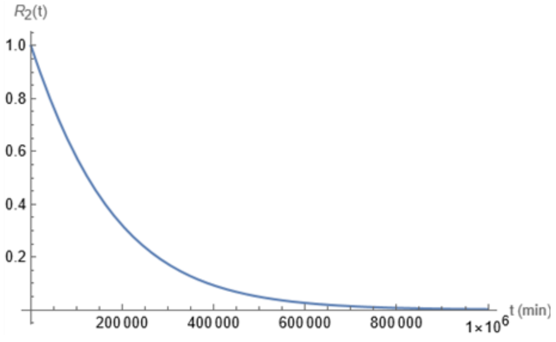
$$R_1(t) = \mathcal{L}^{-1}\{\tilde{R}_1\}(t) =$$
$$(8.2271 \times 10^{-6} + 5.2308 \times 10^{-6}i)e^{(-0.01674170-0.00148814i)t}$$
$$+(8.2271 \times 10^{-6} - 5.2308 \times 10^{-6}i)e^{(-0.01674170+0.00148814i)t}$$
$$-(7.4578 \times 10^{-6} - 1.4309 \times 10^{-5}i)e^{(-0.01363380-0.00164869i)t}$$
$$-(7.4578 \times 10^{-6} + 1.4309 \times 10^{-5}i)e^{(-0.01363380+0.00164869i)t}$$
$$-0.3032e^{-0.00001070t} + 1.3032e^{-6.50536\times10^{-6}t}$$
$$-2.0773 \times 10^{-16}, \quad (48)$$

$$R_2(t) = \mathcal{L}^{-1}\{\tilde{R}_2\}(t) =$$
$$-(0.0194 + 0.0147i)e^{(-0.01674170-0.00148814i)t}$$
$$-(0.0194 - 0.0147i)e^{(-0.01674170+0.00148814i)t}$$
$$+(0.0185 - 0.0269i)e^{(-0.01363380-0.00164869i)t}$$
$$+(0.0185 + 0.0269i)e^{(-0.01363380+0.00164869i)t}$$
$$-0.3040e^{-0.0000107t} + 1.3054e^{-6.50536\times10^{-6}t}$$
$$-2.0773 \times 10^{-16}. \quad (49)$$

Fig. 5 shows graphical waveforms of the reliability functions $R_1(t)$ and $R_2(t)$ for the range of $0 - 10^6$ (min). They satisfy the assumptions about the monotonicity of the reliability function. The function limit at infinity is -2.0773×10^{-16}, which is a value negligibly different from zero.



(a)



(b)

Figure 5. Reliability functions: (a) – $R_1(t)$, (b) – $R_2(t)$.

Fig. 6 shows the difference in the values between the determined reliability functions $R_1(t)$ and $R_2(t)$ for the time domain range of 0 to 10^6 (min). The maximum observed difference in this value does not exceed 0.0014, which means almost identical waveforms of both reliability functions for the analysed time interval. This conclusion prompts the selection of one function as a basis for further reliability analyses. Due to the operational strategy in military transport systems assuming the assignment of fully operational vehicles to the system, it was assumed that at time $t = 0$ the technical object under study was in the S_1 state. This implies determining the remaining reliability characteristics based on the reliability function $R_1(t)$.

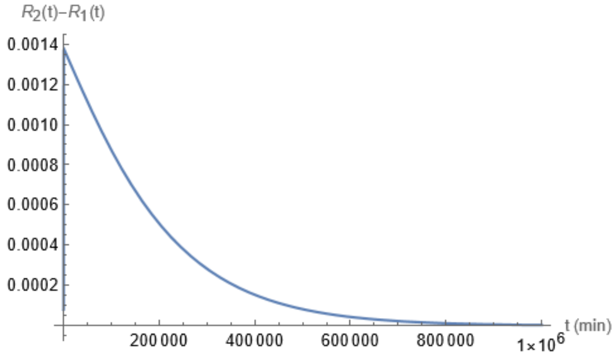


Figure 6. Difference value between reliability functions: $R_2(t) - R_1(t)$.

