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## SELECTED PROBLEMS OF AMBIGUITY OF THE DUAL PRICE OF WATER IN THE POST-OPTIMIZATION ANALYSIS OF THE WATER SUPPLY SYSTEM

### WYBRANE PROBLEMY NIEJEDNOZNACZNOŚCI CENY DUALNEJ WODY W POSTOPTYMALIZACYJNEJ ANALIZIE SYSTEMU WODOCIĄGÓW\*

*In literature it is believed that the dual price of water is an objective premise for shaping the market price of water. However, the authors note that a single vector of dual prices in the distribution of water, when ambiguous, should not become the basis for making decisions both regulating the price of water and affecting the procedures for modernizing the water supply network. This work cautions water management engineers not to duplicate common software errors and indicates how, despite the complete lack of literature tips, the technical problems encountered could be practically solved. The linear dependence of the row vectors of the left-hand parameters of binding constraints in the linear programming model for water consumption is identified here as the reason for the ambiguity of dual price vectors. This ambiguity in the issues of water distribution requires shaping alternative technical scenarios allowing for a variant selection of the method for modifying the water abstraction system. Therefore, the principles for determining the proportionality of simultaneous changes in certain parameters of the right-hand conditions of constraint conditions are described. These principles for the optimal selection of the most productive vectors for the parametric linear programming method were formulated and indicated on a simplified model of water distribution. The methodology developed in the work enables, among others, generating alternative technical scenarios for saving varying amounts of water, resulting in various financial savings.*

**Keywords:** water distribution network, dual price of water, operation process, water management model, post-optimization.

*W literaturze uważa się, że cena dualna wody jest obiektywną przesłanką do kształtowania rynkowej ceny wody. Jednak autorzy zauważają, że pojedynczy wektor cen dualnych w dystrybucji wody, gdy jest niejednoznaczny, nie powinien stać się podstawą do podejmowania decyzji zarówno normującej cenę wody jak i wpływającej na procedury modernizujące sieć wodociągową. Praca uczyła inżynierów gospodarki wodnej by nie powielali powszechnych błędów oprogramowania oraz wskazuje jak, pomimo kompletnego braku literaturowych wskazówek, praktycznie rozwiązywać napotykaną problemy techniczne. Liniowa zależność wektorów wierszowych parametrów lewych stron wiążących warunków ograniczających w modelu programowania liniowego dla zużycia wody identyfikowana jest tu jako przyczyna niejednoznaczności wektorów cen dualnych. Ta niejednoznaczność w zagadnieniach dystrybucji wody wymaga kształtowania alternatywnych scenariuszy technicznych pozwalających na wariantowy wybór sposobu modyfikacji systemu poboru wody. Dlatego opisano zasady wyznaczania proporcjonalności jednoczesnych zmian niektórych parametrów prawych stron warunków ograniczających. Na uproszczonym modelu dystrybucji wody sformułowano i wskazano te zasady optymalnego doboru najbardziej produktywnych wektorów dla metody parametrycznego programowania liniowego. Opracowana w pracy metoda umożliwia m.in. wygenerowanie alternatywnych scenariuszy technicznych oszczędzania różnej ilości wody, skutkującej różnymi oszczędnościami finansowymi.*

**Słowa kluczowe:** sieć dystrybucji wody, cena dualna wody, proces eksploatacji, model gospodarowania wodą, postoptymalizacja.

## 1. Introduction

### 1.1. Fixing the price of water in order to balance water demand

An analysis conducted by the World Bank [64] indicates that water shortages in some regions may reduce GDP by up to 6% and lead to increased migration and in some cases to a greater risk of conflict. Droughts and periods of water scarcity have become a more common and more frequent phenomenon in Europe [17]. The drought experiences in Europe in 2011, 2012, 2015 and 2018 were the worst in a

century and affected not only Southern and Western Europe, but also the countries in Northern Europe (including Great Britain, France, Germany, Sweden and Poland). The difference between water supply and its growing demand also determines the key limitations of China's economic development. It is estimated that before 2005 due to a shortage of water in production, China lost \$ 28 511 million annually. Brown [7] concluded that the shortage of water in China will soon be a threat to global cereal demand. Therefore, water, alongside cereals and crude oil, is referred to as a strategic resource. Hence, it is suggested that public authorities should shape water pricing for users in order to reflect its true shortage or alternative costs [15].

