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Estimation procedures for partially accelerated life test model based on unified hybrid censored sample from the Gompertz distribution

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Highlights

- Statistical inference methods are developed for constant-stress partially accelerated life testing under unified hybrid censoring scheme.
- Component lifetimes are assumed to follow Gompertz distributions.
- Different point estimation methods are discussed using the classical and Bayesian approaches.
- The existence of the maximum likelihood estimate of the parameters of the proposed model is proved.
- Asymptotic and Bootstrap confidence intervals are given for model parameters and accelerated factor.
- Numerical studies show that the MAP estimates perform superior than the MLEs (or MPSs) with respect to the smallest MSE values.

Abstract

The accelerated life testing is the key methodology of evaluating product reliability rapidly. This paper presents statistical inference of Gompertz distribution based on unified hybrid censored data under constant-stress partially accelerated life test (CSPALT) model. We apply the stochastic expectation-maximization algorithm to estimate the CSPALT parameters and to reduce computational complexity. It is shown that the maximum likelihood estimates exist uniquely. Asymptotic confidence intervals and confidence intervals using bootstrap-*p* and bootstrap-*t* methods are constructed. Moreover the maximum product of spacing (MPS) and maximum a posteriori (MAP) estimates of the model parameters and accelerated factor are discussed. The performances of the various estimators of the CSPALT parameters are compared through the simulation study. In summary, the MAP estimates perform superior than MLEs (or MPSs) with respect to the smallest MSE values.

Keywords

constant-stress; maximum a posteriori; maximum product of spacing; stochastic EM algorithm; unified hybrid censoring.

1. Introduction

Due to advanced technology, competitive markets, and consumer demand, most products are highly reliable as these products may work properly for years or even decades. To obtain enough information about the reliability of these products, they should be exposed to higher stresses than normal conditions. Accelerated life test (ALT) is design to overcome the situation. The ALT is more efficient with low cost than the classical reliability testing. However, sometimes the acceleration factor cannot be easily obtained in many situations. To overcome this difficulty, partially accelerated life test (PALT) was proposed. In practice, the constant-stress accelerated life test (CSALT) is one of the most common types in PALT. In CSPALT, the total components are divided into two groups (g1 and g2). Each unit in the two groups is run at constant stress level until it fails. The CSALT model has many applications in various fields. For example, this model comes up in engineering studies such as failure times of electrical insulation, oil breakdown times of insulating fluid and fatigue failure of aircraft structures. Extensive research work has been done on PALT associated with different distributions. Lone and Rahman [20] investigated the estimation of the PALT for competing risk model. Zheng and Fang [28] considered exact confidence intervals for the acceleration factor under CSPALT. Bing and Zhong-zhan [5] used the maximum likelihood method in order to obtain the estimates of the CSPALT parameters with Lomax distribution. Lin et al. [19] addressed the statistical inferences for CSPALT model under log-location-scale lifetime distributions. Lone et al. [21] employed the Bayesian approach to predict CSPALT based on censored data. Maiti and Kayal [22] investigated the estimation of stress-strength parameters using the extended Chen distribution. Yang and Wang [26] considered the characteristics about insulation damage under the ALTs. Yazgan et al. [27] studied the fuzzy reliability model using the weighted exponential distribution and Asadi et al. [3] studied the statistical inferences for CSPALT

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model under Gompertz distribution. Moreover, censoring is a frequent occurrence and strategy in reliability tests and ALT models, either for unavoidable reasons or simply to save experimental time and expense. Type-I and Type-II censoring schemes represent the fundamental censoring schemes ([1,2, 6,10, 13,17, 25]), which allow an experiment to be terminated at a specified time or after a predetermined number of individuals have failed, respectively. The hybrid censoring scheme is a combination of the Type-I and Type-II censored schemes [9]. Chandrasekar et al. [7] improved the hybrid censored schemes of censored sampling by introducing two extensions of this type, named as generalized Type-I hybrid censoring (GIHC) and generalized Type-II hybrid censoring (GIIHC) schemes. Although the GIHC and GIIHC schemes were proposed to avoid the disadvantages of type-I and type-II HCSs, these two have some drawbacks too. In case of the GIHC scheme, the experimenter may not have the m^{th} failure due to the prefixed time. On the other hand, for the GIIHC scheme, there is a possibility of getting effective sample size (here, m) zero or may be very small. To overcome such disadvantages, Type-I and Type-II unified hybrid (UH) censoring scheme were proposed by Balakrishnan et al. [4]. In UH censoring scheme, *let the number of items used in a life testing experiment be n.* In this scheme, $k, r \in \{0,\ldots,n\}$; $k < r < n$ and $T_1 < T_2 \in (0,\infty)$ are decided before hand by the experimenter. If the k^{th} failure occurs before T_1, then the experiment will be stopped at $\min\{\max(X_{r:n}, T_1), T_2\}$; if the k^{th} failure occurs between T_1 and T_2, then the experiment will be terminated at $\min\{X_{r:n}, T_2\}$ and if the k^{th} failure occurs after T_2, then the experiment will be stopped at Y_k (Figure 1). The UH censoring scheme has been studied for many lifetime distributions. For instance, [15] studied Bayesian prediction from the exponentiated Rayleigh distribution. [16] discussed different estimation methods for the half logistic parameter . [18] analyzed the UH censored model with Rayleigh distribution. [23] extensively offered an analysis of Burr type III distribution, then presented the application of the considered model in the fracture toughness data.

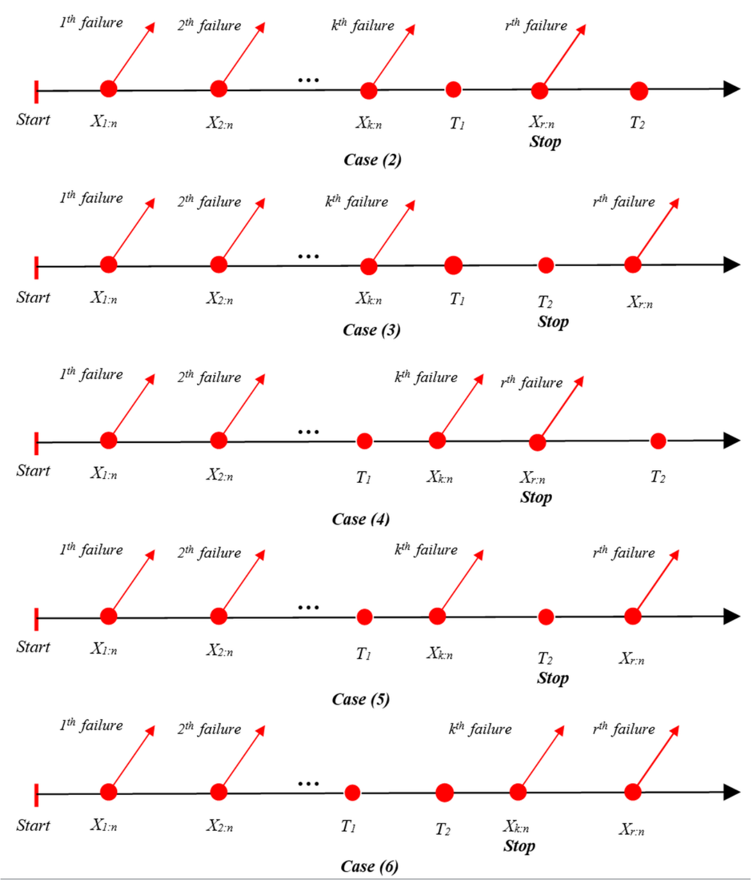

Fig. 1.Diagram of the UHCS

Therefore, under the UH censoring scheme, we have the six cases which are presented in Table 1. To the best of the authors' knowledge, there is not any work related to estimation of CSPALT model based on Gompertz distribution under UH censored sample.

