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Degrading systems availability analysis: analytical semi-Markov approach

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Highlights

- Semi-Markov Process (SMP) is used to model the probabilistic behavior of system.
- SMP takes into account the dependencies and interactions between components of the system.
- System model is developed which is solved using SMP to evaluate system availability.
- The analysis results show that the maintenance policy: perfect repair with opportunistic maintenance is more efficient.

Abstract

This paper deals with modeling and analysis of complex mechanical systems that deteriorate with age. As systems age, the questions on their availability and reliability start to surface. The system is believed to suffer from internal stochastic degradation mechanism that is described as a gradual and continuous process of performance deterioration. Therefore, it becomes difficult for maintenance engineer to model such system. Semi-Markov approach is proposed to analyze the degradation of complex mechanical systems. It involves constructing states corresponding to the system functionality status and constructing kernel matrix between the states. The construction of the transition matrix takes the failure rate and repair rate into account. Once the steady-state probability of the embedded Markov chain is computed, one can compute the steady-state solution and finally, the system availability. System models based on perfect repair without opportunistic and with opportunistic maintenance have been developed and the benefits of opportunistic maintenance are quantified in terms of increased system availability. The proposed methodology is demonstrated for a two-stage reciprocating air compressor with intercooler in between, system in series configuration.

Keywords

availability, perfect repair, opportunistic maintenance, Embedded Markov Chain (EMC), Cumulative Density Function (CDF), Semi-Markov Process (SMP).

Notations

A	Transition probability matrix of the system model	O	Operative
A_s	Steady-state system availability	P_i	Steady-state probability of state ‘i’ of SMP model
D	Degraded	s_i	Steady-state probability of state ‘i’ of Embedded Markov Chain
F	Failed	T_i	Mean sojourn time at state ‘i’ of the SMP model
$F_{ij}(t)$	Cumulative density function (CDF) of transition from state ‘i’ to ‘j’.	i, j	Sequential number of states of the SMP model, where, i, j= 1, 2,…, n
$\bar{F}_{ij}(t)$	Complementary CDF of transition from state ‘i’ to ‘j’.	β_{ij}	Shape parameter of Weibull distribution
$K(t)$	Kernel matrix of the semi-Markov model	θ_{ij}	Scale parameter of Weibull distribution
k_{ij}	Transition probability from state ‘i’ to ‘j’	μ_{ij}	Exponential repair transition rate

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1. Introduction

Degradation modeling of mechanical systems has drawn special attention of plant engineers as it is a crucial aspect of the execution of an effective maintenance plan. Maintenance is the system's design feature that enables the success of various maintenance activities including inspection, repair, replacement and diagnosis [28]. Whenever a system breakdown occurs due to sub-system/component failure, it enters the repair phase. With the capability to repair or restore a system which is under breakdown, a failure-repair-failure-repair cycle is introduced. The restoration cycle can be broken down into a variety of subtasks that include supply delay, maintenance delay, diagnosis, replacement or repair, alignment and verification, etc. Therefore, repair work is a highly skilled and dynamic task in which workers come into close contact with the equipment. There are several factors which affect the maintenance procedure like part layout, ergonomic factors, maintenance skill levels, repair crew size, level of repair, and use and clarity of maintenance procedures and diagrams lay down in the manual. Although, the reliability of the components has improved dramatically, humans still remain messy and unpredictable [1]. There are several measures of maintainability but the most popular and the one discussed in this paper is mean time to repair (MTTR). The growing importance of maintenance has developed an increasing trend in designing and implementing optimal maintenance strategies to enhance system reliability, avoid system failures, and reduce system degradation maintenance cost [27].

The main objective of any maintenance regime is to maintain the system functionality to the maximum extent possible with optimum trade-offs between the down times and cost of maintenance, avoiding any catastrophic failures. Opportunistic maintenance works out to be the suitable remedy, which utilizes the opportunity of system shutdown or module dismantle to perform any maintenance required in the immediate future and saves a substantial amount of system downtime. A system of components working in a random environment is subjected to wear and damage over time and may fail unexpectedly. The components are replaced or repaired upon failure, and such unpleasant events of failure are at the same time also considered in practice as opportunities for maintenance on other components. Opportunistic maintenance basically refers to the scheme in which maintenance is carried out at opportunities, either by choice or based on the physical condition of the system. In this study, the focus is made on the situation in which the opportunities for maintenance are generated by the failure epochs of individual components. At each failure epoch, the failed components are correctively repaired and other components that are still operational are also serviced so that all the subsystems are maintained and restored to certain conditions. For example, when corrective repair on some components requires dismantling of the entire system, a corrective repair on these components combined with opportunistic repair on other or neighboring components might be worthwhile.

System performance evaluation remains a core feature in manufacturing industries as there is a need for efficient method of assessing the efficiency of modern production processes. Availability is one of the most important measures of system performance as it is directly related to the financial returns. It has a broader reach than reliability as it takes into account the measurement of maintenance times [16]. Availability gives the probability of the system being available when called to function at random. Complex mechanical systems work under high reliability and safety standards as such systems deteriorate through distinct mode resulting from different physical phenomenon or different failure attribute of individual subsystems/components with the ageing process [24] which is described by means of increasing failure rate. Certain systems may have lower failure rate, but their down time will be long, hence may interrupt the process at a higher rate as compared to the system with short down time having lower failure rate [22]. The downtime cost of such equipment that progressively degrade with the age, is high. So, it would be more effective to consider multi-state degradation and to take appropriate maintenance actions upon failure of the system. Industries that rely on certain key performance measures have keen interest in being able to model complex mechanical systems and track the availability of such systems. System availability estimation is most frequently done through simulation. Software used for assessment of availability of complex mechanical system usually takes extra time to bring out results by virtue of simulation based technique. Therefore, analytical techniques which are quicker than simulation based techniques are utilized in this paper for system availability analysis. Due to mathematical complications, analytical techniques are hardly used for large and complex mechanical systems but still an attempt is being made to model such complex systems for system availability analysis.

Availability studies for degrading systems have been carried out by numerous researchers, but these are mainly based on Markov model using constant failure and repair rates [19], which is unrealistic in actual operating conditions. Markov model is a stochastic model which is used to model randomly changing systems over time. The basic assumption of a Markov Process is that the behavior of a system in each state is memory less which illustrates that that the future evolution of the process depends only on the present state and not on the past sequence of traversed states prior to current state. Mathematically, the Markov property is stated as:

$$P(X_{t+1} = j \mid X_t = i, X_{t-1} = i_1, ..., X_0 = i_0) = P(X_{t+1} = j \mid X_t = i) \quad (1)$$

However, it has two main limitations: with the complexity of the system, the number of state increases so rapidly that it can lead to state-explosion, restricting the solution to very complex systems [4]. For systems which are subjected to increasing or decreasing failure and repair rates, the Markov approach is inapplicable. Semi-Markov models seem to address this which considers variable failure or repair rates. Therefore, SMP is used to model systems where future state depends on current state as well as sojourn times in this state, which can obey any distribution in turn, not necessarily exponential.

The SMP model however is difficult to approach analytically [30] and no general and detailed procedure for its solution is available. In this respect, several attempts were made to transform the SMP model to a Markov model by approximating non-exponential distributions to exponential distributions in order to get the solution [30, 31]. The SMP has been applied in the areas of software reliability [11, 26, and 32] degradation-dependent reliability [10], fault detection and isolation reliability [18], optimization for condition-based maintenance [3], equipment health management [5], etc. Some integrated methods such as SMP and Bayesian networks were also presented for assessing the availability of fault tolerant systems [21]. The steady-state solution of the SMP model was implemented by a two-stage analytical approach in the field of software reliability but was not applied to mechanical systems [12, 33]. In this article, this method has been extended for availability assessment of repairable mechanical systems. The proposed approach is capable of handling non-exponential distributions i.e. Weibull distribution, as such distribution depicts actual behavior of the systems undergoing degradation. The Weibull distribution has been used for modeling other real life applications such as the deterioration of mechanical systems such as pistons, engine crankshaft, and the breakdown of insulating fluid, etc. [25]. Therefore, to capture the dynamics of real system and to model the dependencies and interactions between components of the system, semi-Markov approach have been adopted which deal with events that are non-exponentially distributed. The primary objective of this paper is to develop a mathematical model for system availability assessment and to quantify the benefits of opportunistic maintenance in terms of increased system availability. The remaining part of the paper is organized as follows: Overview of the analytical semi-Markov approach is described in Section 2. Section 3 deals with the system modelling considering multi-state degradation and its solution is illustrated in detail in Section 4. Section 5 concludes the work while the scope for future work is described in Section 6.