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## 1.2. Methods of optimization and post-optimization in the planned exploitation of water resources

Optimization and post-optimization methods can be used as a tool allowing for dynamic correction and improvement of the operation of complex systems. These methods require two approaches: mathematical and managerial [29-30]. The tools and principles of optimization enabled the development of normative models for optimal management of large-scale water systems, taking into account the ubiquitous uncertainty in forecasting natural processes and economic effects [13]. Numerous optimization models are used to ensure high parameters of network reliability, water quality reliability parameters, appropriate operational schemes with considerations for numerous hydraulic limitations in the form of: *hydraulic head*, leakage, changes in the energy consumption of a pump and sequential discreet pump operation which minimises operational costs. Most optimization models use mathematical techniques such as linear programming (*LP*), dynamic programming (*DP*) and nonlinear programming (*NLP*) or variations thereof [2, 10, 13-14, 19, 22, 25, 53]. Some models of problems with reliability or optimization of the cost and maintenance time of a water supply or sewage system are constructed on the statistical-stochastic basis [3, 38]. Romaniuk [44-46] presents numerous numerical experiments focusing, for example, on optimizing the value of expected costs and duration of exploiting a water supply network, when the decision-making parameter is unconditional exchange time, i.e. when it is better to replace a network fragment instead of performing its another repair in the future.

Models of reliability, readiness and safety in connection with linear programming are useful in the identification and prediction of reliability, readiness and safety of complex technical systems as well as in the optimization and analysis of the operating costs of these systems [28, 32-33, 66]. The uncertainty of the parameters of a typical linear programming model indicated here requires, in addition to an optimization procedure, easy access to post-optimization procedures [56, 58].

Freire-González et al. [18] review literature on the existing research related to input-output models to assess the economic impact of water scarcity during drought and linear input-output (*IO-LP*) methods in the approach to water resource planning in the context of drought and water shortage. Gibbons [22] and Liu et al. [36] point out that correct pricing for water resources, reflecting their real value, is very important for saving water and for mitigating water shortages. They further recognize the dual price of water as the one that should serve to quantify the actual value of water resources, which also reflects their shortage. They do so despite the fact that it is sometimes claimed that it is practically impossible to obtain a dual water price by solving a linear programming model. However, Liu et al. [36] combining the input-output analysis method with the *LP* method, developed a model with limitations imposed on final demand, total production, trade balance and water availability. This model was used to estimate the dual price of water. These results constitute a valuable reference for determining reasonable prices for industrial and productive water in the areas of the nine main Chinese river basins. A review of analogous studies on modelling the value of water in various sectors of the economy in South Africa is carried out by Nieuwoudt and Backeberg [42] and in the world by Conradie and Hoag [11]. These studies were carried out as a result of the emergence of demand for models measuring the willingness to pay for water used, e.g., for irrigating agricultural crops.

## 1.3. Problems of linear programming in the modelling of water resources

A typical system of water resources consists of water reservoirs, hydro power station, irrigated land, artificial and navigational channels, etc. being within the range of a river or basin. Therefore, optimal

planning of a multi-purpose water resource system, i.e. designing the "best" system, which is to be built and used in the planning horizon, is subordinated to, among others, technical, economic, financial, social and political restrictions. These restrictions include seasonal fluctuations in water supply, geographical and geological conditions of selected locations, existence of capital, loans, labour and local services, interest rate (and its trends), regional development plans, etc. [25]. The quoted authors used the *LP* model for a very complex water resources system, taking into account a number of constraints related to the reservoir, irrigation, hydro power station, artificial sewage and navigation limitations. The cases of rivers in southern Argentina were investigated, and these typical problems were described by about 300 constraints and 300 variables. With such a high number of constraints, it is not difficult to find linearly dependent row vectors for left-hand parameters of constraint conditions, and this already generates problems in the post-optimization analysed here.