Table 1. The cases in which a test under UH censoring will be completed

Cases	UH censoring	Terminated time	Number of failure units
I	$0 < X_{k:n} < X_{r:n} < T_1 < T_2$	T_1	d_1
II	$0 < X_{k:n} < T_1 < X_{r:n} < T_2$	$X_{r:n}$	r
III	$0 < X_{k:n} < T_1 < T_2 < X_{r:n}$	T_2	d_2
IV	$0 < T_1 < X_{k:n} < X_{r:n} < T_2$	$X_{r:n}$	r
V	$0 < T_1 < X_{k:n} < T_2 < X_{r:n}$	T_2	d_2
VI	$0 < T_1 < T_2 < X_{k:n} < X_{r:n}$	$X_{k:n}$	k

So, our objective in this study is the development of inference techniques for CSPALT model based on Gompertz distribution under UH censored sample. We suppose that the lifetime X_1 of an item tested for g1 at use conditions has the following probability density function (PDF), cumulative distribution function (CDF), survival and hazard function are respectively given by:

$$f_1(x;\alpha,\gamma) = \gamma \exp\left[\alpha x - \frac{\gamma}{\alpha}(e^{\alpha x} - 1)\right]; \quad x > 0,\ \alpha > 0,\ \gamma > 0 \,, \tag{1}$$

$$F_1(x;\alpha,\gamma) = 1 - \exp\left[-\frac{\gamma}{\alpha}(e^{\alpha x} - 1)\right]; \quad x > 0,\ \alpha > 0,\ \gamma > 0 \,, \tag{2}$$

and:

$$h_1(t;\alpha,\gamma) = \gamma e^{\alpha t}; \quad t > 0,\ \alpha > 0,\ \gamma > 0 \,. \tag{3}$$

By using the condition $h_2(t;\alpha,\gamma) = \lambda h_1(t;\alpha,\gamma)$, the PDF, CDF and hazard functions of g2 (accelerated condition) are:

$$f_2(x;\alpha,\gamma,\lambda) = \gamma\lambda \exp\left[\alpha x - \frac{\gamma\lambda}{\alpha}(e^{\alpha x} - 1)\right]; \quad x > 0,\ \alpha > 0,\ \gamma > 0, \lambda > 1 \,, \tag{4}$$

$$F_2(x;\alpha,\gamma,\lambda) = 1 - \exp\left[-\frac{\gamma\lambda}{\alpha}(e^{\alpha x} - 1)\right]; \quad x > 0,\ \alpha > 0,\ \gamma > 0 \tag{5}$$

and:

$$h_2(t;\alpha,\gamma,\lambda) = \gamma\lambda e^{\alpha t}; \quad t > 0,\ \alpha > 0,\ \gamma > 0 \,. \tag{6}$$

Motivated by these above reasons and statements, we consider the CSPALT model under UHCS, and make statistical inference based on classical and Bayesian approaches. We evaluate the CSPALT parameters using maximum likelihood method. It is observed that the MLEs do not exist in closed form due to the complicated structure of the likelihood function. Therefore, it becomes difficult to evaluate accurate estimates, and the process of obtaining MLEs includes heavy computations. It looks like an alternative to apply other methods, such as EM and SEM algorithms. Further, the implementation of the EM algorithm still requires the numerical techniques. So we employ the stochastic expectation-maximization (SEM) algorithm to reduce complexity and simplify computing. We also, considered the maximum product of spacing (MPS) method as another frequentist estimation approach for estimation of the CSPALT parameters. Based on Bayesian viewpoint, we also apply the maximum a posteriori (MAP) method to obtain the unknown parameters. In the sequel, different confidence intervals are also constructed using maximum likelihood estimates. It is organized as follows for the remainder of the article. In Section 2, we study the model and discuss the point estimation via maximum likelihood method. We also prove the existence and uniqueness of the MLEs. The SEM algorithm is proposed in Section 3. Thereafter, the point estimation using MPS method is investigated in Section 4. In Section 5, the MAP estimates for the CSPALT parameters are proposed based on the UHCS. Different confidence interval methods such as approximate, Boot-p and Boot-t confidence intervals are discussed in Section 6. In Section 7, a simulation study is conducted to compare the proposed procedures. Finally, in Section 8, the concluding remarks are added.

2. Maximum Likelihood Estimates for CSPALT Based On UH Censoring

In this Section, we estimate the model parameters and accelerated factor via maximum likelihood method. By combining two groups (g1 and g2), the likelihood function of the parameters α, γ and λ is given by:

$$L(\alpha,\gamma,\lambda|data) \propto \prod_{j=1}^{D_1} \gamma \exp\left(\alpha x_{1j} - \frac{\gamma\lambda}{\alpha}(e^{\alpha x_{1j}}-1)\right) \times \left(\exp(-\frac{\gamma}{\alpha}(e^{\alpha s_1}-1))\right)^{n_1-D_1} \times \prod_{j=1}^{D_2} \gamma\lambda \exp\left(\alpha x_{2j} - \frac{\gamma\lambda}{\alpha}(e^{\alpha x_{2j}}-1)\right) \times \left(\exp(-\frac{\gamma\lambda}{\alpha}(e^{\alpha s_2}-1))\right)^{n_2-D_2}, \tag{7}$$

where:

$$(D_i, s_i) = \begin{cases} (d_{i1}, T_{i1}) & For\ Case\ (1), \\ (r_i, X_{ir_i}) & For\ Case\ (2), \\ (d_{i2}, T_{i2}) & For\ Case\ (3), \\ (r_i, X_{ir_i}) & For\ Case\ (4), \\ (d_{i2}, T_{i2}) & For\ Case\ (5), \\ (k_i, X_{ik_i}) & For\ Case\ (6). \end{cases}$$

Also, $X_{ij}, i=1,2; j=1,...,n_i$ are the units for the items obtained from Gompertz distribution, where $X_{1j}; j=1,...,n_1$ is the unit in normal condition and $X_{2j}; j=1,...,n_2$ is the lifetime in accelerated condition. Thus, the log-likelihood function (LLF) can be given as:

$$\ln L(\alpha,\gamma,\lambda|data) \propto (D_1+D_2)\ln\gamma + D_2\ln\lambda + \alpha\sum_{j=1}^{D_1} x_{1j} + \alpha\sum_{j=1}^{D_2} x_{2j} - \frac{\gamma}{\alpha}\sum_{j=1}^{D_1}(e^{\alpha x_{1j}}-1) - \frac{\gamma\lambda}{\alpha}\sum_{j=1}^{D_2}(e^{\alpha x_{2j}}-1) - (n_1-D_1)\left(\frac{\gamma}{\alpha}(e^{\alpha s_1}-1)\right) - (n_2-D_2)\left(\frac{\gamma\lambda}{\alpha}(e^{\alpha s_2}-1)\right). \tag{8}$$

The MLEs of α, γ and λ can be evaluated by maximizing the function given in (8) with respect to unknown parameters. Taking partial derivatives of Equation (8) with respect to α, γ and λ, we have:

$$\frac{\partial \ln L(\alpha,\gamma,\lambda|data)}{\partial\alpha} = \left(\sum_{j=1}^{D_1} x_{1j} + \sum_{j=1}^{D_2} x_{2j}\right) + \frac{\gamma}{\alpha^2}\left\{\sum_{j=1}^{D_1}(e^{\alpha x_{1j}}-1) + \lambda\sum_{j=1}^{D_2}(e^{\alpha x_{2j}}-1) + (n_1-D_1)(e^{\alpha s_1}-1) + (n_2-D_2)\lambda(e^{\alpha s_2}-1)\right\} - \frac{\gamma}{\alpha}\left\{\sum_{j=1}^{D_1} x_{1j}e^{\alpha x_{1j}} + \lambda\sum_{j=1}^{D_2} x_{2j}e^{\alpha x_{2j}} + (n_1-D_1)(s_1e^{\alpha s_1}) + (n_2-D_2)\lambda(s_2e^{\alpha s_2})\right\} = 0, \tag{9}$$

$$\frac{\partial \ln L(\alpha,\gamma,\lambda|data)}{\partial\gamma} = \frac{D_1+D_2}{\gamma} - \frac{1}{\alpha}\sum_{j=1}^{D_1}(e^{\alpha x_{1j}}-1) - \frac{\lambda}{\alpha}\sum_{j=1}^{D_2}(e^{\alpha x_{2j}}-1) - (n_1-D_1)\left(\frac{1}{\alpha}(e^{\alpha s_1}-1)\right) - (n_2-D_2)\left(\frac{\lambda}{\alpha}(e^{\alpha s_2}-1)\right) = 0, \tag{10}$$

and:

$$\frac{\partial \ln L(\alpha,\gamma,\lambda|data)}{\partial\lambda} = \frac{D_2}{\lambda} - \frac{1}{\alpha}\sum_{j=1}^{D_1}(e^{\alpha x_{1j}}-1) - \frac{\gamma}{\alpha}\sum_{j=1}^{D_2}(e^{\alpha x_{2j}}-1) - (n_2-D_2)\left(\frac{\gamma}{\alpha}(e^{\alpha s_2}-1)\right) = 0. \tag{11}$$

To obtain the MLEs of α, γ and λ, the nonlinear system of equations given by (9), (10) and (11) are to be solved numerically using a nonlinear optimization algorithm. In the following theorem, it is shown that the MLEs of the parameters α, γ and λ exist and also unique.