The failure probability density function $f(t)$ and the failure intensity function $\lambda(t)$ are identity reliability characteristics, which are related to the probability of a failure at a given time t. The function $f(t)$ defines the failure probabilities at time t [34] per unit of time. The probability calculus defines that the failure occurrence density $f(t)$ is a derivative of the unreliability function $F(t)$, as demonstrated by dependence:

$$f(t) = \frac{dF(t)}{dt} = \frac{d(1-R(t))}{dt}. \tag{50}$$

After substituting the function $R_1(t)$ into the formula (50), the authors obtained the following $f_1(t)$:

$$f_1(t) = (1.2995 \times 10^{-7} + 9.9815 \times 10^{-8} i) e^{(-0.01674169-0.00148813i)t} + (1.2995 \times 10^{-7} - 9.9815 \times 10^{-8} i) e^{(-0.01674169+0.00148813i)t} - (1.2527 \times 10^{-7} + 1.8279 \times 10^{-7} i) e^{(-0.01363376+0.00164869i)t} - (1.2527 \times 10^{-7} - 1.8279 \times 10^{-7} i) e^{(-0.01363376-0.001648692i)t} - (3.2440 \times 10^{-6} + 4.6129 \times 10^{-22} i) e^{-0.00001070t} + (8.4776 \times 10^{-6} + 1.6602 \times 10^{-21} i) e^{-0.00000651t} \tag{51}$$

Whereas the function $\lambda(t)$ defines the value of conditional probability for a technical object failure at time t, provided that it was not damaged during the interval $(0, t)$. According to the properties of conditional probability, the value of function $\lambda(t)$ at time t is expressed through the formula (52):

$$\lambda(t) = \frac{f(t)}{R(t)}. \tag{52}$$

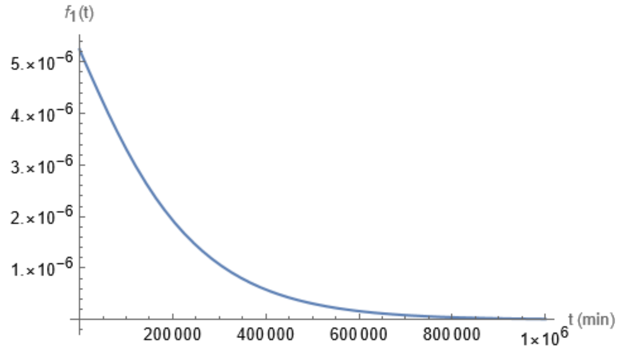
After substituting the function $R_1(t)$ and $f_1(t)$ into the formula (52), the authors obtained the following form of function $\lambda_1(t)$:

$$\lambda_1(t) = \begin{pmatrix} (1.2995 \times 10^{-7} + 9.9815 \times 10^{-8} i) e^{(-0.01674169-0.00148813i)t} + \\ (1.2995 \times 10^{-7} - 9.9815 \times 10^{-8} i) e^{(-0.01674169+0.00148813i)t} - \\ (1.2527 \times 10^{-7} + 1.8279 \times 10^{-7} i) e^{(-0.01363376+0.00164869i)t} - \\ (1.2527 \times 10^{-7} - 1.8279 \times 10^{-7} i) e^{(-0.01363376-0.001648692i)t} - \\ (3.2440 \times 10^{-6} + 4.6129 \times 10^{-22} i) e^{-0.00001070t} + \\ (8.4776 \times 10^{-6} + 1.6602 \times 10^{-21} i) e^{-0.00000651t} \end{pmatrix} \Big/ \begin{pmatrix} (8.2271 \times 10^{-6} + 5.2308 \times 10^{-6} i) e^{(-0.01674170-0.00148814i)t} + \\ (8.2271 \times 10^{-6} - 5.2308 \times 10^{-6} i) e^{(-0.01674170+0.00148814i)t} - \\ (7.4578 \times 10^{-6} - 1.4309 \times 10^{-5} i) e^{(-0.01363380-0.00164869i)t} - \\ (7.4578 \times 10^{-6} + 1.4309 \times 10^{-5} i) e^{(-0.01363380+0.00164869i)t} - \\ 0.3032 e^{-0.00001070t} + 1.3032 e^{-6.50536\times10^{-6}t} - 2.0773 \times 10^{-16} \end{pmatrix}. \tag{53}$$