2. Research Methodology: Analytical Semi-Markov Approach

Semi-Markov approach is an extension of Markov process, which is used to model such systems that degrade non-exponentially. Semi-Markov process is more suitable to present the deterioration process of physical system than continuous time Markov chain in terms of the mathematical generality and tractability. Semi-Markov process (Semi-MP) is of two types: discrete and continuous time chain. Availability assessment problems are usually encountered in continuous time chain for the system. However, to avoid the complexity involved in the solution of continuous time chain model, it is converted into an Embedded Markov chain (EMC) which is a discrete time chain. Hence, EMC is an ideal process for modeling and analysis of degrading systems. The solution of the Semi-Markov model involves complex integration and hence more complicated than the Markov process. Semi-Markov process is characterized as an arrangement of two dimensional random variables, $\{(X_k,T_k): k \in 1, 2, 3, 4, 5, \ldots\ldots,m\}$, with the properties mentioned below:

- X_k represents the system state after k transitions in a discrete-time Markov chain (DTMC).
- T_k represents the sojourn time i.e., the amount of time the system is expected to spend at a particular state [6].

The complete framework of implementing Semi-MP to model system availability is summarized using a two stage method. In stage 1, transition probability matrix of the EMC of SMP model is determined while in stage two, availability of the system is evaluated by using sojourn time and steady-state probability of each state of the SMP model.

The steps of the semi-Markov approach expressed in the flow chart given below:

Consider a system state space with m possible states, m being a finite natural number, which is represented as I = {1, 2, 3, 4, 5, ……, m}. The state space is the set of all feasible states where each state represents a different configuration of system. On the basis of this, semi-Markov model is developed which can be conveniently represented by a labeled directed graph which consists of all feasible states connected by the transition lines showing the appropriate distribution. The semi-Markov process is decomposed into two stages which are discussed below:

In stage 1, the transition probability matrix **A** of the SMP model's embedded Markov chain (EMC) is evaluated and this matrix helps in determining steady-state solution, s_i, of all the feasible EMC states.

In stage 2, mean sojourn time T_i, has to be evaluated for each state. The equation expressed below helps in determining the steady-state solution, P_i, of all feasible states of the SMP model [13].

$$P_i = \frac{s_i T_i}{\sum\limits_{i=I} s_i T_i}, i \in I \tag{2}$$

where, s_i is Steady state probability of the EMC and T_i depicts Sojourn time of each system state.

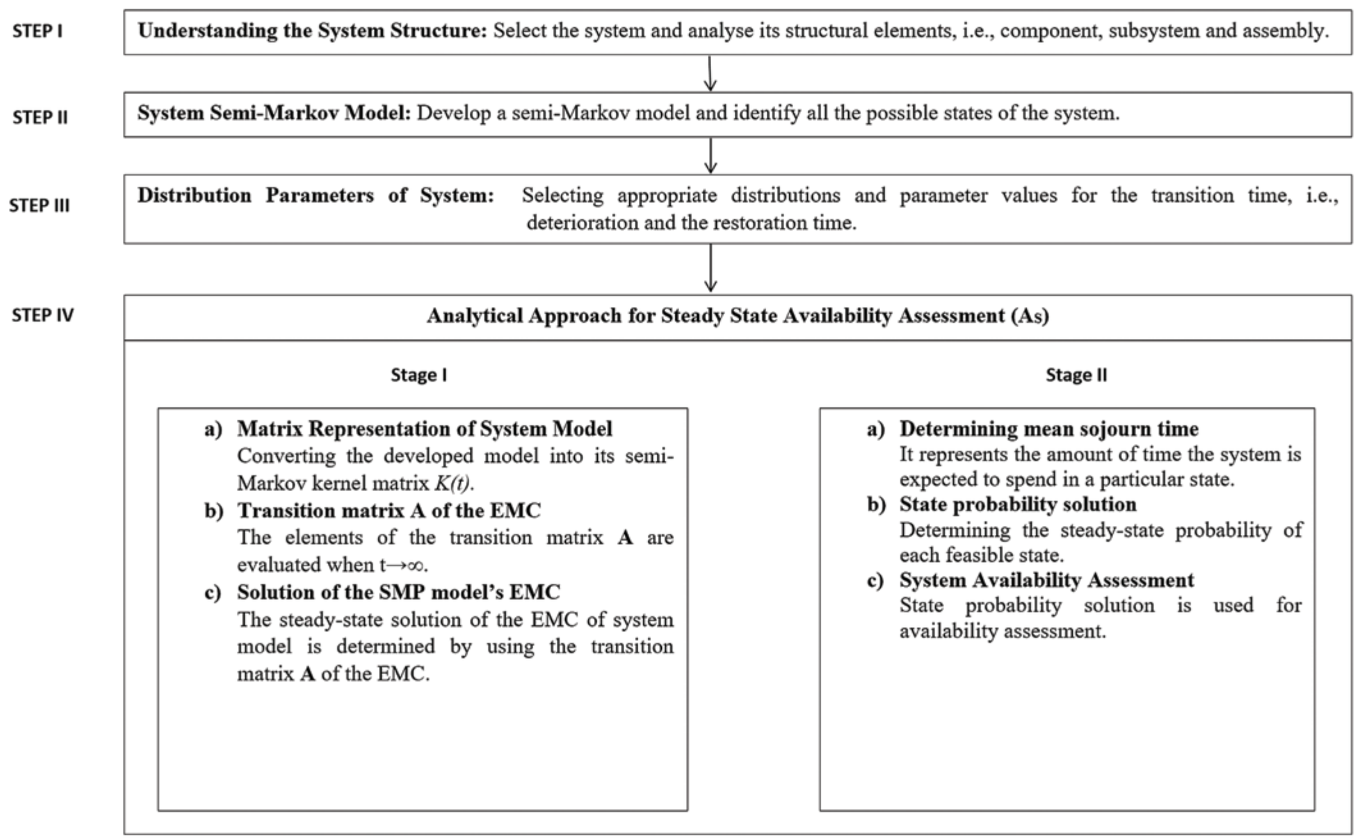
The equation expressed below is utilized for determining the System Availability (A_S).

$$A_s = \sum_{i=I} P_i, i \in I \tag{3}$$

where, P_i is the steady-state probability of the system in working state only.

3. System modeling

Degradation modelling of mechanical systems for availability analysis has drawn special attention of researchers in the recent years. Whenever a system is in working state, it is always degrading with



time representing gradual degradation. Therefore, the degradation process for the system must be interpreted in such a manner that an effective model is built and implemented in operation. In this section, semi-Markov model is developed based on subsystem degradation.

3.1. System Description

A system is characterized as a set of subsystems/components working together towards achieving some logical end. Coupling relationships occurs between various parts of the system and also between various fault types, resulting in several routes for the propagation of faults [7]. The two stage reciprocating air compressor system consists of low pressure compressor (LPC) and high pressure compressor (HPC) with intercooler in between, system in series configuration, is selected for this paper as shown in the Fig. 1. This series system is widely used in power plants to increase the air pressure without increasing its temperature as this result in power saving in compressing the air.

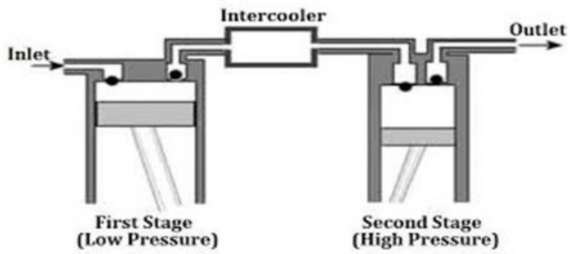


Fig. 1. Multi-stage reciprocating air compressor system

Being a series system, it should be highly available when called to function so that the power plant operations do not suffer. There is a huge loss in case of continuous operating units when unexpected shutdown occurs and their economic competitiveness implies an effective maintenance strategy to improve system availability and reduce operating cost [17]. Therefore, it becomes important to evaluate availability of such system so that overall system availability can be analyzed and appropriate maintenance policy can be suggested to improve it.