McKee et al. [41] developed a model of an aquifer exploitation process through more than 900 wellbores, mainly for the needs of industry, municipal supply and crop irrigation in Arkansas. They took into account three variants of the *LP* models to simulate optimized outflows of surface and ground water, while simultaneously retaining the stream flow rate and numerous hydraulic limitations. Also this complex issue poses the above-mentioned problems with linear dependence.

Techniques for multi-criteria optimization, e.g. for a contaminated aquifer, come down to a single-criterion optimization through the use of the weighted sum method or the method of constraints [16, 27, 40]. In the latter of these methods, one of the objective functions is optimized by using other objective functions as a constraint, i.e. including them into the constraint conditions of the model, thereby increasing the number of constraints. This usually leads to the problem of linear dependence, analysed in this work and associated with the excessive number of constraint conditions.

Abdy Sayyed et al. [1] optimize the water distribution network by minimising network costs under constraint conditions resulting from pressure requirements at all nodes. Due to the high number of constraint conditions, which complicates the issue, already in the optimal design some constraint conditions are replaced there by an additional penalty in the objective function. The penalty is applied for failing to meet pressure constraints. In the cited work, three methods of penal inference were applied. This denotes a tendency, which sometimes emerges in the literature, to escape from an excess of constraint conditions. But this requires proper selection of the penalty function.

Frizzone et al. [19] in order to maximise the net income for several crops subjected to water and crop area access constraints, perform linearization of the non-linear objective function. Linearization is a typical tool used to carry out the optimization process [37].

## 1.4. Critical evaluation of some of the results presented in the literature

Numerous studies [26, 29-30, 40, 48, 55-58] indicate the necessity of caution when using *LP* methods. This is mainly due to the ambiguity of sensitivity reports containing dual price vectors [55-58]. In this work, it is noted that the ambiguity of dual price vectors is a consequence of the linear dependence of the row vectors of the parameters of the left-hand sides binding the constraint conditions of the *LP* model. This dependence must take place when the number of  $m_0$  of binding constraint conditions exceeds the number of decision variables  $n$ . But this dependence is not usually controlled by analysts. Koltai and Terlaki [29] indicate that it appears almost always and for a small number  $m$  of constraint conditions. If the problem of the ambiguity of the dual price vector does not occur in *dual model (DM)*, then the cyclicity of post-optimization procedures will quickly lead to it. It is noted that in the literature cited, water management models

contain  $m$  and  $n$  values, very often numbering even hundreds or thousands. And yet, already for their small values, this should inspire great caution in analysts when operating with dual prices. A lack of necessary criticism of the authors of many works who use dual prices for economic and technical issues is noticeable. And so, a single ambiguous vector of dual prices in water distribution should not become the basis for making decisions both regulating the market price of water and affecting the procedures modernising the water supply network. In this work, the need to consider several alternative technical decision scenarios based on some nodal solutions in the dual model to the original  $LP$  model, is signalled. This is where the methodology for creating these alternative technical decision scenarios is formulated. The authors note that the dubiousness in the literature on the usefulness of dual prices that can be seen so far is probably related to both the lack of knowledge about their ambiguity and the lack of practical methods for utilising this fact. The works [24, 55-58] may be an exception here. In section 2, on a simple example of a water supply network with infinitely many dual price vectors, it is suggested how to propose the framework for alternative modernization scenarios of a previously optimized network.

### 1.5. Interpretation of dual prices. Problems in the sensitivity analysis

Sensitivity analysis reports for  $LP$  indicate the features determining the choice of the optimal decision variant:

1. they describe some of the simple effects caused by deviation from the optimal plan (i.e. the height of the marginal increment, that is the amount by which the optimal value of the objective function and the scope of the correction in this amount should be adjusted);
2. they indicate how long one should refrain from changing the variant of the optimal decision when changing individual parameters of the linear objective function;
3. they indicate, by means of dual prices, how the optimal value of the objective function will change when changing (not necessarily single) parameters of right-hand sides of particular constraint conditions in a certain scope;
4. they indicate whether it is profitable for the company to increase the availability of a certain resource by a certain number of units (analysis of "more for less" Arsham [4]).