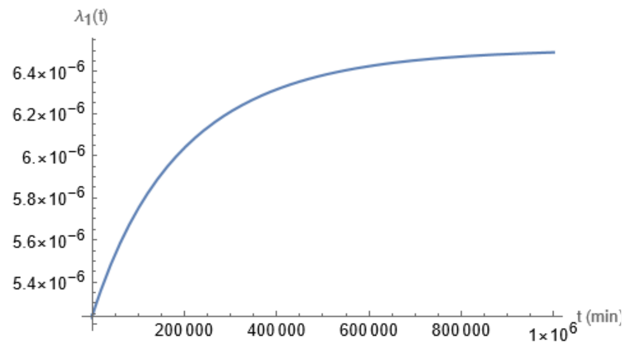
Function waveforms $f_1(t)$ and $\lambda_1(t)$ have been graphically presented using graphs within the time domain range of $0 - 10^6$ (min) in Fig. 7. The function $f_1(t)$ decreases and asymptotically tends to 0, while function $\lambda_1(t)$ is increasing and stabilizes at a level of approximately 6.5×10^{-6} (min^{-1}) after 5×10^5 (min).

The mean time to failure ($MTTF$) is an important measure of the reliability of technical objects that defines the correct operating time of the vehicle. It is determined as a definite integral of the reliability function, within a range from 0 to ∞, which is expressed in mathematical notation by equation (54):

$$MTTF = \int_0^\infty t \cdot f(t) dt = \int_0^\infty R(t) dt. \tag{54}$$



(a)



(b)

Figure 7. Reliability characteristics: (a) – PDF of failure $f_1(t)$, (b) – intensity of failure $\lambda_1(t)$.

In the analysed case study of light utility vehicles, the calculated $MTTF$ based on the base of the $R_1(t)$ function is 171,989.0 (min). Converted to calendar days, this amounts to a value of 119.44 (days).

4.4. Readiness and suitability

According to the methodology adopted to determine vehicle readiness and suitability indices, the authors studied the developed 3-state semi-Markov model. The conditional probability matrix $\tilde{\boldsymbol{p}}(s)$ in the domain of the Laplace operator s is calculated using equation (55):

$$\tilde{\mathbf{p}}(s) = \frac{1}{s}\left(\mathbf{I} - \tilde{\mathbf{q}}(s)\right)^{-1}\left(\mathbf{I} - \tilde{\mathbf{h}}(s)\right), \tag{55}$$

where elements of the matrix $\tilde{\boldsymbol{q}}(s)$ are represented by the formula (46), while non-zero elements of the matrix $\tilde{\boldsymbol{h}}(s)$ have been determined based on systems of equations (56) and (57):

$$\begin{cases} G_1(t) = Q_{12}(t) + Q_{13}(t) = p_{12}\left(1 - e^{-\lambda_{12}t}\right) + p_{13}\left(1 - e^{-\lambda_{13}t}\right) \\ G_2(t) = Q_{21}(t) = p_{21}\frac{1}{\Gamma(k_{21})}\gamma\left(k_{21}, \frac{t}{\theta_{21}}\right) \\ G_3(t) = Q_{31}(t) + Q_{32}(t) = p_{31}\left(1 - e^{-\lambda_{31}t}\right) + p_{32}\left(1 - e^{-\lambda_{32}t}\right) \end{cases} \tag{56}$$

$$\begin{cases} h_{11}(t) = g_1(t) = \frac{dG_1(t)}{dt} = p_{12}\lambda_{12}e^{-\lambda_{12}t} + p_{13}\lambda_{13}e^{-\lambda_{13}t} \\ h_{22}(t) = g_2(t) = \frac{dG_2(t)}{dt} = p_{21}\frac{1}{\Gamma(k_{21})\theta_{21}{}^{k_{21}}}t^{k_{21}-1}e^{-\frac{t}{\theta_{21}}} \\ h_{33}(t) = g_3(t) = \frac{dG_3(t)}{dt} = p_{31}\lambda_{31}\,e^{-\lambda_{31}t} + p_{32}\lambda_{32}e^{-\lambda_{32}t} \end{cases} \tag{57}$$