3.2. System Model Development

A system is characterized as a set of subsystems/components working together and its performance depends on how individual components work. Configuration of a series production system is such that failure of any machine may cause the entire system breakdown [34]. If proper attention is not given to the system in operation, it can result in a catastrophic failure that needs significant repair time and expense [35].To scientifically research it, we also have to make a series of assumptions that typically take the form of mathematical or logical relationships that constitutes a model. Variables are required at a given time to explain the state of system. A system's failure is due to inappropriate functional interplay between its components and subsystems [29]. Many real-world systems are therefore too complicated to be evaluated analytically but still an attempt is being made to model such complex mechanical system. A system consisting of three subsystems in series configuration, with three degradation states, namely operational, degraded and failed, is considered for availability analysis. When any of its subsystems or part fails, it enters the repair process. The system is expected to perform its intended functions when an appropriate maintenance policy is adopted, however in practice it may unexpectedly fail. The corrective maintenance approach is more suitable in such conditions for failed subsystems or components as it is believed that shutdown and repair cost in case of breakdown will be less than the investment required for preventive maintenance. According to the level of repair, the corrective maintenance approach can be categorized into perfect and imperfect repair but in this paper, only perfect repair is incorporated in the system which restores the system from failed state to operative state whenever the system enters the repair process. For mechanical systems and subsystems, the maintenance actions are varied with the age of components which is described by means of increasing failure rate therefore; it becomes evident for the study of system availability to accept multi-state degradation at the subsystem/component level, depending on their life span. The Weibull distribution, which is more appropriate for degrading systems, is considered [9].

For each subsystem/component, O, D and F are assumed in order for the system undergoing gradual degradation as shown in the Fig. 2. Each component is assumed to be in any of the three states throughout, where; O: Operative; D: Degraded; F: Failed. The states O and D are considered as the working states while F is considered as the repair state.

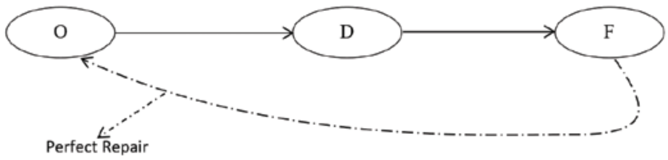


Fig. 2. Multi-state degradation of a subsystem and perfect repair

Models are developed with corrective maintenance approach for system availability assessment. Various maintenance models find the maintenance actions to be done perfectly. In fact, the effectiveness of maintenance staff to restore the failed component/subsystem usually lies between two extreme limits ("as good as new" and "as bad as old"), commonly referred as imperfect repair [23]. However, only perfect repair module is discussed in this study. The recoverability dimension quantifies how quickly and how well a system can recover after interruption to its normal state [8]. In the system model development, as mentioned earlier state O is considered as the original operating state, i.e. "as good as new" state and D as degraded state and F as the failed state at the subsystem level. Refer Fig. 2, a maintenance action, i.e. perfect repair is considered, which restores the subsystem from its failed state F to the operating state O. The states O and D of the subsystem is considered as working states, while F is the 'repair state' due to performance below the unacceptable level. Let a system is considered with its three subsystems in series, with each subsystem having one original operating state, one degraded state and one failed state. It is assumed that only one subsystem is changing its state at a particular time. State of a system is dependent on the state of its subsystems. All the feasible states and their corresponding transitions are identified for the system to develop SMP model as shown in the Table 1. In developing the system model simultaneous failure of two or more subsystems is not considered as it is assumed to be an infeasible state.

The criteria to be followed in decision making for corrective maintenance and/or opportunistic maintenance are discussed here. Table 1 shows system states for three subsystems '1', '2', and '3' in series, each with three states as O, D and F. The last column of the table gives the maintenance option or possibility of 'Corrective Maintenance and/or opportunistic maintenance' for the subsystems '1', '2', and '3', which is decided from the system state status (Column 2); 'Under Repair', i.e. 3, 6, 7, 8, 11, 14 to 20. The rationale behind the decision making 'yes' or 'no' is illustrated by considering the first system state status 'Under Repair' (Column 2) in the Table 1, i.e. at S. No. 3 (Column 1). In this state, states of subsystems '1', '2' and '3' (Column 3, 4 and 5) are O, O and F respectively. For the subsystem '3' in state, F i.e. the failed state, a corrective maintenance (perfect repair) is the best choice as one needs considerable time to perform the maintenance task, including use of resources needed which are also expected to be on a higher side. In view of this, it is 'yes' logic for corrective maintenance (perfect repair) of subsystem '3'. With subsystem '3' under maintenance, there is need to check if opportunistic maintenance for the other subsystems, i.e. subsystem '1' and '2' can be undertaken. For this, one needs to check their state, which is O and O, i.e. 'Original Operating State', and they does not need it. In view of this, it is 'no' logic for opportunistic maintenance. In a similar way, decision 'yes' or 'no' for other system states 6, 7, 8, 11, 14 to 20, which are 'Under Repair' are carried out. In all, two models for the system have been developed as discussed in the following subsections.

3.2.1. System model based on corrective maintenance (perfect repair) without opportunistic maintenance

In a system with three subsystems in series configuration, out of the twenty feasible states of the system, twelve states (3, 11, 6,7, 8, 14, 15, 16, 17, 18, 19 and 20) are the 'under repair' system states, while the remaining eight states falls under the 'working states' category. Considering subsystem degradation and perfect repair, a system model is developed as shown in the Figure 3. The system model features two types of transition edges. Refer Fig. 3, a black line depicts the degradation of the subsystem from O to D and from D to F while a continuous dotted black line represents perfect repair, i.e. from F to O. Perfect repair is carried out for under repair states and such states are represented in the model by transition lines 3-1, 7-1, 8-2, 11-9, 14-12, 15-9, 16-10, 17-1, 18-2, 19-4 and 20-5 for the subsystems restoring the subsystem which is under repair from state F to O.

3.2.2. System Model based on corrective maintenance (perfect repair) with opportunistic maintenance

In opportunistic maintenance, whenever a system or module is grounded for corrective or preventive maintenance, that opportunity is utilized to do maintenance on other parts of the module, which are found to be damaged or have started to deteriorate. On one hand, this improves the safety and reliability of the system, and on the other hand it reduces the downtime by avoiding unscheduled maintenance. This in turn reduces the cost of maintenance and loss of revenue due to extra groundings. In this model, opportunistic maintenance is considered along with perfect repair. Following the similar pattern, a system model is developed considering subsystem degradation and opportunistic maintenance with perfect repair, as shown in the Figure 4. In the system model, the transition lines for degradation are same as in the previous model, Fig. 3. The system being in the 'under repair' states, 3, 7 and 17, only perfect repair is carried out for these as the opportunistic maintenance is not possible because the other subsystems are still in state 'O' i.e. 'as good as new' state. In the system model, perfect repair for these states is represented by transition line 3-1, 7-1, and 17-1 for the subsystems '3', '2', and '1' respectively, restoring the system which is under repair from its state, F to O. For the remaining nine 'under repair' states, opportunistic maintenance is possible because the other subsystems are in state D, i.e. degraded state. In the system model, opportunistic maintenance with perfect repair is represented by transition lines 6→1, 8→1, 11→1, 14→1, 15→1, 16→1, 18→1, 19→1and 20→1, restoring the system which is under repair from its state, D to O and F to O. It is assumed that the time to restore a subsystem in perfect repair from state F to O is more than the restoration time of other partially degraded states of the subsystems in opportunistic maintenance from state D to O.

4. Solution of the System Model

The solution of the SMP model is divided into two stages as discussed below. Stage 1 deals with matrix representation of the system model while stage 2 deals with system availability assessment using sojourn time and state probability values.

Table 1. States of a two stage reciprocating compressor system and their corresponding transitions

System State		State of subsystem 1	State of subsystem 2	State of subsystem 3	Transition to	Maintenance Possibility	
S.NO	Status					C.M*	O.M*
1	Working	O	O	O	2 4 9	-	-
2	Working	O	O	D	3 5 10	-	-
3	Under Repair	O	O	F	1	Yes(S_3)	No
4	Working	O	D	O	5 7 12	-	-
5	Working	O	D	D	6 8 13	-	-
6	Under Repair	O	D	F	4	Yes(S_3)	Yes(S_2)
7	Under Repair	O	F	O	1	Yes(S_2)	No
8	Under Repair	O	F	D	2	Yes(S_2)	Yes(S_3)
9	Working	D	O	O	17 12 10	-	-
10	Working	D	O	D	18 13 11	-	-
11	Under Repair	D	O	F	9	Yes(S_3)	Yes(S_1)
12	Working	D	D	O	19 15 13	-	-
13	Working	D	D	D	20 16 14	-	-
14	Under Repair	D	D	F	12	Yes(S_3)	Yes($S_{1,2}$)
15	Under Repair	D	F	O	9	Yes(S_2)	Yes(S_1)
16	Under Repair	D	F	D	10	Yes(S_2)	Yes($S_{1,3}$)
17	Under Repair	F	O	O	1	Yes(S_1)	No
18	Under Repair	F	O	D	2	Yes(S_1)	Yes(S_3)
19	Under Repair	F	D	O	4	Yes(S_1)	Yes(S_2)
20	Under Repair	F	D	D	5	Yes(S_1)	Yes($S_{2,3}$)