In general, the dual price measures the change in the value of the objective function resulting from the increased availability of a specific resource by a unit, usually with a clearly understated and implied assumption that the remaining "deficit" resources will not change. Each limited resource is then accounted for by a separate dual price. Each time, the impact of the change in the amount of each single "deficit" factor is examined (i.e. the one for which the constraint is binding). The dual prices remain constant until the set of binding limitations of the optimal solution changes. Each of these prices, measures the value of benefits from expanding production capacity or losses resulting from their reduction. In other words, the dual price corresponding to the right-hand side of a particular constraint condition indicates how much the value of the objective function will change when the limitation is relaxed. If a specific production factor, i.e. a certain resource, is not fully utilized in the optimal solution (i.e. it is not „deficit“, and it does not constitute a binding), then it has a dual price equal to zero. It may be partially used after increasing other resources, but it does not have to be deficient. However, as it has been shown in the present work, any other factor with a dual price equal to zero, once it has been fully consumed (i.e. it has become „deficit“ because it constitutes a binding constraints), may be additionally required in the proportion subjected to estimation in order to increase the amount of certain resources with a non-zero dual price. The analysis of changes not only for a single parameter of the right-

hand sides of constraint conditions is the subject of numerous studies. A broad review of the literature in this area is presented by Shahin et al. [48]. The classification performed in this work allows for distinguishing – apart from ordinary sensitivity analysis – 7 other types of post-optimization analyses: 1) the rule of 100% [6], 2) "symmetric tolerance" – [59-61], 3) "non-symmetric tolerance" which is an extension of the symmetric tolerance, and introduced by Arsham and Oblak [5], Wondolowski [63] and Wendell [62], 4) ( $PLP$ ) parametric linear programming [21, 47], 5) multiparametric linear programming [54], 6) sensitivity analysis with the functional dependence of the parameters of right-hand sides of constraint conditions or the objective function coefficients [23], 7) sensitivity analysis with the correlation of the above mentioned parameters [48]. Arsham's [4] work compares most of the above-mentioned methods through the construction of the largest sensitivity region for the general  $LP$ . Thus, Arsham [4] indicates most of these types as special cases in his analysis. Nevertheless, the question of many special cases, especially degenerated ones, remains unresolved, as Arsham [4] clearly indicates. It should be noted here that the preservation of certain (preferably optimal) proportions when increasing resources is the basis of the (optimal)  $PLP$ . Otherwise, part of the increased resources may remain unused, i.e. unproductive. Thus, the need to formulate the principles of optimal selection of the most productive vectors for  $PLP$  is recognized, which is the subject of the present work. Another problem, unresolved in the literature, is the practical usefulness of ambiguous sensitivity analysis reports, or even the fact that they are shown by popular calculation packages [26]. The fact that there are infinitely many solutions to the dual model [55-56] makes a sensitivity report most often unhelpful for an average analyst due to problems with the interpretation of ambiguous reports obtained. So far, this last problem has been described only partially and only in individual cases of transport models [57-58]. The transport models presented there are a special case of sensitivity analysis with the functional dependence of the parameters of the right-hand sides of constraint conditions [23].