Theorem 3.1. The MLEs of the parameters α, γ and λ for $(\alpha,\gamma,\lambda) \in (0,\infty)\times(0,\infty)\times(1,\infty)$ exist and also unique.

Proof: *See Appendix A.*

For more discussion about the existence and uniqueness of maximum likelihood estimates, we propose to consider the contour and 3D plot (Figure 2) using the following steps:

1. Generate a sample size n_1 from the Gompertz distribution based on the group 1 (X_1).
2. Consider the unified hybrid censoring scheme.
3. Estimate parameters α and γ using the maximum likelihood method.
4. Use the "*rsm*" package in *R* software to present the contour and 3D profile plot. The Figure indicates that the MLEs of the α and γ are exist and are also unique.

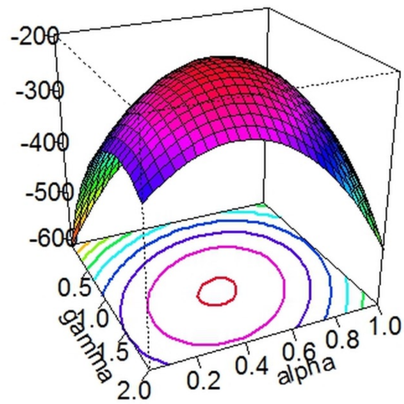

Fig. 2. The contour and 3D profile plot for log-likelihood function

We also show the existence and uniqueness of the MLEs using the following Steps:

1. Generate a sample size (n=50) from the Gopertz distribution.
2. Consider the censoring scheme, r=40, k=15, T_1=0.6, T_2=1.2.

3. Plot $\partial L(\alpha,\beta|data)/\partial\alpha=0$ and $\partial L(\alpha,\beta|data)/\partial\gamma=0$ in Figure 3.
4. Plot the profile loglikelihood function $l(\alpha,\hat{\gamma}|data)$ and $l(\hat{\alpha},\gamma|data)$ in Figures 4 and 5 respectively.

From Figure 3, we can observe that there exists one intersection point (1.3, 0.6). Also based on Figures 4 and 5, we observed that the intersection point maximize the LLF of α and γ . Consequently, we conclude that the maximum likelihood estimates of the α and γ exist and also unique.

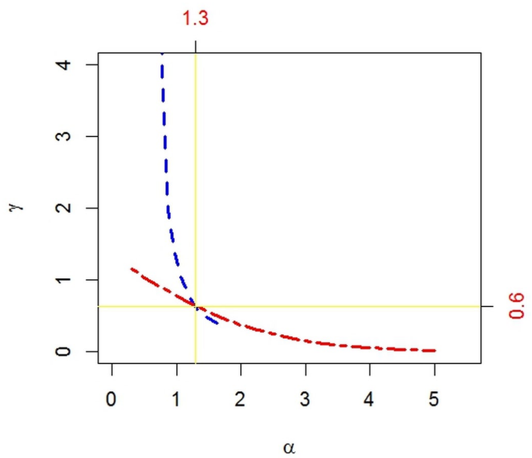

Fig. 3. The plot of the MLE's of the parameters

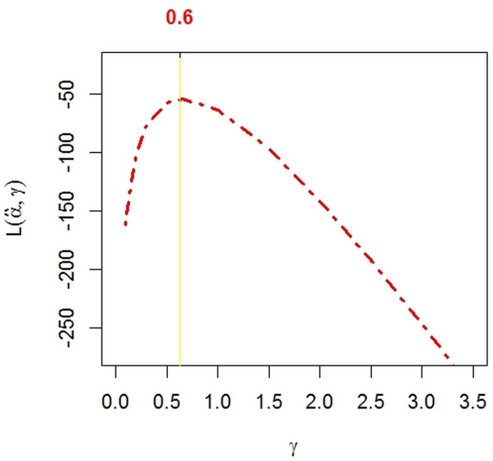

Fig. 4. The plot of $L(\hat{\alpha},\gamma)$

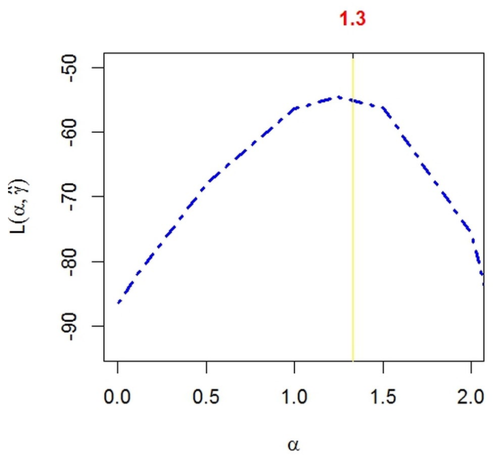

Fig. 3. The plot of $L(\alpha,\hat{\gamma})$

3. Stochastic EM Algorithm

In this Section, we consider the SEM algorithm by using the data obtained from UH censoring scheme as a missing data (see, Panahi and Asadi [24]). The missing values in this case are the lifetimes of the censored units. Denote the unobserved censored data by $Z_{i1}, Z_{i2},, Z_{n_1-D_i}; i=1,2$. Combining the observed and missing data, we obtain the complete data. Based on the complete data (W), the LLF of the complete sample is taken as follows:

$$\ln L_W = \log(L_{W_1}.L_{W_2}) = Constant + n\ln\gamma + n_2\ln\lambda + \alpha\Big(\sum_{j=1}^{D_1} x_{1j} + \sum_{j=1}^{n_1-D_1} z_{1j}\Big) + \alpha\Big(\sum_{j=1}^{D_2} x_{2j} + \sum_{j=1}^{n_2-D_2} z_{2j}\Big)$$
$$-\frac{\gamma}{\alpha}\Big(\sum_{j=1}^{D_1}(e^{\alpha x_{1j}}-1) + \sum_{j=1}^{n_1-D_1}(e^{\alpha z_{1j}}-1)\Big) - \frac{\gamma\lambda}{\alpha}\Big(\sum_{j=1}^{D_2}(e^{\alpha x_{2j}}-1) + \sum_{j=1}^{n_2-D_2}(e^{\alpha z_{2j}}-1)\Big). \quad (12)$$

At the E-step of the EM algorithm, we should evaluate these conditional expectations $E(Z_{1j}|Z_{1j}>s_1)$, $E(Z_{2j}|Z_{2j}>s_2)$, $E(e^{\alpha Z_{1j}}-1|Z_{1j}>s_1)$ and $E(e^{\alpha Z_{2j}}-1|Z_{2j}>s_2)$ which are intractable and complex. Based on the SEM algorithm, the E-step of EM algorithm has been replaced by a stochastic step and it can be executed by simulation. Thus, we apply SEM algorithm to approximate proposed conditional expectations by using the data obtained from UH censoring as a missing data problem. The details of the SEM steps are:

I Generate the missing samples $Z_{ij};\ i=1,2$ whose conditional distribution function is given by:

$$\zeta_1(z_{1j}|z_{1j}>s_1) = \frac{F(z_{1j})-F(s_1)}{1-F(s_1)};\ z_{1j}>s_1, \zeta_2(z_{2j}|z_{2j}>s_2) = \frac{F(z_{2j})-F(s_2)}{1-F(s_2)};\ z_{2j}>s_2$$

Here, the conditional expectations can be approximated as:

$$E(Z_{1j}|Z_{1j}>s_1) \sim Z_{1j},\quad E(Z_{2j}|Z_{2j}>s_2) \sim Z_{2j},$$
$$E(e^{\alpha Z_{1j}}-1|Z_{1j}>s_1) \sim e^{\alpha Z_{1j}}-1,\ E(e^{\alpha Z_{2j}}-1|Z_{2j}>s_2) \sim e^{\alpha Z_{2j}}-1.$$

II Assume that at the t^{th} stage, estimates of generate Z_{1j} and Z_{2j} through the condition density functions $\zeta_1(z_{1j}|z_{1j}>s_1)$ and $\zeta_2(z_{2j}|z_{2j}>s_2)$ respectively.