C.M- Corrective Maintenance, O.M- Opportunistic Maintenance

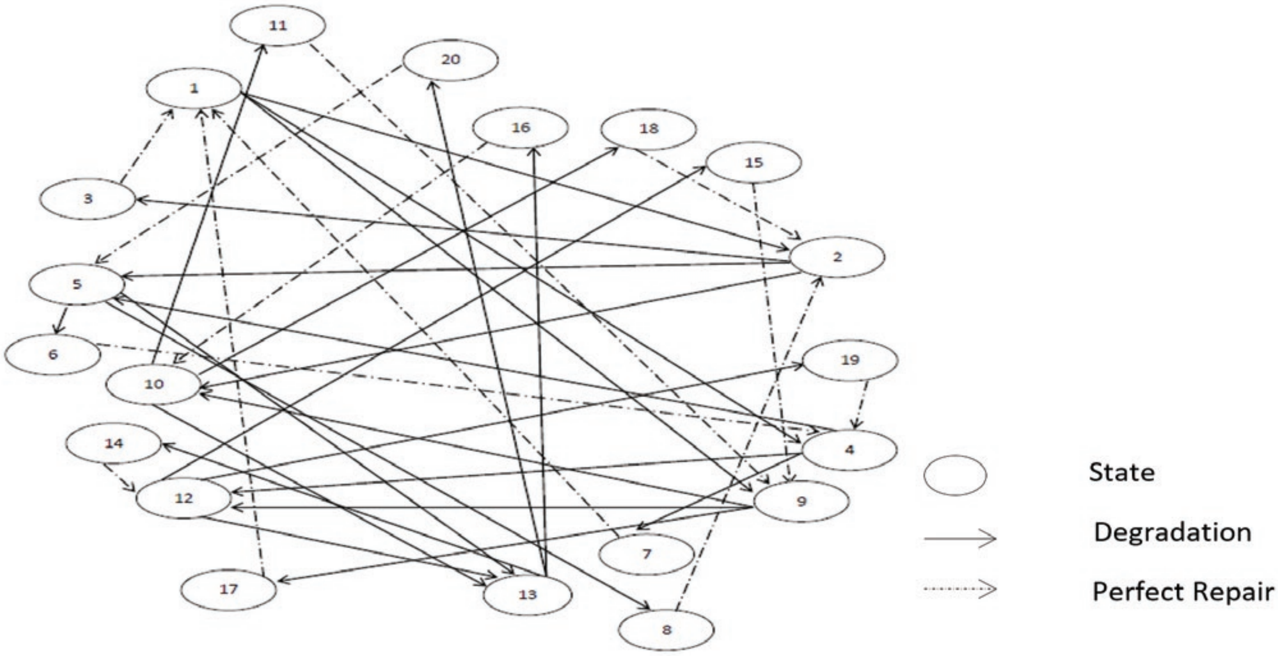


Fig. 3. Semi-Markov model for the system undergoing perfect repair without opportunistic maintenance

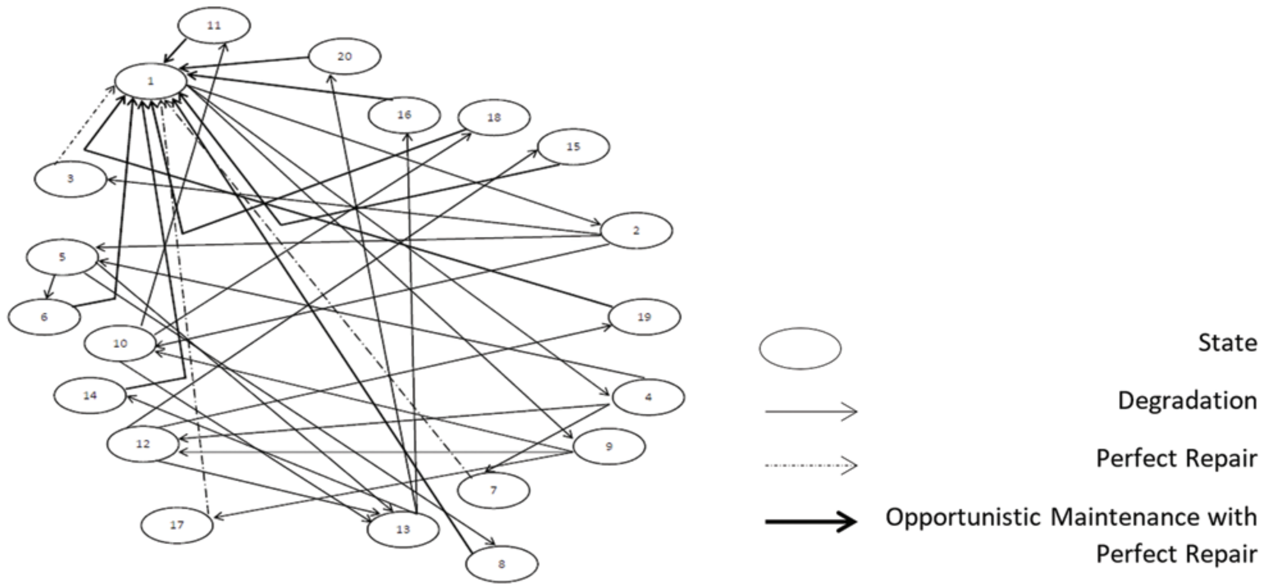


Fig. 4. Semi-Markov model for the system undergoing perfect repair coupled with opportunistic maintenance

Stage I

a) The SMP model is described by its Kernel matrix, $K(t)$ as shown below.

$$K(t)=\begin{bmatrix} k_{11}(t) & k_{12}(t) & \cdots & \cdots & k_{1N}(t) \\ k_{21}(t) & k_{22}(t) & \cdots & \cdots & k_{2N}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ k_{N1}(t) & k_{N2}(t) & \cdots & \cdots & k_{NN}(t) \end{bmatrix}$$

The matrix $K(t) = [k_{ij}(t)]$ is called kernel of the SMP model and the matrix elements gives the probability of jumping from one state to another. The matrix's rows and columns correspond to the number of feasible system states. The matrix elements, $k_{ij}(t)$ are defined as [14]:

$$k_{ij}(t) = \begin{cases} 0 & \text{When there is no possible transition from state } i \text{ within time } t. \\ F_{ij}(t) & \text{When there is a single possible transition from state } i \text{ to state j within time } t. \\ \int_0^t \bar{F}_{ik}\bar{F}_{im}dF_{ij}\,(t) & \text{When there are multiple possible transition from state } i \text{ to state } j,\, k \text{ and m within time } t. \end{cases}$$

where, $F_{ij}(t)$ is the CDF and $\bar{F}_{ij}(t)=1-F_{ij}(t)$ represents complement of CDF associated with transition from state i to j. Therefore, the kernel matrix $K(t)$ of the SMP model developed as shown in the Fig. 3 is obtained as:

$$K(t)=\begin{bmatrix}
0 & k_{12} & 0 & k_{14} & 0 & 0 & 0 & 0 & k_{19} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & k_{23} & 0 & k_{25} & 0 & 0 & 0 & 0 & k_{210} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
k_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & k_{45} & 0 & k_{47} & 0 & 0 & 0 & 0 & k_{412} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & k_{56} & 0 & k_{58} & 0 & 0 & 0 & 0 & k_{513} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & k_{64} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
k_{71} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & k_{82} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{910} & 0 & k_{912} & 0 & 0 & 0 & 0 & k_{917} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{1011} & 0 & k_{1013} & 0 & 0 & 0 & 0 & k_{1018} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{119} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{1213} & 0 & k_{1215} & 0 & 0 & 0 & k_{1219} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{1314} & 0 & k_{1316} & 0 & 0 & 0 & k_{1320}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{1412} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{159} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{1610} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
k_{171} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & k_{182} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & k_{194} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & k_{205} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Table 2. Distribution parameters of compressor system [2]