Therefore, the authors of this work formulate the principles of determining the proportionality of simultaneous changes in the parameters of the right-hand sides of the constraint conditions in the case of ambiguous sensitivity analysis reports. At the same time, they identify the linear dependence of the row vectors of the left-hand side parameters of binding conditions as the reason for the ambiguity of dual prices. Under these conditions, they use various reports of ordinary sensitivity analysis available through commonly accessible software. They indicate the difference in the interpretation of the dual price corresponding to the first constraint condition for each of the sensitivity analysis reports (tables 2, 3, 6). This difference consists in the fact that the unit change of the right-hand side of the first constraint condition for each report forces the simultaneous, respectively proportional change of the right-hand side of other constraint conditions. In the example considered, the authors indicate that savings in water consumption by one unit may require simultaneous execution of one of two alternative scenarios of technical activities. At the same time, each of these two technical scenarios saves a different amount of water and results in other financial savings. The choice between these scenarios requires the inclusion of additional information not specified in this example. In the purpose of a clear presentation of the new methodology, useful in the operational processes of many complex technical systems, consideration was limited to conducting a study of a simplified model of water management.

## 2. A study of a simplified water management model

The authors have already indicated above that each  $LP$  model for a water supply network, with a single ambiguous solution of a dual task, should not be the basis for shaping market water prices or modifying parameters of this network without constructing several alternative

technical scenarios. And a full analysis of the already small *LP* model leads to many side threads, which are not significant for the presentation of the methodology of creating the foundations of alternative scenarios of technical procedure. Thus, in the simplified example below, a detailed interpretation was attached to only two constraint conditions with a partial interpretation of the majority of other constraint conditions of the *LP* model. It is a deliberate effort of the authors to point out the wide range of suitability of the methodology below, highlighting the importance of reducing water consumption and the costs of obtaining it. Making small changes in interpretations, the following example may equally well refer to a local water supply system, which is connected with three water abstraction nodes from a larger water supply system of a large urban and industrial agglomeration [cf. 38]. Such a model of water demand can be supplemented, among others, with the dynamics of seasonal changes and weekly rhythm [cf. 35]. Then, some parameters of the model change and the interpretation of particular constraint conditions change. In each of these cases the problem of a detailed interpretation of individual constraints will have to be solved individually, but each time the same problems resolved here will be revisited: 1) clear description of an incalculable set of all ambiguous dual prices, 2) attaching a practical interpretation to this ambiguity, e.g. through formulation of alternative post-optimization scenarios for the water supply network. In order to overcome these difficulties, an analyst of urban logistics systems should familiarize himself with the mathematical formalism presented here, describing the methodology of the transition from the answer reports and ambiguous sensitivity reports typical for *LP* to postulated scenarios. To make it clear, it is necessary to sketch a model for a small and not too complicated example of water distribution, which does not expose technical problems unnecessarily, but focuses on improving the methodology of engineering inference.

Therefore, in order to obtain simplicity and fix the attention on methodologies expanding the post-optimization, let us assume that the analysed urban-industrial agglomeration is supplied with water by three water abstractions located along the same watercourse in water abstraction sites  $P_i$  for  $i=1,2,3$  (as in Fig. 1).



Fig. 1. Three water abstractions for an urban and industrial agglomeration located along the same watercourse, water gauge cross-section

The daily distribution of water in the agglomeration is a variable. Therefore, the amount of water sourced from the intake No.  $i$  is a decision variable  $x_i \geq 0$  for  $i=1,2,3$  and periodically it can be determined in different units, e.g.  $[\text{dm}^3 \cdot \text{s}^{-1}]$ ,  $[\text{dm}^3 \cdot \text{h}^{-1}]$ ,  $[\text{m}^3 \cdot \text{d}^{-1}]$ ,  $[\text{m}^3 \cdot \text{month}^{-1}]$ ,  $[\text{m}^3 \cdot \text{year}^{-1}]$ . Moreover, let us assume that the current periodic agglomeration demand for water varies and is ranging between 200 and 300 units, which can be interpreted as constraints in the form of:

$$CO_1: x_1 + x_2 + x_3 \geq 200 \quad CO_2: x_1 + x_2 + x_3 \leq 300$$

Due to the principle of the inviolability of water flow in the river through the water gauge cross-section "S", or technical considerations, the dependencies between intake amounts are expressed as the following limitations:

$$CO_3: 9 \cdot x_1 + 11 \cdot x_2 + 7 \cdot x_3 \leq 1800 \quad CO_4: 5 \cdot x_1 + 6 \cdot x_2 + 4 \cdot x_3 \geq 1000 \\ CO_5: 2 \cdot x_1 + 4 \cdot x_2 + 3 \cdot x_3 \geq 600 \quad CO_6: 0 \cdot x_1 + 2 \cdot x_2 + 1 \cdot x_3 \geq 200$$