and take the form consistent with the set of equations (58):

$$\begin{cases} \tilde{h}_{11}(s) = \mathcal{L}\{h_{11}\}(s) = p_{12}\left(\frac{\lambda_{12}}{s+\lambda_{12}}\right) + p_{13}\left(\frac{\lambda_{13}}{s+\lambda_{13}}\right) \\ \tilde{h}_{22}(s) = \mathcal{L}\{h_{22}\}(s) = p_{21}\left(\frac{1}{1+\theta_{21}s}\right)^{k_{21}} \\ \tilde{h}_{33}(s) = \mathcal{L}\{h_{33}\}(s) = p_{31}\left(\frac{\lambda_{31}}{s+\lambda_{31}}\right) + p_{32}\left(\frac{\lambda_{32}}{s+\lambda_{32}}\right) \end{cases} \tag{58}$$

The elements of the values of the conditional probability matrix in the time domain t are calculated as the inverse Laplace transform of matrix elements $\tilde{\boldsymbol{p}}(s)$, as demonstrated by the formula (59):

$$\mathrm{p}(t) = \mathcal{L}^{-1}\{\tilde{\mathrm{p}}\}(t). \tag{59}$$

Instantaneous probabilities of a technical object staying in individual states are determined as a product of the initial distribution vector and conditional probability matrices for the semi-Markov process. It was assumed that at time $t = 0$ a vehicle was in full technical suitability, therefore, it remained in state S_1. The initial distribution vector $\boldsymbol{p}_j(0)$, which describes the assumed adopted, has been presented using formula (60):

$$\mathbf{p}_j(0) = [1 \quad 0 \quad 0]. \tag{60}$$

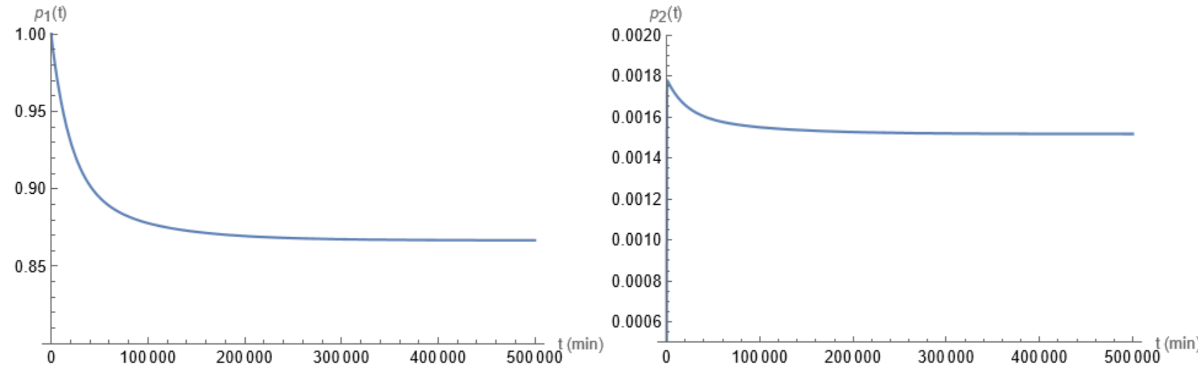
Fig. 8 shows the waveform of changes in the values of instantaneous probabilities $p_j(t)$ over the range from 0 to 5×10^5 (min). After 10^5 (min), the value stabilizes, and the probabilities $p_j(t)$ tend to ergodic values.