CDFs	Distribution	Parameter	CDFs	Distribution	Parameter
F_{12}	Weibull	β_{12}=3.78,θ_{12}=10000	F_{917}	Weibull	β_{917}= 2.5,θ_{917}=6776
F_{14}	Weibull	β_{14}=1.2,θ_{12}=6000	F_{1011}	Weibull	β_{1011}= 3.78,θ_{1011}=4892
F_{19}	Weibull	β_{19}=2.5,θ_{19}=16000	F_{1013}	Weibull	β_{1013}= 1.2,θ_{1013}=6000
F_{23}	Weibull	β_{23}=3.78,θ_{23}=4892	F_{1018}	Weibull	β_{1018}= 2.5,θ_{1018}=6776
F_{25}	Weibull	β_{25}= 1.2,θ_{25}=6000	F_{119}	Exponential	μ_{119}=0.001667
F_{210}	Weibull	β_{210}=2.5,θ_{210}=16000	F_{1213}	Weibull	β_{1213}= 3.78,θ_{1213}=10000
F_{31}	Exponential	μ_{31}=0.001667	F_{1215}	Weibull	β_{1215}= 1.2 ,θ_{1215}=4000
F_{45}	Weibull	β_{45}=3.78,θ_{45}=10000	F_{1219}	Weibull	β_{1219}= 2.5,θ_{1219}=6776
F_{47}	Weibull	β_{47}= 1.2,θ_{47}=4000	F_{1314}	Weibull	β_{1314}= 3.78 ,θ_{1314}=4892
F_{412}	Weibull	β_{412}= 2.5,θ_{412}=16000	F_{1316}	Weibull	β_{1316}= 1.2 ,θ_{1316}=4000
F_{56}	Weibull	β_{56}= 3.78,θ_{56}=4892	F_{1320}	Weibull	β_{1320}= 2.5,θ_{1320}=6776
F_{58}	Weibull	β_{58}= 1.2,θ_{58}=4000	F_{1412}	Exponential	μ_{1412}=0.001667
F_{513}	Weibull	β_{513}= 2.5,θ_{513}=16000	F_{159}	Exponential	μ_{159}=0.001667
F_{64}	Exponential	μ_{64} =0.001667	F_{1610}	Exponential	μ_{1610}=0.001667
F_{71}	Exponential	μ_{71}=0.001667	F_{171}	Exponential	μ_{171}=0.001667
F_{82}	Exponential	μ_{82}=0.001667	F_{182}	Exponential	μ_{182}=0.001667
F_{910}	Weibull	β_{910}=3.78,θ_{910}=10000	F_{194}	Exponential	μ_{194}=0.001667
F_{912}	Weibull	β_{912}= 1.2,θ_{912}=6000	F_{205}	Exponential	μ_{205}=0.001667

The empirical estimators of the kernel matrix *K(t)*, are determined for the semi-Markov model as shown in the Fig. 3 using the distribution available in Table 2.

For example, for the non-zero element of the kernel matrix, k_{12}, see the details for its distribution available in Table 2. It is clear from the Fig. 3 that there are three outgoing transitions from State 1 to State 2, 4 and 9 respectively. The distribution of these three transitions is Weibull, therefore the expression for the non-zero element of the kernel matrix, K_{12} in terms of Weibull parameters is expressed as follows:

$$k_{12}=\int_0^t \overline{F_{14}F_{19}}\,dF_{12}=\frac{\beta_{12}}{(\theta_{12})^{\beta_{12}}}\int_0^t t^{(\beta_{12}-1)}e^{-\left[\left(\frac{t}{\theta_{12}}\right)^{\beta_{12}}+\left(\frac{t}{\theta_{14}}\right)^{\beta_{14}}+\left(\frac{t}{\theta_{19}}\right)^{\beta_{19}}\right]}dt \quad (4)$$

Similarly, for the non-zero element of the kernel matrix, k_{31}, see the details for its distribution available in Table 2. It is clear from the Fig. 3 that there is only one outgoing transition from State 3. The distribution of this transition is exponential; therefore the expression for the matrix element, k_{31} in terms of exponential parameters is expressed as follows:

$$k_{31}=F_{31}(t)=1-e^{-(\mu_{31}t)} \quad (5)$$

Remaining elements of the kernel matrix *K(t)* are expressed as explained above, and all these expressions are listed in "Appendix: Table 8".

Table 3: Elements of the matrix, A

k_{ij}	Value	k_{ij}	Value	k_{ij}	Value	k_{ij}	Value
k_{12}	0.1885	k_{45}	0.0896	k_{910}	0.0753	k_{1213}	0.0431
k_{14}	0.7367	k_{47}	0.8678	k_{912}	0.5687	k_{1215}	0.7238
k_{19}	0.0748	k_{412}	0.0426	k_{917}	0.3560	k_{1219}	0.2331
k_{23}	0.4915	k_{56}	0.3066	k_{1011}	0.3743	k_{1314}	0.2610
k_{25}	0.4823	k_{58}	0.6741	k_{1013}	0.4397	k_{1316}	0.5993
k_{210}	0.0262	k_{513}	0.0193	k_{1018}	0.1860	k_{1320}	0.1397

The transition matrix A of the Embedded Markov Chain (EMC), obtained is:

$$A=K(\infty)=\begin{bmatrix}
0 & 0.1885 & 0 & 0.7367 & 0 & 0 & 0 & 0 & 0.0748 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0.4915 & 0 & 0.4823 & 0 & 0 & 0 & 0 & 0.0262 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0.0896 & 0 & 0.8678 & 0 & 0 & 0 & 0 & 0.0426 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0.3066 & 0 & 0.6741 & 0 & 0 & 0 & 0 & 0.0193 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0753 & 0 & 0.5687 & 0 & 0 & 0 & 0 & 0.3560 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3743 & 0 & 0.4397 & 0 & 0 & 0 & 0 & 0.1860 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0431 & 0 & 0.7238 & 0 & 0 & 0 & 0.2331 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2610 & 0 & 0.5993 & 0 & 0 & 0 & 0.1397\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

b) Transition matrix **A** of the EMC

The state transition matrix **A** has to be determined such that the system spends a prescribed amount of time in each state before making transition. The system transits from the operative state to the failed state in a strongly correlated manner with the time. To imitate this behavior of system model, some coupling is required with the matrix *K(t)*.

The developed matrix *K(t)* helps in determining the matrix, **A** = *K(∞),* of the EMC considering t→∞, which is necessary condition for the steady-state analysis. As it is clear from the kernel matrix, *K(t)*, that there is only one element in the 3rd, 6th, 7th, 8th, 11th, 14th, 15th, 16th, 17th, 18th, 19th, and 20th row, therefore, the values of elements $k_{31}(\infty)$, $k_{64}(\infty)$, $k_{71}(\infty)$, $k_{82}(\infty)$, $k_{119}(\infty)$, k_{1412}, $k_{159}(\infty)$, $k_{1610}(\infty)$, $k_{171}(\infty)$, $k_{182}(\infty)$, $k_{194}(\infty)$ and $k_{205}(\infty)$ *are all equal to 1*. Distribution parameter values shown in Table 2 are substituted in the expressions of kernel matrix elements and these are solved using MATLAB [20]. Table 3 shows the non-zero elements of matrix **A**. By substituting the values of non-zero elements in the kernel matrix, K(t), the complete transition probability matrix A is obtained.

c) Probability of the states of EMC

The matrix **A** helps in determining the state probability of EMC. The steady-state solution of the EMC is determined using the equation expressed below which is to be solved using MATLAB and the values of s_i, i ϵ I are listed in Table 4.

$$\left[s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_{11} s_{12} s_{13} s_{14} s_{15} s_{16} s_{17} s_{18} s_{19} s_{20}\right]=\left[s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_{10} s_{11} s_{12} s_{13} s_{14} s_{15} s_{16} s_{17} s_{18} s_{19} s_{20}\right]*A \quad (6)$$

Table 4. Probability of the states of EMC

s_i	Probability Value	s_i	Probability Value	s_i	Probability Value	s_i	Probability Value
s_1	0.2289	s_6	0.0191	s_{11}	0.0038	s_{16}	0.0043
s_2	0.0820	s_7	0.1705	s_{12}	0.0376	s_{17}	0.0171
s_3	0.0403	s_8	0.0370	s_{13}	0.0071	s_{18}	0.0019
s_4	0.1964	s_9	0.0481	s_{14}	0.0019	s_{19}	0.0088
s_5	0.0571	s_{10}	0.0100	s_{15}	0.0272	s_{20}	0.0009

Stage 2

a) Mean sojourn time evaluation of all the feasible states of the system model

The mean sojourn time, T_i, is the amount of time the system is expected to spend at state i, before leaving for another state depending upon the configuration of the system. This helps in determining the steady-state probability of all the feasible states of the system model.