III In t^{th} iteration, obtain the estimation of α, γ and λ as $\hat{\alpha}^k$, $\hat{\gamma}^k$ and $\hat{\lambda}^k$.

IV Repeat steps I-III, t times.

V The SEM estimates of α, γ and λ can be evaluated as:

$$\hat{\alpha}_{SEM} = \frac{\sum_{i=B}^{t}\hat{\alpha}^i}{t-B},\quad \hat{\gamma}_{SEM} = \frac{\sum_{i=B}^{t}\hat{\gamma}^i}{t-B} \text{ and } \hat{\lambda}_{SEM} = \frac{\sum_{i=B}^{t}\hat{\lambda}^i}{t-B}.$$

where B is burn-in period.

4. Maximum Product of Spacing

The method of the MPS was introduced by Cheng and Amin [8] as an alternative to the method of maximum likelihood. The MPS method performs better than the MLEs in the case of small samples for heavy tailed or skewed distributions. The MPS estimates are evaluated by selecting the parameter values that maximize the product of the distances between the values of the distribution function at adjacent ordered points. Based on UH censored sample and Gompertz distribution, the maximum product function of PALT is given by:

$$M(\alpha,\gamma,\lambda) = A\prod_{i=1}^{D_1+1}\left[F_1(x_i;\alpha,\gamma)-F_1(x_{i-1};\alpha,\gamma)\right]\prod_{i=1}^{D_2+1}\left[F_2(x_i;\alpha,\gamma,\lambda)-F_2(x_{i-1};\alpha,\gamma,\lambda)\right]$$
$$\times\left[1-F_1(s_1;\alpha,\gamma)\right]^{n_1-D_1}\times\left[1-F_2(s_2;\alpha,\gamma,\lambda)\right]^{n_2-D_2}. \tag{13}$$

Using Equations (1), (2), (4) and (5), the equation (13) can be written as:

$$M(\alpha,\gamma,\lambda) = \prod_{i=1}^{D_1+1}\left[\exp(-\frac{\gamma}{\alpha}(e^{\alpha x_{i-1}}-1))-\exp(-\frac{\gamma}{\alpha}(e^{\alpha x_i}-1))\right]\times\left[\exp(-\frac{\gamma}{\alpha}(e^{\alpha s_1}-1))\right]^{n_1-D_1}$$
$$\times\prod_{i=1}^{D_2+1}\left[\exp(-\frac{\gamma\lambda}{\alpha}(e^{\alpha x_{i-1}}-1))-\exp(-\frac{\gamma\lambda}{\alpha}(e^{\alpha x_i}-1))\right]\times\left[\exp(-\frac{\gamma\lambda}{\alpha}(e^{\alpha s_2}-1))\right]^{n_2-D_2}. \tag{14}$$

the natural logarithm of Equation (14) is given by:

$$\ln M(\alpha,\gamma,\lambda) = \sum_{i=1}^{D_1+1}\ln\left[\exp(-\frac{\gamma}{\alpha}(e^{\alpha x_{i-1}}-1))-\exp(-\frac{\gamma}{\alpha}(e^{\alpha x_i}-1))\right]-(n_1-D_1)\frac{\gamma}{\alpha}(e^{\alpha s_1}-1)$$
$$+\sum_{i=1}^{D_2+1}\ln\left[\exp(-\frac{\gamma\lambda}{\alpha}(e^{\alpha x_{i-1}}-1))-\exp(-\frac{\gamma\lambda}{\alpha}(e^{\alpha x_i}-1))\right]-(n_2-D_2)\frac{\gamma\lambda}{\alpha}(e^{\alpha s_2}-1). \tag{15}$$

The MPS estimators of α, γ and λ can be computed by differentiating (15) with respect to α, γ and λ and equating them to zero. We consider the numerical method to get the MPS estimates of α, γ and λ .

5. Maximum a Posteriori (MAP) Estimation

In contrast to traditional frequentist methods, the Bayesian approaches take advantage of available data information and incorporate prior information of parameters, thereby attracting much attention in statistical inference. The MAP method can be applied to obtain point estimates with a Bayesian flavor. The MAP estimate indicates the mode of the posterior distribution. In this Section, we explore the MAP estimates of the unknown parameters of CSPALT under UHCS. The MAP estimates are much faster to evaluate than Bayesian estimates, as they do not require the estimation of integrals [11,14]. The MAP estimator can be written as:

$$\begin{aligned}\hat{\Theta}_{MAP} &= \arg\max_{\Theta}\left(\pi(\Theta|data)\right)\\ &= \arg\max_{\Theta}\left(\log\pi(data|\Theta)+\log\pi(\Theta)\right).\end{aligned} \tag{16}$$

Where, $\Theta=(\alpha,\gamma,\lambda)$ and also, $\pi(\Theta|data)$, $\pi(data|\Theta)$ and $\pi(\Theta)$ are the posterior distribution, joint distribution of the data and the joint prior distribution respectively. Based on Gamma priors for $\alpha(\alpha\sim Gamma(a,b))$ and $\gamma(\gamma\sim Gamma(c,d))$ and Jeffery prior density function for $\lambda(\lambda\propto 1/\lambda)$. The MAP estimator is given by:

$$\hat{\Theta}_{MAP} = \arg\max_{\Theta}\left(\pi(\Theta|data)\right)$$
$$=\arg\max_{\Theta}\left((D_1+D_2+c-1)\log\gamma+(D_2-1)\log\lambda+(a-1)\log\alpha-\alpha(b-\sum_{j=1}^{D_1}x_{1j}-\sum_{j=1}^{D_2}x_{2j})\right.$$
$$\left.-\gamma(d+\frac{1}{\alpha}\sum_{j=1}^{D_1}(e^{\alpha x_{1j}}-1)+\frac{(n_1-D_1)(e^{\alpha s_1}-1)}{\alpha})-\lambda(\frac{\gamma}{\alpha}\sum_{j=1}^{D_2}(e^{\alpha x_{2j}}-1)+\frac{\gamma(n_2-D_2)(e^{\alpha s_2}-1)}{\alpha}\right). \tag{17}$$

To obtain $\hat{\alpha}_{MAP}$, $\hat{\gamma}_{MAP}$ and $\hat{\lambda}_{MAP}$, differentiate (17) with respect to α, γ and λ respectively and then equating to zero, we have:

$$\frac{a-1}{\alpha}+b-\sum_{j=1}^{D_1}x_{1j}-\sum_{j=1}^{D_2}x_{2j}$$
$$+\frac{\gamma}{\alpha^2}\left\{\sum_{j=1}^{D_1}(e^{\alpha x_{1j}}-1)+\lambda\sum_{j=1}^{D_2}(e^{\alpha x_{2j}}-1)+(n_1-D_1)(e^{\alpha s_1}-1)+(n_2-D_2)\lambda(e^{\alpha s_2}-1)\right\}$$
$$-\frac{\gamma}{\alpha}\left\{\sum_{j=1}^{D_1}x_{1j}e^{\alpha x_{1j}}+\lambda\sum_{j=1}^{D_2}x_{2j}e^{\alpha x_{2j}}+(n_1-D_1)(s_1e^{\alpha s_1})+(n_2-D_2)\lambda(s_2e^{\alpha s_2})\right\}=0, \tag{18}$$

$$\frac{D_1+D_2+d-1}{\gamma}-c-\frac{1}{\alpha}\sum_{j=1}^{D_1}(e^{\alpha x_{1j}}-1)-\frac{\lambda}{\alpha}\sum_{j=1}^{D_2}(e^{\alpha x_{2j}}-1)-\frac{(n_1-D_1)(e^{\alpha s_1}-1)}{\alpha}-\frac{\lambda(n_2-D_2)(e^{\alpha s_2}-1)}{\alpha}=0, \tag{19}$$

and

$$\frac{D_2-1}{\lambda}-\frac{1}{\alpha}\sum_{j=1}^{D_1}(e^{\alpha x_{1j}}-1)-\frac{\gamma}{\alpha}\sum_{j=1}^{D_2}(e^{\alpha x_{2j}}-1)-(n_2-D_2)\left(\frac{\gamma}{\alpha}(e^{\alpha s_2}-1)\right)=0. \tag{20}$$

It is observed that Equations (18)-(20), can not be solved explicitly. So, numerical computations like Newton-Raphson algorithm are used to evaluate $\hat{\alpha}_{MAP}$, $\hat{\gamma}_{MAP}$ and $\hat{\lambda}_{MAP}$. Moreover, we can use the "*nleqslv*" package in *R* software for the computation of MAP estimates of the parameters.