At the same time, *instream flow* or *minimum acceptable flow* is defined as the amount of water, which should be left in the cross-section of a given stream due to biological, ecological and social considerations. The necessity of preserving this flow should not be subject to economic assessments. Therefore, the resources taken into account in water balance should be reduced by the amount of instream flow. The hydrobiological criterion determines the minimum flow needed for supporting the life of flora and fauna in the aquatic environment. The environmental criterion determines the minimum flow to maintain the level balance of surface and underground waters within national parks, nature reserves and landscape protection areas. The fishing and angling criterion defines the minimum flow allowing fish to develop. The criterion of sport and water tourism determines the minimum water levels and corresponding flows making water tourism feasible [9-10, 43]. What also plays a significant role in the discussed hydrological issues, is the technical criterion relating to the technological capabilities of the water abstraction system which requires specialised equipment. Also, some of such limitations can be related to the reliable functioning of key economic areas, such as the power generation industry and water transport.

The total cost of water abstraction from these three intakes is described by the minimised objective function  $OF: c \cdot x \rightarrow \min$ . Thereby, the vector of cost factors assumes the form of:

$$c = [c_1 \quad c_2 \quad c_3] = [3 \quad 4 \quad 3] \quad \text{and} \quad x = [x_1 \quad x_2 \quad x_3]^T$$

determines the vector of decision variables. In order to optimize the decision variables, the *Solver* plugin in the *Excel* application was used (see tables 1 and 2). With the constraint conditions  $CO_j$  for  $j=1, \dots, 6$  the vector of optimal decision variables:

$$x^* = [x_1^* \quad x_2^* \quad x_3^*]^T = [66.(6) \quad 66.(6) \quad 66.(6)]^T$$

indicates that three identical water abstraction amounts from three intakes determine the minimum value of the objective function at the level of 666.67 monetary units. The solution obtained will remain unchanged when the unit cost of water abstraction from the  $P_1$  intake remains in the range of 2 to 3.5 monetary units and the cost of water consumption from other sources does not undergo change. Therefore, for example, when the unit cost of water abstraction from this intake increases by 0.3 monetary units, the optimal  $x^*$  solution will not change, but the total cost will increase by 20 monetary units. Similarly, in the upper part of the table 2, we can observe acceptable changes of the remaining singular parameters of the total function of costs that allow the optimal  $x^*$  solution to remain unchanged and for calculation of the corresponding variable of the total cost of water abstraction.

Let us assume that the critical condition of the system forces a decrease in water abstraction, i.e. an infringement of the right-hand of  $CO_1$ . Which other parameters of the model should be used and how should they be modified in order to make the reduction in water abstraction feasible and associated with optimal cost reduction?

Due to the 6 constraint conditions with 3 decision variables, we conclude that the row vectors for the parameters of the left-hand side of the constraints are linearly dependent. In addition, due to the 5 binding constraint conditions and 1 non-binding one, shown in table 1, with 3 decision variables, we conclude that the solution to the primary task is unambiguous, but the dual solutions form a certain

Table 1. Optimum water abstraction for an urban and industrial agglomeration from three water abstractions – PM. A report on results in the Excel application

Objective Cell (Min)				
Cell	Name	Original Value	Final Value	
\$E\$10	Total Cost	10	666.6666667	

Variable Cells				
Cell	Name	Original Value	Final Value	Integer
\$A\$2	x1	1	66.66666667	Contin
\$B\$2	x2	1	66.66666667	Contin
\$C\$2	x3	1	66.66666667	Contin