See the details for its distribution available in Table 2 for sojourn time expression for state 1, T_1. It is clear from the Fig. 3 that there are three outgoing transitions from State 1 to State 2, 4 and 9 respectively. The distribution of these three transitions is Weibull, therefore the expression for the mean sojourn time, T_1 in terms of Weibull parameters is expressed as follows [14]:

$$T_1 = \int_0^\infty \overline{F_{12} F_{14} F_{19}} dt = \int_0^\infty e^{-\left[\left(\frac{t}{\theta_{12}}\right)^{\beta_{12}} + \left(\frac{t}{\theta_{14}}\right)^{\beta_{14}} + \left(\frac{t}{\theta_{19}}\right)^{\beta_{19}}\right]} dt \quad (7)$$

See the details for its distribution available in Table 2 for sojourn time expression for state 1, T_3. From Figure 3, it is evident that only one state is reachable from state 3 i.e. to state 1. The distribution is exponential for this transition; therefore the expression of the sojourn time, T_3, in terms of exponential parameters is expressed as follows [14]:

$$T_3 = \int_0^\infty \overline{F_{31}} dt = \int_0^\infty e^{-(\mu_{31} t)} dt \quad (8)$$

Likewise, the time values are represented in "Appendix: Table 9" for the remaining feasible states of the system model. Using the sojourn time expressions of all the states and by substituting the distribution parameter values listed down in Table 2, the sojourn time for all the states are determined. To solve sojourn time expressions, MATLAB is used and its value is shown in Table 5.

Table 5. Mean Sojourn Time of System States

T_i	Value(hr)	T_i	Value(hr)	T_i	Value(hr)	T_i	Value(hr)
T_1	4458.2	T_6	600	T_{11}	600	T_{16}	600
T_2	3176.8	T_7	600	T_{12}	2922.8	T_{17}	600
T_3	600	T_8	600	T_{13}	2519.4	T_{18}	600
T_4	3379	T_9	3611.9	T_{14}	600	T_{19}	600
T_5	2683.9	T_{10}	2941.2	T_{15}	600	T_{20}	600

b) Steady-state probability of states of the system model

The probability 'P_i' of all the feasible states of the system model is determined by Eq. (1). The evaluated values of the states are listed in Table 6.

Table 6. Steady-state probability of the states of System model

P_i	Probability values	P_i	Probability values	P_i	Probability values	P_i	Probability values
P_1	0.3882	P_6	0.0044	P_{11}	0.00085842	P_{16}	0.00097674
P_2	0.0991	P_7	0.0384	P_{12}	0.0418	P_{17}	0.0039
P_3	0.0092	P_8	0.0084	P_{13}	0.0068	P_{18}	0.00042657
P_4	0.2525	P_9	0.0661	P_{14}	0.00042538	P_{19}	0.0020
P_5	0.0583	P_{10}	0.0112	P_{15}	0.0062	P_{20}	0.00022768

c) Availability Assessment

The state probability solutions are used for availability assessment. Availability is the summation of state probabilities in which the system is operational or available [15]. It is determined by using Eq. (3); A_s = 0.9241. As mentioned earlier system availability is a performance measure and is defined as a measure of the percentage of time the equipment is in operable state. The steady–state availability of a two stage reciprocating air compressor system by analytical semi-Markov approach is 0.9241. It represents that the system is available 92.41% of the time and working at 92.41% of the system's technical limit. Availability measure is typically equal to the financial output of the system. Therefore, availability modeling and analysis is crucial for degrading system as it will give insights for its improvement. All our efforts should be focused on improving system availability in order to achieve the planned service life. The analytical approach utilized for availability assessment typically requires less time in computation but involves complex integrals. The suggested approach quantifies the impact of corrective maintenance policy in terms of system availability.

Table 7. Steady-state availability analysis

S.No.	Corrective Maintenance	System Availability	
		Without Opportunistic Maintenance	With Opportunistic Maintenance
1.	Perfect Repair	0.9241	0.9342

Steps explained in Section 4 are repeated for the SMP model based on corrective maintenance (perfect repair) with opportunistic maintenance. The results obtained are tabulated in Table 7.

As the results obtained provide a definite indication of the trend in the availability for different maintenance policies, these numeric results can be analyzed quantitatively to compare the relative improvement in the performance of the system in different scenarios. For the same mission time, moving from perfect repair without opportunistic maintenance to perfect repair with opportunistic maintenance, the availability shows the increasing trend. This clearly establishes that the maintenance policy: perfect repair without opportunistic maintenance is inefficient and should be seldom used unless cost of maintenance is the only dictating factor.

5. Conclusions

Considering the importance of maintenance, this research has attempted to establish a framework for availability modelling and analysis of degrading system. Using analytical semi-Markov methodology, a method for determining the availability based on corrective maintenance approach applied to complex mechanical system is suggested. By recognizing the configuration of the system, different subsystems/components are identified and allocated at different stratified levels of mechanical system experiencing deterioration with ageing. The states and states transitions are identified. The state transition rates for deterioration and restoration are expressed by Weibull and exponential distribution respectively. The main contribution of this research is to develop mathematical model for system availability assessment which will give insights for its improvement. The steady-state availability of the system based on corrective maintenance (perfect repair) without opportunistic maintenance is low (A_s= 0.9241). So, there lies a scope of improving it further. Therefore, the system needs attention on revising existing maintenance policies towards further improvement. Another model is added to investigate the gain in system availability when corrective maintenance (perfect repair) is combined with opportunistic maintenance. Hence, it is concluded that corrective maintenance (perfect repair) with opportunistic maintenance should be preferred and certain redundant strategies may also be applied in the designing of the system to enhance its availability further. The suggested approach is valuable for maintenance personnel allowing them to establish repair and replacement policies. The designers can also use the methodology to make a decision on the implementation of the repairs and the degree of repair to achieve the required system availability values.

6. Practical Utility and Future Directions

In spite of the reality that the suggested technique provides numerous merits, it involves significant calculations for developing its detailed solution. The proposed approach is useful for plant engineers and maintenance personnel in designing a system with high availability by incorporating appropriate maintenance policies. The suggested methodology can be applied to variety of systems as it incorporates the varying deterioration and restoration rates. In this paper, steady-state availability of the system is evaluated by employing the semi-Markov technique. Availability assessment under transient conditions case can also be carried out for measuring system performance. The suggested approach can also be extended for system availability assessment of mechanical systems considering various maintenance approaches like condition based maintenance, preventive maintenance, etc. The effect of human error can also be incorporated in the maintenance procedure in order to analyse its effect on system availability. After, one find a suitable way of quantifying system availability, it is possible to carry out further modifications of system design and operations thereby enhancing availability. Moreover, other performance parameters of complex mechanical systems, including resilience, maintainability and reliability will also be evaluated using this technique.

Appendix

Table 8. Expressions of the elements of kernel matrix, K(t)