6. Confidence Intervals (CIs)

6.1. Asymptotic CI

Here, we discussed the construction of asymptotic confidence intervals using the concept of observed Fisher information matrix (FIM). The FIM of $\theta=(\alpha,\beta,\gamma)$ is given by:

$$\hat{I}(\hat{\theta}) = -\left(\frac{\partial^2\hat{l}(\alpha,\beta,\gamma)}{\partial\theta_i\partial\theta_j}\right)_{i,j=1,2} = \left[I_{ij}\right]_{2\times 2}.$$

The elements o f the Fisher information matrix are:

$$\frac{\partial^2\ln L(\alpha,\gamma,\lambda|data)}{\partial\alpha^2}=$$
$$-\frac{2\gamma}{\alpha^3}\left\{\sum_{j=1}^{D_1}(e^{\alpha x_{1j}}-1)+\lambda\sum_{j=1}^{D_2}(e^{\alpha x_{2j}}-1)+(n_1-D_1)(\frac{\gamma}{\alpha^2}(e^{\alpha s_1}-1))+(n_2-D_2)(\frac{\gamma\lambda}{\alpha^2}(e^{\alpha s_2}-1))\right\}$$
$$+\frac{\gamma}{\alpha^2}\left\{\sum_{j=1}^{D_1}x_{1j}e^{\alpha x_{1j}}+\lambda\sum_{j=1}^{D_2}x_{2j}e^{\alpha x_{2j}}-(n_1-D_1)(-\frac{\gamma}{\alpha}(s_1e^{\alpha s_1}))-(n_2-D_2)(-\frac{\gamma\lambda}{\alpha}(s_2e^{\alpha s_2}))\right\}$$
$$-\frac{\gamma}{\alpha}\left\{\sum_{j=1}^{D_1}x_{1j}^2e^{\alpha x_{1j}}+\lambda\sum_{j=1}^{D_2}x_{2j}^2e^{\alpha x_{2j}}-(n_1-D_1)\left((\frac{\gamma}{\alpha^2}(s_1e^{\alpha s_1}))-\frac{\gamma}{\alpha}(s_1^2e^{\alpha s_1})\right)\right.$$
$$\left.-(n_2-D_2)\left((\frac{\gamma\lambda}{\alpha^2}(s_2e^{\alpha s_2}))-\frac{\gamma\lambda}{\alpha}(s_2^2e^{\alpha s_2})\right)\right\}$$
$$+\frac{\gamma}{\alpha^2}\left\{\sum_{j=1}^{D_1}x_{1j}e^{\alpha x_{1j}}+\lambda\sum_{j=1}^{D_2}x_{2j}e^{\alpha x_{2j}}+(n_1-D_1)\left((-\frac{2\gamma}{\alpha^3}(e^{\alpha s_1}-1))+\frac{\gamma s_1}{\alpha^2}e^{\alpha s_1}\right)\right.$$
$$\left.+(n_2-D_2)\left((-\frac{2\gamma\lambda}{\alpha^3}(e^{\alpha s_2}-1))+\frac{\gamma\lambda s_2}{\alpha^2}e^{\alpha s_2}\right)\right\}$$

$$\frac{\partial^2\ln L(\alpha,\gamma,\lambda|data)}{\partial\gamma^2}=-\frac{(D_1+D_2)}{\gamma^2},$$

$$\frac{\partial^2\ln L(\alpha,\gamma,\lambda|data)}{\partial\gamma\partial\alpha}=\frac{\partial^2\ln L(\alpha,\gamma,\lambda|data)}{\partial\alpha\partial\gamma}=\frac{1}{\alpha^2}\sum_{j=1}^{D_1}(e^{\alpha x_{1j}}-1)-\frac{1}{\alpha}\sum_{j=1}^{D_1}x_{1j}e^{\alpha x_{1j}}+\frac{\lambda}{\alpha^2}\sum_{j=1}^{D_2}(e^{\alpha x_{2j}}-1)-\frac{\lambda}{\alpha}\sum_{j=1}^{D_2}x_{2j}e^{\alpha x_{2j}}$$
$$-(n_1-D_1)\left(-\frac{1}{\alpha}(e^{\alpha s_1}-1)+\frac{1}{\alpha}(s_1e^{\alpha s_1})\right)-(n_2-D_2)\left(-\frac{1}{\alpha}(e^{\alpha s_2}-1)+\frac{1}{\alpha}(s_2e^{\alpha s_2})\right),$$

$$\frac{\partial^2\ln L(\alpha,\gamma,\lambda|data)}{\partial\lambda^2}=-\frac{D_2}{\lambda^2},$$

$$\frac{\partial^2 \ln L(\alpha,\gamma,\lambda|data)}{\partial\gamma\partial\lambda} = \frac{\partial^2 \ln L(\alpha,\gamma,\lambda|data)}{\partial\lambda\partial\gamma} = -\frac{1}{\alpha}\sum_{j=1}^{D_2}(e^{\alpha x_{2j}} - 1) - (n_2 - D_2)\left(\frac{1}{\alpha}(e^{\alpha s_2} - 1)\right),$$

$$\frac{\partial^2 \ln L(\alpha,\gamma,\lambda|data)}{\partial\lambda\partial\alpha} = \frac{\partial^2 \ln L(\alpha,\gamma,\lambda|data)}{\partial\alpha\partial\lambda} = \frac{1}{\alpha^2}\sum_{j=1}^{D_1}(e^{\alpha x_{1j}} - 1) - \frac{1}{\alpha}\sum_{j=1}^{D_1} x_{1j} e^{\alpha x_{1j}} + \frac{\gamma}{\alpha^2}\sum_{j=1}^{D_2}(e^{\alpha x_{2j}} - 1) - \frac{\gamma}{\alpha}\sum_{j=1}^{D_2} x_{2j} e^{\alpha x_{2j}}$$
$$-(n_2 - D_2)\left(-\frac{\gamma}{\alpha^2}(e^{\alpha s_2} - 1) + \frac{\gamma}{\alpha}(s_2 e^{\alpha s_2})\right),$$

Since maximum likelihood estimator has asymptotic normality property under certain regularity conditions, the estimator $\theta = (\alpha,\gamma,\lambda)$ has asymptotic distribution $\hat{\theta} - \theta \to N(0, I^{-1}(\theta))$. Also, $I^{-1}(\theta)$ is the variance-covariance matrix. Therefore, the $100(1-\tau)\%$ asymptotic confidence interval (ACI) of $\theta = (\alpha,\gamma,\lambda)$ is constructed as:

$$\left(\hat{\theta} - z_{\tau/2}\sqrt{var(\hat{\alpha})}\,, \hat{\theta} + z_{\tau/2}\sqrt{var(\hat{\alpha})}\right), \tag{21}$$

where, $z_{\tau/2}$ is the upper $(\tau/2)^{th}$ quantile of $N(0,1)$. Also, the coverage probability (CP) of $\theta = (\alpha,\gamma,\lambda)$ is given by:

$$CP = \left[\left|\frac{\hat{\theta} - \theta}{\hat{\text{v}}\text{ar}(\hat{\theta})}\right| \le z_{\tau/2}\right].$$

6.2. Bootstrap CIs

In this Section, we use two bootstrap methods [12, 26], which is simpler than ACI method.