The approximate values of instantaneous probabilities $p_1(t)$, $p_2(t)$ and $p_3(t)$ are represented by formulas (61-63):

$$\begin{aligned} p_1(t) = {} & 0.0912e^{-0.00005131t} + 0.0177e^{-0.00001486t} + 0.0226e^{-0.00001233t} + 0.8667e^{9.84533644\times10^{-22}t} + \\ & e^{(-0.01674167-0.00148812i)t}\left((0.0248 + 0.0190i) + (0.0248 - 0.0190i)e^{0.00297623it}\right) + \\ & e^{(-0.01363379-0.00164865i)t}\left((-0.0239 + 0.0349i) - (0.0239 + 0.0349i)e^{(0.00329731i)t}\right), \end{aligned} \tag{61}$$

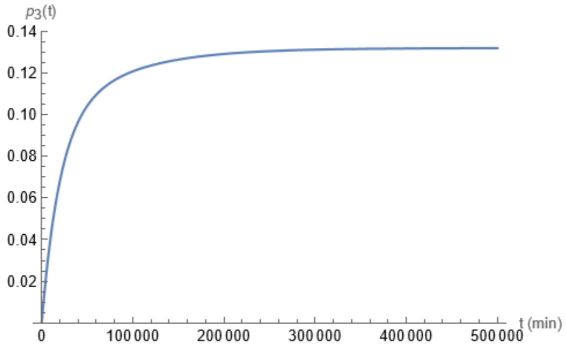
$$\begin{aligned} p_2(t) = {} & 0.7277 + 1.6486 \times 10^{-4}e^{-0.00005131t} - 2.8573 \times 10^{-5}e^{-0.00001486t} + 1.3228 \times 10^{-4}e^{-0.00001233t} - \\ & 0.7262e^{9.84533644\times10^{-22}t} + e^{(-0.016741665-0.00148811i)t}\left((-0.0248 - 0.0190i) - (0.0248 - 0.0190i)e^{(0.00329731i)t}\right) + \\ & e^{(-0.01363379-0.00164865i)t}\left((0.0239 - 0.0349i) + (0.0239 + 0.0349i)e^{(0.00329731i)t}\right), \end{aligned} \tag{62}$$

$$p_3(t) = 5.2430 \times 10^{-6}\begin{pmatrix} -148274.1947 - 17434.5692e^{-0.00005131t} - 3349.5998e^{-0.00001486t} - \\ 4360.4149e^{-0.00001233t} + 173419.0738e^{9.84533644\times10^{-22}t} + \\ e^{(-0.01674167i)t}\left((-1.5734 - 0.9999i) - (1.5734 - 0.9999i)e^{(0.00297623i)t}\right) + \\ e^{(-0.01363379-0.00164865i)t}\left((1.4259 - 2.7382i) + (1.4259 + 2.7382i)e^{(0.00329731i)t}\right) \end{pmatrix}. \tag{63}$$



(a) (b)

Figure 8. Instantaneous probabilities of states: (a) – $p_1(t)$, (b) – $p_2(t)$, (c) – $p_3(t)$.



(c)

Figure 8. (continued)

Dependence (31) was used to determine the ergodic probabilities π_j for an embedded Markov chain. The dependence, after substituting appropriate values for the developed model, adopted the form of a matrix equation (64):

$$\left(\begin{bmatrix} 0 & p_{12} & p_{13} \\ p_{21} & 0 & 0 \\ p_{31} & p_{32} & 0 \end{bmatrix}^{T} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (64)$$

assuming that the sum π_j is equal to 1.

The solution of Equation (64) is shown in Table 3. Based on Equation (35), the authors calculated the expected durations for the vehicle to stay in individual operational states. The system should tend to maximize the duration of the S_1 state. The time $E(T_2)$ depends on the adopted periodic maintenance strategy, the scope of maintenance activities, and the technical capabilities of the system. The S_3 state is undesirable, since it reduces the capabilities of a transport system, and, thus, tends to minimize its duration.

Table 3. Ergodic probabilities of embedded Markov chain and semi-Markov process.

	S_1	S_2	S_3
π_j	0.4882	0.2750	0.2368
$E(T_j)$ (min)	84140.3	263.4	26121.3
$E(T_j)$ (days)	58.43	0.18	18.14
p_j	0.8678	0.0015	0.1307

Probabilities π_j and the expected times $E(T_j)$ were used to determine the ergodic probabilities of the semi-Markov process p_j. They are the basis for calculating technical suitability and readiness indices.