k_{ij}	$k_{ij}(t)$	$k_{ij}(t)$ with specific distribution
k_{12}	$\int_0^t \overline{F_{14}}\,\overline{F_{19}}dF_{12}$	$\frac{\beta_{12}}{(\theta_{12})^{\beta_{12}}}\int_0^t t^{(\beta_{12}-1)}e^{-\left[\left(\frac{t}{\theta_{12}}\right)^{\beta_{12}}+\left(\frac{t}{\theta_{14}}\right)^{\beta_{14}}+\left(\frac{t}{\theta_{19}}\right)^{\beta_{19}}\right]}dt$
k_{14}	$\int_0^t \overline{F_{12}}\,\overline{F_{19}}dF_{14}$	$\frac{\beta_{14}}{(\theta_{14})^{\beta_{14}}}\int_0^t t^{(\beta_{14}-1)}e^{-\left[\left(\frac{t}{\theta_{12}}\right)^{\beta_{12}}+\left(\frac{t}{\theta_{14}}\right)^{\beta_{14}}+\left(\frac{t}{\theta_{19}}\right)^{\beta_{19}}\right]}dt$
k_{19}	$\int_0^t \overline{F_{14}}\,\overline{F_{12}}dF_{19}$	$\frac{\beta_{19}}{(\theta_{19})^{\beta_{19}}}\int_0^t t^{(\beta_{19}-1)}e^{-\left[\left(\frac{t}{\theta_{12}}\right)^{\beta_{12}}+\left(\frac{t}{\theta_{14}}\right)^{\beta_{14}}+\left(\frac{t}{\theta_{19}}\right)^{\beta_{19}}\right]}dt$
k_{23}	$\int_0^t \overline{F_{25}}\,\overline{F_{210}}dF_{23}$	$\frac{\beta_{23}}{(\theta_{23})^{\beta_{23}}}\int_0^t t^{(\beta_{23}-1)}e^{-\left[\left(\frac{t}{\theta_{23}}\right)^{\beta_{23}}+\left(\frac{t}{\theta_{25}}\right)^{\beta_{25}}+\left(\frac{t}{\theta_{210}}\right)^{\beta_{210}}\right]}dt$
k_{25}	$\int_0^t \overline{F_{23}}\,\overline{F_{210}}dF_{25}$	$\frac{\beta_{25}}{(\theta_{25})^{\beta_{25}}}\int_0^t t^{(\beta_{25}-1)}e^{-\left[\left(\frac{t}{\theta_{23}}\right)^{\beta_{23}}+\left(\frac{t}{\theta_{25}}\right)^{\beta_{25}}+\left(\frac{t}{\theta_{210}}\right)^{\beta_{210}}\right]}dt$
k_{210}	$\int_0^t \overline{F_{25}}\,\overline{F_{23}}dF_{210}$	$\frac{\beta_{210}}{(\theta_{210})^{\beta_{210}}}\int_0^t t^{(\beta_{210}-1)}e^{-\left[\left(\frac{t}{\theta_{23}}\right)^{\beta_{23}}+\left(\frac{t}{\theta_{25}}\right)^{\beta_{25}}+\left(\frac{t}{\theta_{210}}\right)^{\beta_{210}}\right]}dt$
k_{31}	$F_{31}(t)$	$1-e^{-(\mu_{31}t)}$
k_{45}	$\int_0^t \overline{F_{47}}\,\overline{F_{412}}dF_{45}$	$\frac{\beta_{45}}{(\theta_{45})^{\beta_{45}}}\int_0^t t^{(\beta_{45}-1)}e^{-\left[\left(\frac{t}{\theta_{45}}\right)^{\beta_{45}}+\left(\frac{t}{\theta_{47}}\right)^{\beta_{47}}+\left(\frac{t}{\theta_{412}}\right)^{\beta_{412}}\right]}dt$
k_{47}	$\int_0^t \overline{F_{45}}\,\overline{F_{412}}dF_{47}$	$\frac{\beta_{47}}{(\theta_{47})^{\beta_{47}}}\int_0^t t^{(\beta_{47}-1)}e^{-\left[\left(\frac{t}{\theta_{45}}\right)^{\beta_{45}}+\left(\frac{t}{\theta_{47}}\right)^{\beta_{47}}+\left(\frac{t}{\theta_{412}}\right)^{\beta_{412}}\right]}dt$
k_{412}	$\int_0^t \overline{F_{47}}\,\overline{F_{45}}dF_{412}$	$\frac{\beta_{412}}{(\theta_{412})^{\beta_{412}}}\int_0^t t^{(\beta_{412}-1)}e^{-\left[\left(\frac{t}{\theta_{45}}\right)^{\beta_{45}}+\left(\frac{t}{\theta_{47}}\right)^{\beta_{47}}+\left(\frac{t}{\theta_{412}}\right)^{\beta_{412}}\right]}dt$

k_{56}	$\int_0^t \overline{F_{58}}\,\overline{F_{513}}\,dF_{56}$	$\frac{\beta_{56}}{(\theta_{56})^{\beta_{56}}}\int_0^t t^{(\beta_{56}-1)} e^{-\left[\left(\frac{t}{\theta_{56}}\right)^{\beta_{56}}+\left(\frac{t}{\theta_{58}}\right)^{\beta_{58}}+\left(\frac{t}{\theta_{513}}\right)^{\beta_{513}}\right]}dt$
k_{56}	$\int_0^t \overline{F_{56}}\,\overline{F_{513}}\,dF_{58}$	$\frac{\beta_{58}}{(\theta_{58})^{\beta_{58}}}\int_0^t t^{(\beta_{58}-1)} e^{-\left[\left(\frac{t}{\theta_{56}}\right)^{\beta_{56}}+\left(\frac{t}{\theta_{58}}\right)^{\beta_{58}}+\left(\frac{t}{\theta_{513}}\right)^{\beta_{513}}\right]}dt$
k_{513}	$\int_0^t \overline{F_{58}}\,\overline{F_{56}}\,dF_{513}$	$\frac{\beta_{513}}{(\theta_{513})^{\beta_{513}}}\int_0^t t^{(\beta_{513}-1)} e^{-\left[\left(\frac{t}{\theta_{56}}\right)^{\beta_{56}}+\left(\frac{t}{\theta_{58}}\right)^{\beta_{58}}+\left(\frac{t}{\theta_{513}}\right)^{\beta_{513}}\right]}dt$
k_{64}	$F_{64}(t)$	$1-e^{-(\mu_{64}t)}$
k_{71}	$F_{71}(t)$	$1-e^{-(\mu_{71}t)}$
k_{82}	$F_{82}(t)$	$1-e^{-(\mu_{82}t)}$
k_{910}	$\int_0^t \overline{F_{912}}\,\overline{F_{917}}\,dF_{910}$	$\frac{\beta_{910}}{(\theta_{910})^{\beta_{910}}}\int_0^t t^{(\beta_{910}-1)} e^{-\left[\left(\frac{t}{\theta_{910}}\right)^{\beta_{910}}+\left(\frac{t}{\theta_{912}}\right)^{\beta_{912}}+\left(\frac{t}{\theta_{917}}\right)^{\beta_{917}}\right]}dt$
k_{912}	$\int_0^t \overline{F_{910}}\,\overline{F_{917}}\,dF_{912}$	$\frac{\beta_{912}}{(\theta_{912})^{\beta_{912}}}\int_0^t t^{(\beta_{912}-1)} e^{-\left[\left(\frac{t}{\theta_{910}}\right)^{\beta_{910}}+\left(\frac{t}{\theta_{912}}\right)^{\beta_{912}}+\left(\frac{t}{\theta_{917}}\right)^{\beta_{917}}\right]}dt$
k_{917}	$\int_0^t \overline{F_{912}}\,\overline{F_{910}}\,dF_{917}$	$\frac{\beta_{917}}{(\theta_{917})^{\beta_{917}}}\int_0^t t^{(\beta_{917}-1)} e^{-\left[\left(\frac{t}{\theta_{910}}\right)^{\beta_{910}}+\left(\frac{t}{\theta_{912}}\right)^{\beta_{912}}+\left(\frac{t}{\theta_{917}}\right)^{\beta_{917}}\right]}dt$
k_{1011}	$\int_0^t \overline{F_{1013}}\,\overline{F_{1018}}\,dF_{1011}$	$\frac{\beta_{1011}}{(\theta_{1011})^{\beta_{1011}}}\int_0^t t^{(\beta_{1011}-1)} e^{-\left[\left(\frac{t}{\theta_{1011}}\right)^{\beta_{1011}}+\left(\frac{t}{\theta_{1013}}\right)^{\beta_{1013}}+\left(\frac{t}{\theta_{1018}}\right)^{\beta_{1018}}\right]}dt$
k_{1013}	$\int_0^t \overline{F_{1011}}\,\overline{F_{1018}}\,dF_{1013}$	$\frac{\beta_{1013}}{(\theta_{1013})^{\beta_{1013}}}\int_0^t t^{(\beta_{1013}-1)} e^{-\left[\left(\frac{t}{\theta_{1011}}\right)^{\beta_{1011}}+\left(\frac{t}{\theta_{1013}}\right)^{\beta_{1013}}+\left(\frac{t}{\theta_{1018}}\right)^{\beta_{1018}}\right]}dt$
k_{1018}	$\int_0^t \overline{F_{1013}}\,\overline{F_{1011}}\,dF_{1018}$	$\frac{\beta_{1018}}{(\theta_{1018})^{\beta_{1018}}}\int_0^t t^{(\beta_{1018}-1)} e^{-\left[\left(\frac{t}{\theta_{1011}}\right)^{\beta_{1011}}+\left(\frac{t}{\theta_{1013}}\right)^{\beta_{1013}}+\left(\frac{t}{\theta_{1018}}\right)^{\beta_{1018}}\right]}dt$
k_{119}	$F_{119}(t)$	$1-e^{-(\mu_{119}t)}$
k_{1213}	$\int_0^t \overline{F_{1215}}\,\overline{F_{1219}}\,dF_{1213}$	$\frac{\beta_{1213}}{(\theta_{1213})^{\beta_{1213}}}\int_0^t t^{(\beta_{1213}-1)} e^{-\left[\left(\frac{t}{\theta_{1213}}\right)^{\beta_{1213}}+\left(\frac{t}{\theta_{1215}}\right)^{\beta_{1215}}+\left(\frac{t}{\theta_{1219}}\right)^{\beta_{1219}}\right]}dt$
k_{1215}	$\int_0^t \overline{F_{1213}}\,\overline{F_{1219}}\,dF_{1215}$	$\frac{\beta_{1215}}{(\theta_{1215})^{\beta_{1215}}}\int_0^t t^{(\beta_{1215}-1)} e^{-\left[\left(\frac{t}{\theta_{1213}}\right)^{\beta_{1213}}+\left(\frac{t}{\theta_{1215}}\right)^{\beta_{1215}}+\left(\frac{t}{\theta_{1219}}\right)^{\beta_{1219}}\right]}dt$
k_{1219}	$\int_0^t \overline{F_{1215}}\,\overline{F_{1213}}\,dF_{1219}$	$\frac{\beta_{1219}}{(\theta_{1219})^{\beta_{1219}}}\int_0^t t^{(\beta_{1219}-1)} e^{-\left[\left(\frac{t}{\theta_{1213}}\right)^{\beta_{1213}}+\left(\frac{t}{\theta_{1215}}\right)^{\beta_{1215}}+\left(\frac{t}{\theta_{1219}}\right)^{\beta_{1219}}\right]}dt$