6.2.1. Parametric Bootstrap-p CI

1) Based on the UH censored sample, compute the MLE $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\gamma}$ of α, λ and γ respectively.
2) Generate random samples from two independent Gompertz distributions of sizes n_1 and n_2, respectively. Then, generate a bootstrap unified hybrid censored sample.
3) Compute bootstrap estimates of $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{\gamma}$ say, $\hat{\alpha}^*$, $\hat{\lambda}^*$ and $\hat{\gamma}^*$.
4) Repeat Steps 2-3 B times and obtain B bootstrap samples.
5) Arrange all $\hat{\alpha}^*$, $\hat{\lambda}^*$ and $\hat{\gamma}^*$ in ascending order and denote $\hat{\xi}_k^{*[1]}, \hat{\xi}_k^{*[2]}, ..., \hat{\xi}_k^{*[B]}; k = 1,2,3$, where, $\xi_1^* = \alpha^*$, $\xi_2^* = \beta^*$ and $\xi_3^* = \gamma^*$.
6) Then, the $100(1-\tau)\%$ Boot-p confidence interval for α, λ and γ are given by:

$$\left(\hat{\alpha}^{*[B_{\tau/2}]}, \hat{\alpha}^{*[B_{(1-\tau/2)}]}\right),\ \left(\hat{\lambda}^{*[B_{\tau/2}]}, \hat{\lambda}^{*[B_{(1-\tau/2)}]}\right) \text{ and } \left(\hat{\gamma}^{*[B_{\tau/2}]}, \hat{\gamma}^{*[B_{(1-\tau/2)}]}\right). \tag{22}$$

respectively.

6.2.2. Parametric Bootstrap-t CI

1) Repeat the Steps 1 to 3 of the Boot-p method.
2) Compute the t-statistic for parameters as:
3) $T_1^* = \dfrac{\hat{\alpha}^* - \hat{\alpha}}{\sqrt{Var(\hat{\alpha}^*)}}$, $T_2^* = \dfrac{\hat{\lambda}^* - \hat{\lambda}}{\sqrt{Var(\hat{\lambda}^*)}}$ and $T_3^* = \dfrac{\hat{\gamma}^* - \hat{\gamma}}{\sqrt{Var(\hat{\gamma}^*)}}$.
4) Repeat Steps 2-4 B times and obtain $T_k^{*(1)}, T_k^{*(2)}, ..., T_k^{*(B)}; k = 1,2,3$.
5) Arrange $T_k^{*(1)}, T_k^{*(2)}, ..., T_k^{*(B)}; k = 1,2,3$ in ascending order and denote $T_k^{*[1]}, T_k^{*[2]}, ..., T_k^{*[B]};\ k = 1,2,3$.
6) Then, the $100(1-\tau)\%$ Boot-t confidence interval for α, λ and γ are given by:

$$\left(\hat{\alpha} - T_1^{*[B_{(1-\tau/2)}]}\sqrt{Var(\hat{\alpha})}, \hat{\alpha} - T_1^{*[B_{(\tau/2)}]}\sqrt{Var(\hat{\alpha})}\right),$$
$$\left(\hat{\lambda} - T_1^{*[B_{(1-\tau/2)}]}\sqrt{Var(\hat{\lambda})}, \hat{\lambda} - T_1^{*[B_{(\tau/2)}]}\sqrt{Var(\hat{\lambda})}\right),$$
$$\left(\hat{\gamma} - T_1^{*[B_{(1-\tau/2)}]}\sqrt{Var(\hat{\gamma})}, \hat{\gamma} - T_1^{*[B_{(\tau/2)}]}\sqrt{Var(\hat{\gamma})}\right). \tag{23}$$

respectively.

7. Simulation Studies

Since all methods mentioned above cannot be compared theoretically, numerical simulation studies are carried out to evaluate their performance. We compare the performance of ML, MPS and MAP estimators in terms of the mean square errors (MSEs) under different UH censoring scheme (different values of r, k, T_1, T_2). We consider various combinations of (n,r,k) as $(40,20,10), (40,20,16), (40,30,10), (40,30,16)$. The predetermined termination times (T_1,T_2) are also taken as $(0.2,0.5), (0.2,1.0)$, $(0.7,0.5)$. To run the experiment according to a UH censored sampling from the CSPALT model, we propose the following algorithm:

Step 1: Set the parameter values of α, λ and γ.

Step 2: Carried out a unified hybrid censored sample by choosing the values of n_1, n_2, r_1, r_2, k_1, k_2, T_{11}, T_{12}, T_{21} and T_{22}.

Step 3: Generated unified hybrid censored samples from $Gompertz(\alpha,\gamma)$ and $Gompertz(\alpha,\gamma\lambda)$ by inverting the CDFs in (2) and (5) respectively. Under unified hybrid censoring scheme, the sample data will consists of one of the following cases such as:

- If $0 < X_{ik_i:n_i} < X_{ir_i:n_i} < T_{i1} < T_{i2};\ i = 1,2,$ the experiment stops at T_{i1} and the number of failure units is d_{i1}, that is Case I.
- If $0 < X_{ik_i:n_i} < T_{i1} < X_{ir_i:n_i} < T_{i2};\ i = 1,2,$ the experiment stops at $X_{ir_i:n_i}$ and the number of failure units is r_i, that is Case II.
- If $0 < X_{ik_i:n_i} < T_{i1} < T_{i2} < X_{ir_i:n_i};\ i = 1,2,$ the experiment stops at T_{i2} and the number of failure units is d_{i2}, that is Case III.
- If $0 < T_{i1} < X_{ik_i:n_i} < X_{ir_i:n_i} < T_{i2};\ i = 1,2,$ the experiment stops at $X_{ir_i:n_i}$ and the number of failure units is r_i, that is Case IV.
- If $0 < T_{i1} < X_{ik_i:n_i} < T_{i2} < X_{ir_i:n_i};\ i = 1,2,$ the experiment stops at T_{i2} and the number of failure units is d_{i2}, that is Case V.
- If $0 < T_{i1} < T_{i2} < X_{ik_i:n_i} < X_{ir_i:n_i};\ i = 1,2,$ the experiment stops at $X_{ik_i:n_i}$ and the number of failure units is k_i, that is Case VI.
- For each sample, the MLEs of CSPALT model parameters are obtained using the stochastic EM algorithm.
- Different confidence intervals are constructed, through asymptotic properties of the MLEs and the Bootstrap methods. We compared the interval estimates using the interval lengths and the coverage probabilities (CP).
- The maximum product of spacing (MPS) and maximum a posteriori (MAP) estimates of the model parameters and accelerated factor are also computed.
- The above steps were repeated 10^4 times for different sample sizes and different UH censoring schemes. The performance of various point and interval estimates are obtained in terms of the following criteria quantities:

- Mean square error (MSE) of the point estimate $\hat{\theta} = (\hat{\alpha}, \hat{\gamma}, \hat{\lambda})$, which is calculated by $\frac{1}{N}\sum(\hat{\theta}-\theta)^2$ where N is the number of replications.

A. Average length (AL) of $100(1-\tau)\%$ approximate, Bootstrap-p and Bootstrap-p confidence intervals of $\theta = (\alpha, \gamma, \lambda)$.

B. Coverage probability (CP) of $100(1-\tau)\%$ confidence intervals of $\theta = (\alpha, \gamma, \lambda)$, which is considered as the probability that the estimated confidence interval contains the true parameters. Moreover, the significance level is considered as $\tau = 0.05$.

Moreover, for the SEM algorithm, we set the length of the sequence as 10000 with burn-in period as B= 1000, and the estimates are obtained based on averaging the 9000 iterations. Based on MAP estimations, we consider the informative prior for α and γ which they are obtained with equating the mean and the variance of $\alpha^{(j)}$ and $\gamma^{(j)}$; $j = 1,..,N$ to the mean and variance of the corresponding gamma density priors, respectively, as:

$$\frac{1}{N}\sum_{j=1}^{N}\hat{\alpha}^{(j)} = \frac{a}{b} \quad \text{and} \quad \frac{1}{N-1}\sum_{j=1}^{N}\left(\hat{\alpha}^{(j)} - \frac{1}{N}\sum_{j=1}^{N}\hat{\alpha}^{(j)}\right)^2 = \frac{a}{b^2},$$

and:

$$\frac{1}{N}\sum_{j=1}^{N}\hat{\gamma}^{(j)} = \frac{c}{d} \quad \text{and} \quad \frac{1}{N-1}\sum_{j=1}^{N}\left(\hat{\gamma}^{(j)} - \frac{1}{N}\sum_{j=1}^{N}\hat{\gamma}^{(j)}\right)^2 = \frac{c}{d^2}.$$