Constraints					
Cell	Name	Cell Value	Formula	Status	Slack
\$E\$4	Left Side Constraint 1	200	\$E\$4 >= \$G\$4	Binding	0
\$E\$5	Left Side Constraint 2	200	\$E\$5 <= \$G\$5	Not Binding	100
\$E\$6	Left Side Constraint 3	1800	\$E\$6 <= \$G\$6	Binding	0
\$E\$7	Left Side Constraint 4	1000	\$E\$7 >= \$G\$7	Binding	0
\$E\$8	Left Side Constraint 5	600	\$E\$8 >= \$G\$8	Binding	0
\$E\$9	Left Side Constraint 6	200	\$E\$9 >= \$G\$9	Binding	0

Table 2. A report on optimization sensitivity in the primary model for example in the Excel application

Variable cells						
Cell	Name	Final Value	Reduced Cost	Coefficient Objective	Allowable Increase	Allowable Decrease
\$A\$2	x1	66.66666667	0	3	0.5	1
\$B\$2	x2	66.66666667	0	4	0.285714286	1
\$C\$2	x3	66.66666667	0	3	1E+30	0.25

Constraints						
Cell	Name	Final Value	Price dual	Constraint R. H. Side	Allowable Increase	Allowable Decrease
\$E\$4	Left Side Constraint 1	200	0	200	0	1E+30
\$E\$5	Left Side Constraint 2	200	0	300	1E+30	100
\$E\$6	Left Side Constraint 3	1800	-0.66666667	1800	0	1.42109E-14
\$E\$7	Left Side Constraint 4	1000	1.66666667	1000	7.10543E-15	0
\$E\$8	Left Side Constraint 5	600	0.33333333	600	100	2.84217E-14
\$E\$9	Left Side Constraint 6	200	0	200	2.84217E-14	1E+30

subset of 2-dimensional space. Therefore, in order to carry out correct post-optimization, the dual model should be thoroughly examined in comparison with the primary model (PM) presented here [24, 52]. This is due to the ambiguity of the results of the dual optimization model and the resulting problematic nature of inference. This means that the infinite number of results of the dual optimization model poses problems in the interpretation of business and technical issues [55-56]. Obtaining many nodal solutions of the dual model on the basis of PM constitutes a certain problem. One method to obtain more nodal solutions is to change the order in which the constraint conditions are introduced [24]. However, it requires examination of a significant number of permutations out of as many as  $6! = 720$ . It is helpful to optimize the dual model to PM. The DM takes the form of [12, 62]:

$$OF_{DM}: 200 \cdot y_1 + 300 \cdot y_2 + 1800 \cdot y_3 + 1000 \cdot y_4 + 600 \cdot y_5 + 200 \cdot y_6 \rightarrow \max$$

$$CO_{DM1}: 1 \cdot y_1 + 1 \cdot y_2 + 9 \cdot y_3 + 5 \cdot y_4 + 2 \cdot y_5 + 0 \cdot y_6 \leq 3$$

$$CO_{DM2}: 1 \cdot y_1 + 1 \cdot y_2 + 11 \cdot y_3 + 6 \cdot y_4 + 4 \cdot y_5 + 2 \cdot y_6 \leq 4$$

$$CO_{DM3}: 1 \cdot y_1 + 1 \cdot y_2 + 7 \cdot y_3 + 4 \cdot y_4 + 3 \cdot y_5 + 1 \cdot y_6 \leq 3$$

$$BR_{DM}: y_j \leq 0 \text{ for } j = 2, 3, y_j \geq 0 \text{ for } j = 1, 4, 5, 6.$$

$CO_2$  and  $CO_3$  are non-standard inequalities in PM, therefore, the corresponding dual decision variables in DM are non-positive [49, p. 104]. Since  $CO_2$  is a non-binding constraint condition, the corresponding value of the dual decision variable  $y_2$  is zero. The sensitivity report for PM obtained in the Excel application (table 2) the dual

price column contains one of the optimal solutions in DM and it takes the form of:

$$y_A^* = [y_{A1}^*, y_{A2}^*, y_{A3}^*, y_{A4}^*, y_{A5}^*, y_{A6}^*]^T = [0 \ 0 \ -0.(6) \ 1.(6) \ 0.(3) \ 0]^T = 1/3 \cdot \bar{y}_A^*$$