k_{1314}	$\int_0^t \overline{F_{1316}F_{1320}}dF_{1314}$	$\frac{\beta_{1314}}{(\theta_{1314})^{\beta_{1314}}}\int_0^t t^{(\beta_{1314}-1)}e^{-\left[\left(\frac{t}{\theta_{1314}}\right)^{\beta_{1314}}+\left(\frac{t}{\theta_{1316}}\right)^{\beta_{1316}}+\left(\frac{t}{\theta_{1320}}\right)^{\beta_{1320}}\right]}dt$
k_{1316}	$\int_0^t \overline{F_{1314}F_{1320}}dF_{1316}$	$\frac{\beta_{1316}}{(\theta_{1316})^{\beta_{1316}}}\int_0^t t^{(\beta_{1316}-1)}e^{-\left[\left(\frac{t}{\theta_{1314}}\right)^{\beta_{1314}}+\left(\frac{t}{\theta_{1316}}\right)^{\beta_{1316}}+\left(\frac{t}{\theta_{1320}}\right)^{\beta_{1320}}\right]}dt$
k_{1320}	$\int_0^t \overline{F_{1316}F_{1314}}dF_{1320}$	$\frac{\beta_{1320}}{(\theta_{1320})^{\beta_{1320}}}\int_0^t t^{(\beta_{1320}-1)}e^{-\left[\left(\frac{t}{\theta_{1314}}\right)^{\beta_{1314}}+\left(\frac{t}{\theta_{1316}}\right)^{\beta_{1316}}+\left(\frac{t}{\theta_{1320}}\right)^{\beta_{1320}}\right]}dt$
k_{1412}	$F_{1412}(t)$	$1-e^{-(\mu_{1412}t)}$
k_{159}	$F_{159}(t)$	$1-e^{-(\mu_{159}t)}$
k_{1610}	$F_{1610}(t)$	$1-e^{-(\mu_{1619}t)}$
k_{171}	$F_{171}(t)$	$1-e^{-(\mu_{171}t)}$
k_{182}	$F_{182}(t)$	$1-e^{-(\mu_{182}t)}$
k_{194}	$F_{194}(t)$	$1-e^{-(\mu_{194}t)}$
k_{205}	$F_{205}(t)$	$1-e^{-(\mu_{205}t)}$

Table 9. Sojourn time expressions for each system state

T_i	Sojourn time expressions	Sojourn time expressions with parameter values
T_1	$\int_0^\infty \overline{F_{12}F_{14}F_{19}}dt$	$\int_0^\infty e^{-\left[\left(\frac{t}{\theta_{12}}\right)^{\beta_{12}}+\left(\frac{t}{\theta_{14}}\right)^{\beta_{14}}+\left(\frac{t}{\theta_{19}}\right)^{\beta_{19}}\right]}dt$
T_2	$\int_0^\infty \overline{F_{23}F_{25}F_{210}}dt$	$\int_0^\infty e^{-\left[\left(\frac{t}{\theta_{23}}\right)^{\beta_{23}}+\left(\frac{t}{\theta_{25}}\right)^{\beta_{25}}+\left(\frac{t}{\theta_{210}}\right)^{\beta_{210}}\right]}dt$
T_3	$\int_0^\infty \overline{F_{31}}dt$	$\int_0^\infty e^{-(\mu_{31}t)}dt$
T_4	$\int_0^\infty \overline{F_{45}F_{47}F_{412}}dt$	$\int_0^\infty e^{-\left[\left(\frac{t}{\theta_{45}}\right)^{\beta_{45}}+\left(\frac{t}{\theta_{47}}\right)^{\beta_{47}}+\left(\frac{t}{\theta_{412}}\right)^{\beta_{412}}\right]}dt$
T_5	$\int_0^\infty \overline{F_{56}F_{58}F_{513}}dt$	$\int_0^\infty e^{-\left[\left(\frac{t}{\theta_{56}}\right)^{\beta_{56}}+\left(\frac{t}{\theta_{58}}\right)^{\beta_{58}}+\left(\frac{t}{\theta_{513}}\right)^{\beta_{513}}\right]}dt$
T_6	$\int_0^\infty \overline{F_{64}}dt$	$\int_0^\infty e^{-(\mu_{64}t)}dt$
T_7	$\int_0^\infty \overline{F_{71}}dt$	$\int_0^\infty e^{-(\mu_{71}t)}dt$

T_8	$\int_0^\infty \overline{F_{82}}dt$	$\int_0^\infty e^{-(\mu_{82}t)}dt$
T_9	$\int_0^\infty \overline{F_{910}F_{912}F_{917}}dt$	$\int_0^\infty e^{-\left[\left(\frac{t}{\theta_{910}}\right)^{\beta_{910}}+\left(\frac{t}{\theta_{912}}\right)^{\beta_{912}}+\left(\frac{t}{\theta_{917}}\right)^{\beta_{917}}\right]}dt$
T_{10}	$\int_0^\infty \overline{F_{1011}F_{1013}F_{1018}}dt$	$\int_0^\infty e^{-\left[\left(\frac{t}{\theta_{1011}}\right)^{\beta_{1011}}+\left(\frac{t}{\theta_{1013}}\right)^{\beta_{1013}}+\left(\frac{t}{\theta_{1018}}\right)^{\beta_{1018}}\right]}dt$
T_{11}	$\int_0^\infty \overline{F_{119}}dt$	$\int_0^\infty e^{-(\mu_{119}t)}dt$
T_{12}	$\int_0^\infty \overline{F_{1213}F_{1215}F_{1219}}dt$	$\int_0^\infty e^{-\left[\left(\frac{t}{\theta_{1213}}\right)^{\beta_{1213}}+\left(\frac{t}{\theta_{1215}}\right)^{\beta_{1215}}+\left(\frac{t}{\theta_{1219}}\right)^{\beta_{1219}}\right]}dt$
T_{13}	$\int_0^\infty \overline{F_{1314}F_{1316}F_{1320}}dt$	$\int_0^\infty e^{-\left[\left(\frac{t}{\theta_{1314}}\right)^{\beta_{1314}}+\left(\frac{t}{\theta_{1316}}\right)^{\beta_{1316}}+\left(\frac{t}{\theta_{1320}}\right)^{\beta_{1320}}\right]}dt$
T_{14}	$\int_0^\infty \overline{F_{1412}}dt$	$\int_0^\infty e^{-(\mu_{1412}t)}dt$
T_{15}	$\int_0^\infty \overline{F_{159}}dt$	$\int_0^\infty e^{-(\mu_{159}t)}dt$
T_{16}	$\int_0^\infty \overline{F_{1610}}dt$	$\int_0^\infty e^{-(\mu_{1619}t)}dt$
T_{17}	$\int_0^\infty \overline{F_{171}}dt$	$\int_0^\infty e^{-(\mu_{171}t)}dt$
T_{18}	$\int_0^\infty \overline{F_{182}}dt$	$\int_0^\infty e^{-(\mu_{182}t)}dt$
T_{19}	$\int_0^\infty \overline{F_{194}}dt$	$\int_0^\infty e^{-(\mu_{194}t)}dt$
T_{20}	$\int_0^\infty \overline{F_{205}}dt$	$\int_0^\infty e^{-(\mu_{205}t)}dt$

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