By solving the above equations, we have:

$$\hat{a} = \frac{(N^{-1}\sum_{j=1}^{N}\hat{\alpha}^{(j)})^2}{\frac{1}{N-1}\sum_{j=1}^{N}\left(\hat{\alpha}^{(j)} - 1/N\sum_{j=1}^{N}\hat{\alpha}^{(j)}\right)^2} \quad \text{and} \quad b = \frac{N^{-1}\sum_{j=1}^{N}\hat{\alpha}^{(j)}}{\frac{1}{N-1}\sum_{j=1}^{N}\left(\hat{\alpha}^{(j)} - 1/N\sum_{j=1}^{N}\hat{\alpha}^{(j)}\right)^2},$$

and:

$$\hat{c} = \frac{(N^{-1}\sum_{j=1}^{N}\hat{\gamma}^{(j)})^2}{\frac{1}{N-1}\sum_{j=1}^{N}\left(\hat{\gamma}^{(j)} - 1/N\sum_{j=1}^{N}\hat{\gamma}^{(j)}\right)^2} \quad \text{and} \quad d = \frac{N^{-1}\sum_{j=1}^{N}\hat{\gamma}^{(j)}}{\frac{1}{N-1}\sum_{j=1}^{N}\left(\hat{\gamma}^{(j)} - 1/N\sum_{j=1}^{N}\hat{\gamma}^{(j)}\right)^2}.$$

From the simulation results in Tables 2- 7, one could consider the following conclusions:

- For fixed , and , the values of MSE's of the MLEs, MPSs and MAPs increase, when the values of decrease.
- When increases, the MSE's of the MLEs, MPSs and MAPs decrease, by keeping , and fixed.
- For fixed , and , the MSE's of the MLEs, MPSs and MAPs increase, when the values ofdecreases.
- The increasing on , for other fixed values, decrease the values of MSE's of the MLEs, MPSs and MAPs.
- For all the censoring schemes, it is observed that the MAP estimates perform better than classical methods (MLEs and MPSs) for and
- For fixed , and , the approximate/Bootstrap interval lengths increase, when the value ofgets to be decreased.
- For fixed , and , the approximate/Bootstrap interval lengths decrease, when the values ofincreases.
- For fixed , and , the approximate/Bootstrap interval lengths increase, when the value ofgets to be decreased.
- For fixed , and , the approximate/Bootstrap interval lengths decrease, when the values ofincreases.
- In most cases, the coverage probabilities (CPs) of all confidence intervals approach to the desired level of 0.95.
- The approximate interval lengths are shorter than the Bootstrap interval lengths in most censoring schemes. Moreover, based on CP values, the approximate confidence intervals can be used as a better choice than other intervals.
- As for the Bootstrap method, there is no remarkable difference between Bootstrap-p and Bootstrap-t approaches.
- All obtained results can be considered to other censoring schemes. For example the results can be specialized to Type I, Type II, Type I hybrid, Type II hybrid, generalized Type I hybrid and generalized Type II hybrid censoring schemes.

8. Conclusions

The accelerated life test is one of the important methods in the research of applied sciences. In this paper, statistical inferences for the parameters of the CSPALT model have been developed under UHCS. The main reason for considering this censoring scheme is that it provides at least a specific number of failures. Moreover, the UH censoring scheme contains other censoring schemes, such as, Type I, Type II, hybrid and generalized hybrid censoring schemes. Using the SEM algorithm and asymptotic normality properties, the MLEs and the approximate confidence intervals of the model parameters have been evaluated. The existence and uniqueness of the MLEs have been proved. The maximum product of spacing as another classical estimation method is proposed. The maximum a posteriori estimations for model parameters and accelerated factor are also presented. Moreover, the Bootstrap-p and Bootstrap-t confidence intervals of the unknown parameters are constructed. A Monte Carlo simulation study has been employed to compare the performance of the proposed estimates. We have compared the MLE, MPS and MAP estimators using the MSEs and the approximate, Bootstrap-p and Bootstrap-t confidence intervals via the average interval length and coverage probability. The results indicated that the MAP estimates perform better than the classical estimates. The flexibility of the unified hybrid censoring scheme provides many different options for reliability studies and help to overcome many difficulties in engineering problems. Some extensions of this censoring scheme will be studied for future work, for instance, inference for stress-strength model and the design of censoring schemes. These topics will be reported in the future.

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Appendix A:

Proof of Theorem 3.1

We want to show that *i)* for given α , and λ , the MLE of γ exist and unique. *ii)* for given α , and γ , the MLE of λ exist and unique and *iii)* for given γ , and λ , the MLE of α exist and unique.

Proof: Firstly, taking the derivatives of the log-likelihood function which is given in Equation (8) with respect to α , γ and λ , respectively we have,

$$\hat{\alpha} = \frac{D_1 + D_2}{\frac{D_1 + D_2}{\Upsilon}\left(\sum_{j=1}^{D_1} x_{1j} e^{\alpha x_{1j}} + \lambda \sum_{j=1}^{D_2} x_{2j} e^{\alpha x_{2j}} + (n_1 - D_1)(s_1 e^{\alpha s_1}) + (n_2 - D_2)\lambda(s_2 e^{\alpha s_2})\right) - (\sum_{j=1}^{D_1} x_{1j} + \sum_{j=1}^{D_2} x_{2j})}, \tag{A.1}$$

$$\hat{\gamma} = \frac{(D_1 + D_2)\alpha}{\sum_{j=1}^{D_1} (e^{\alpha x_{1j}} - 1) + \lambda \sum_{j=1}^{D_2} (e^{\alpha x_{2j}} - 1) + (n_1 - D_1)(e^{\alpha s_1} - 1) + (n_2 - D_2)\lambda(e^{\alpha s_2} - 1)}, \tag{A.2}$$

and:

$$\hat{\lambda} = \frac{D_2}{\frac{\gamma}{\alpha} \sum_{j=1}^{D_2} (e^{\alpha x_{2j}} - 1) + (n_2 - D_2)\left(\frac{\gamma}{\alpha}(e^{\alpha s_2} - 1)\right)}, \tag{A.3}$$

i) Using the Equations (8), (A.2), we can write

$$\ln L(\alpha,\gamma,\lambda|data)=(D_1+D_2)\ln\gamma+\Re \overset{\textit{logarithm properties}}{\leq} \frac{\Upsilon\gamma}{\alpha}-(D_1+D_2)+(D_1+D_2)\ln\hat{\gamma}+\Re,$$

where:

$$\Re = D_2\ln\lambda+\alpha\sum_{j=1}^{D_1}x_{1j}+\alpha\sum_{j=1}^{D_2}x_{2j}-\frac{\gamma}{\alpha}\sum_{j=1}^{D_1}(e^{\alpha x_{1j}}-1)$$
$$-\frac{\gamma\lambda}{\alpha}\sum_{j=1}^{D_2}(e^{\alpha x_{2j}}-1)-(n_1-D_1)\left(\frac{\gamma}{\alpha}(e^{\alpha s_1}-1)\right)-(n_2-D_2)\left(\frac{\gamma\lambda}{\alpha}(e^{\alpha s_2}-1)\right)$$

and:

$$\Upsilon=\sum_{j=1}^{D_1}(e^{\alpha x_{1j}}-1)+\lambda\sum_{j=1}^{D_2}(e^{\alpha x_{2j}}-1)+(n_1-D_1)(e^{\alpha s_1}-1)+(n_2-D_2)\lambda(e^{\alpha s_2}-1).$$

Therefore, based on Equation (A.1), we have:

$$\ln L(\alpha,\gamma,\lambda|data)\leq(D_1+D_2)\ln\hat{\gamma}+D_2\ln\lambda+\alpha\sum_{j=1}^{D_1}x_{1j}+\alpha\sum_{j=1}^{D_2}x_{2j}-\frac{\hat{\gamma}}{\alpha}\sum_{j=1}^{D_1}(e^{\alpha x_{1j}}-1)$$
$$-\frac{\hat{\gamma}\lambda}{\alpha}\sum_{j=1}^{D_2}(e^{\alpha x_{2j}}-1)-(n_1-D_1)\left(\frac{\hat{\gamma}}{\alpha}(e^{\alpha s_1}-1)\right)-(n_2-D_2)\left(\frac{\hat{\gamma}\lambda}{\alpha}(e^{\alpha s_2}-1)\right)$$
$$=\log L(\alpha,\hat{\gamma},\lambda|data)$$
$$\Rightarrow \log L(\alpha,\gamma,\lambda|data)\overset{\textit{iff }\gamma=\hat{\gamma}}{=}\log L(\alpha,\hat{\gamma},\lambda|data).$$