Readiness means that a vehicle reamins in the S_1 state and the value of the readiness coefficient corresponds to the ergodic probability $p_1 = 0.8678$. In turn, the set $\{S_1, S_2\}$ is a subset of the technical suitability states, which implies a value of the technical suitability coefficient equal to $p_1+p_2 = 0.8693$.

5. Conclusions

The operational process of light utility vehicles was modelled using the semi-Markov process theory. The phase space was identified on the basis of the analysis of the empirical waveform of operation and previously developed models. The 9-state model was aggregated into 3 main operational states. S_1 – operation, S_2 – periodic maintenance, S_3 – unsuitability. Based on actual data acquired from a military transport system, the authors estimated the values of interstate transition conditional probability matrices of an embedded Markov chain and matched time characteristic distribution functions. The standard estimation errors of the $SE(p_{ij})$ probabilities and the determination coefficient R^2 between the empirical and theoretical distribution functions obtained values that were satisfactory from the engineering applicability perspective. Given all the above, it can be concluded that the 3-state semi-Markov model is a credible representation of the studied military vehicle operation process. Reliability functions were determined as complements to the variable distribution function Θ_A, which denotes the time of the first transition to the subset of unsuitable states. The considerations were carried out under two assumptions regarding a technical object that remained at time $t = 0$ in states S_1 and S_2, respectively. Based on the graph showing the difference between functions $R_1(t)$ and $R_2(t)$, it was concluded that the waveform of the reliability function negligibly depends on the initial state of the operation process (assuming that this state belongs to a subset of states of technical suitability). The reliability function was used to determine other characteristics of the objects studied, that is, the failure probability density function and the failure intensity. The analytical form of the determined characteristics was not directly interpreted. However, using advanced IT software, the authors obtained their graphical form, which facilitated the interpretation of results in the form of dependence graphs in the time domain. The failure probability density function is decreasing, while the failure intensity is increasing, and stabilizes after 5×10^5 (min) at a level of approx. 6.5×10^{-6} (min^{-1}). The expected time to failure was calculated using a definite integral reliability function in the range from 0 to ∞ and amounted to approx. 119.44 (days).

The final stage of the research involved determining instantaneous probabilities that a vehicle would remain in the operational state. The solution of a matrix equation in the domain of the Laplace operator s, followed by the implementation of an inverse Laplace transform, allowed us to obtain the analytical form of instantaneous probabilities $p_j(t)$ in the time domain t. Based on graphical interpretations, the authors concluded that the process stabilizes after a time of approximately 10^5 (min). The ergodic probabilities of the semi-Markov process have been determined using the ergodic probabilities of the embedded Markov chain and values of expected times of residence in operational states. The vehicle technical suitability and readiness indices adopted values of 0.8678 and 0.8693, respectively, which, however, compared to the 9-state model [33] means a reduction of approximately 4.6%. Therefore, the 3-state model is a less accurate reflection of the actual operation process. However, it significantly reduces and simplifies the computations performed in relation to reliability analyses.

The proposed approach enables a comprehensive reliability analysis of technical systems and objects for a multi-state phase space of the operation process. This paper presents all the stages of developing a semi-Markov model, its validation, and its application to determine the most important reliability characteristics and readiness indices of an object. In turn, the potential for further research directions may be the employing of a 3-state semi-Markov model to optimize the periodic maintenance and repair process.

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Author Contributions
Conceptualization, M.O.; methodology, M.O. and J.Z.; software, M.O.; validation, M.O.; formal analysis, M.O.; investigation, M.O.; resources, M.O. and J.Z.; data curation, M.O. and J.Z.; writing—original draft preparation, M.O.; writing—review and editing, M.O., J.Z. and J.M.; visualization, M.O.; supervision, J.Z. and J.M.; project administration, M.O., J.Z. and J.M.; funding acquisition, J.M. All authors have read and agreed to the published version of the manuscript.

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