Table 2. The MSEs of unknown parameters based on ML method

n	r	k	T_1	T_2	MSEs of the MLE		
					α	γ	λ
40	20	10	0.2	0.5	0.70617	0.55483	1.05627
40	20	16	0.2	0.5	0.62528	0.53076	0.95940
40	30	10	0.2	0.5	0.47937	0.39633	0.80258
40	30	16	0.2	0.5	0.42593	0.38596	0.75437
40	20	10	0.7	0.5	0.66619	0.53361	1.03496
40	20	16	0.7	0.5	0.61114	0.51985	0.87941
40	30	10	0.7	0.5	0.46891	0.34118	0.71352
40	30	16	0.7	0.5	0.39392	0.32383	0.66616
40	20	10	0.2	1.0	0.61072	0.50826	0.89097
40	20	16	0.2	1.0	0.58885	0.47597	0.79541
40	30	10	0.2	1.0	0.44608	0.30975	0.63998
40	30	16	0.2	1.0	0.36482	0.29541	0.61157

ii) For fixed α, and γ, the MLE of λ exist and unique. Moreover, we have:

$$\log L(\alpha,\gamma,\lambda|data)=\log L(\alpha,\gamma,\hat{\lambda}|data)\xrightarrow{\textit{Equality holds iff}}\lambda=\hat{\lambda}.$$

iii) By considering $\hat{\gamma}$ and $\hat{\lambda}$ in Equation (10) and taking the derivative of α and equation it to zero, we obtain Equation (A.1) as:

$$\hat{\alpha}=\frac{D_1+D_2}{\frac{D_1+D_2}{\Upsilon}\left(\sum_{j=1}^{D_1}x_{1j}e^{\alpha x_{1j}}+\lambda\sum_{j=1}^{D_2}x_{2j}e^{\alpha x_{2j}}+(n_1-D_1)(s_1e^{\alpha s_1})+(n_2-D_2)\lambda(s_2e^{\alpha s_2})\right)-(\sum_{j=1}^{D_1}x_{1j}+\sum_{j=1}^{D_2}x_{2j})}.$$

Table 3. The MSEs of unknown parameters based on MPS method

n	r	k	T_1	T_2	MSEs of the MPS		
					α	γ	λ
40	20	10	0.2	0.5	0.59862	0.48073	0.79213
40	20	16	0.2	0.5	0.51914	0.46397	0.70773
40	30	10	0.2	0.5	0.43729	0.26880	0.68442
40	30	16	0.2	0.5	0.36864	0.23993	0.65280
40	20	10	0.7	0.5	0.52582	0.46741	0.72986
40	20	16	0.7	0.5	0.50243	0.45603	0.68633
40	30	10	0.7	0.5	0.42702	0.24514	0.65598
40	30	16	0.7	0.5	0.33747	0.20431	0.62734
40	20	10	0.2	1.0	0.50737	0.45373	0.69692
40	20	16	0.2	1.0	0.48604	0.44793	0.63985
40	30	10	0.2	1.0	0.41549	0.21607	0.59901
40	30	16	0.2	1.0	0.30482	0.16374	0.57346

Table 4. The MSEs of unknown parameters based on MAP estimator

n	r	k	T_1	T_2	MSEs of the MAP		
					α	γ	λ
40	20	10	0.2	0.5	0.39584	0.29943	0.52368
40	20	16	0.2	0.5	0.28940	0.28331	0.50911
40	30	10	0.2	0.5	0.26773	0.23432	0.46805
40	30	16	0.2	0.5	0.24724	0.21421	0.43287
40	20	10	0.7	0.5	0.29094	0.27643	0.50036
40	20	16	0.7	0.5	0.28328	0.25999	0.57995
40	30	10	0.7	0.5	0.25199	0.19854	0.43287
40	30	16	0.7	0.5	0.22909	0.17954	0.42730
40	20	10	0.2	1.0	0.28043	0.27764	0.40543
40	20	16	0.2	1.0	0.25998	0.24632	0.37851
40	30	10	0.2	1.0	0.27002	0.18232	0.36430
40	30	16	0.2	1.0	0.21160	0.12040	0.33429

Table 5. The %95 approximate interval lengths and CPs of unknown parameters

n	r	k	T_1	T_2	%95 Approximate interval lengths		
					α	γ	λ
40	20	10	0.2	0.5	2.67432(92.67)	1.66743(93.25)	2.64067(91.84)
40	20	16	0.2	0.5	2.21342(94.41)	1.32685(94.77)	2.39852(93.89)
40	30	10	0.2	0.5	2.11062(93.75)	1.42182(93.42)	2.49532(93.66)
40	30	16	0.2	0.5	2.078234(96.00)	1.19743(96.08)	2.14764(95.18)
40	20	10	0.7	0.5	2.47832(93.62)	1.54004(94.87)	2.28995(94.62)
40	20	16	0.7	0.5	2.10943(93.65)	1.24808(94.84)	2.22895(94.44)
40	30	10	0.7	0.5	1.98639(94.16)	1.34065(94.28)	2.31650(92.76)
40	30	16	0.7	0.5	1.89964(95.69)	1.09543(94.87)	2.06886(96.32)
40	20	10	0.2	1.0	2.28964(95.09)	1.41063(94.73)	2.18639(95.34)
40	20	16	0.2	1.0	1.88609(94.78)	1.16283(95.59)	1.99755(96.00)
40	30	10	0.2	1.0	1.11073(95.18)	1.29995(95.14)	1.97535(94.94)
40	30	16	0.2	1.0	1.54387(94.83)	1.12025(95.12)	1.72640(95.19)

Table 6. The %95 Bootstrap-p interval lengths and CPs of unknown parameters

n	r	k	T_1	T_2	%95 Bootstrap interval lengths		
					α	γ	λ
40	20	10	0.2	0.5	2.75324(91.22)	1.73284(92.89)	2.74392(91.23)
40	20	16	0.2	0.5	2.28943(94.32)	1.38974(94.72)	2.45673(93.74)
40	30	10	0.2	0.5	2.16732(93.76)	1.49543(93.34)	2.54392(93.32)
40	30	16	0.2	0.5	2.11054(96.17)	1.29996(96.11)	2.28654(95.26)
40	20	10	0.7	0.5	2.59432(93.56)	1.58994(94.73)	2.52004(94.54)
40	20	16	0.7	0.5	2.18432(93.49)	1.29948(95.37)	2.36807(95.75)
40	30	10	0.7	0.5	2.06750(94.86)	1.37402(94.66)	2.338119(93.57)
40	30	16	0.7	0.5	1.93452(95.28)	1.12543(95.11)	2.12339(96.00)
40	20	10	0.2	1.0	2.43206(94.89)	1.49629(95.54)	2.29905(95.43)
40	20	16	0.2	1.0	1.94328(94.57)	1.22546(94.22)	2.20426(96.09)
40	30	10	0.2	1.0	1.45391(95.31)	1.30274(94.81)	2.18932(94.89)
40	30	16	0.2	1.0	1.42988(94.77)	1.15730(94.86)	1.91173(95.21)

Table 7. The %95 Bootstrap-t interval lengths and CPs of unknown parameters

n	r	k	T_1	T_2	%95 Bootstrap interval lengths		
					α	γ	λ
40	20	10	0.2	0.5	2.98547(90.05)	1.85435(92.56)	2.75667(91.18)
40	20	16	0.2	0.5	2.31462(94.19)	1.42043(94.69)	2.47845(93.68)
40	30	10	0.2	0.5	2.19875(92.97)	1.54270(93.17)	2.62564(92.89)
40	30	16	0.2	0.5	2.17643(96.54)	1.22457(96.18)	2.39875(95.53)
40	20	10	0.2	1.0	2.85433(92.72)	1.76548(94.21)	2.71097(93.99)
40	20	16	0.2	1.0	2.27854(93.35)	1.38875(95.66)	2.43527(96.15)
40	30	10	0.2	1.0	2.16578(94.71)	1.48954(94.86)	2.55673(93.69)
40	30	16	0.2	1.0	1.99769(94.68)	1.19753(95.38)	2.35268(94.11)
40	20	10	0.7	0.5	2.65738(95.37)	1.56995(94.23)	2.62348(95.89)
40	20	16	0.7	0.5	2.22314(95.43)	1.29965(94.10)	2.38056(93.96)
40	30	10	0.7	0.5	1.63146(95.56)	1.45357(94.75)	2.37976(95.57)
40	30	16	0.7	0.5	1.49654(95.87)	1.17654(94.89)	1.99895(94.55)