And in the sensitivity report for DM obtained in the Excel application (table 3) the final value column contains another optimal solution in DM and takes the form of:

$$y_B^* = [y_{B1}^*, y_{B2}^*, y_{B3}^*, y_{B4}^*, y_{B5}^*, y_{B6}^*]^T = [0.(6) \ 0 \ 0 \ 0.(3) \ 0.(3) \ 0]^T = 1/3 \cdot \bar{y}_B^*$$

The limits of acceptable increase and decrease in the value  $b$  of the right-hand side of PM constraint conditions can be read respectively from the lower part of the table 2 or from the upper part of the table 3.

Table 3. Sensitivity report in the dual model obtained in the Excel application

Variable cells						
Cell	Name	Final Value	Reduced Cost	Coefficient Objective	Allowable Increase	Allowable Decrease
\$A\$1	y1	0.666666667	0	200	0	0
\$B\$1	y2	0	100	300	1E+30	100
\$C\$1	y3	0	0	1800	1E+30	0
\$D\$1	y4	0.333333333	0	1000	0	200
\$E\$1	y5	0.333333333	0	600	100	0
\$F\$1	y6	0	0	200	0	1E+30

Constraints						
Cell	Name	Final Value	Price dual	Constraint R. H. Side	Allowable Increase	Allowable Decrease
\$H\$3	Left Side Constraint 1	3	66.66666667	3	0.5	1
\$H\$4	Left Side Constraint 2	4	66.66666667	4	0.285714286	1
\$H\$5	Left Side Constraint 3	3	66.66666667	3	0.5	0.25

Table 4. The vector  $y_c^*$  of dual prices and the corresponding limits of the acceptable increase and decrease of the value of  $b$  the right-hand side of the constraint conditions in PM; based on the last matrix of the simplex method obtained according to [65]

Name	Price dual	Constraint R. H. Side	Allowable Increase	Allowable Decrease
Left Side Constraint 1	0	200	0	1E+30
Left Side Constraint 2	0	300	1E+30	100
Left Side Constraint 3	-1.3333333	1800	0	0
Left Side Constraint 4	3	1000	0	0
Left Side Constraint 5	0	600	0	1E+30
Left Side Constraint 6	0.3333333	200	0	0

A slightly different vector of dual prices (see table 4):

$$y_C^* = [y_{C1}^*, y_{C2}^*, y_{C3}^*, y_{C4}^*, y_{C5}^*, y_{C6}^*]^T = [0 \ 0 \ -1.(3) \ 3 \ 0 \ 0.(3)]^T = 1/3 \cdot \bar{y}_C^*$$

was obtained using the application available on the website [65] for the PM model. In addition, from the last matrix of the simplex method obtained there, the inverse matrix  $B^{-1}$  to the so-called base was read (supplemented with additional variables, in accordance with the simplex method rule) and the limits of acceptable increase and decrease of the value of  $b$  the right-hand side of PM constraints contained in the table 4 were determined by solving an appropriate inequality:

$$[x_3^* \ x_5^* \ x_1^* \ x_8^* \ x_4^* \ x_2^*]^T = B^{-1} (b + \Delta b) \geq 0$$

due to the vector  $\Delta \mathbf{b}$  of changes on the right-hand side of the constraints [49, p. 79].

For all  $\alpha, \beta, \gamma \in \langle 0, 1 \rangle$  such as  $\alpha + \beta + \gamma = 1$  any convex linear combination  $\alpha \cdot \mathbf{y}_A^* + \beta \cdot \mathbf{y}_B^* + \gamma \cdot \mathbf{y}_C^*$  of three vectors  $\mathbf{y}_A^*, \mathbf{y}_B^*, \mathbf{y}_C^*$  constitutes also a vector of dual solutions. Moreover, for the column vector  $\mathbf{b}$  of the right-hand side constraints  $PM$  in accordance with the Gale-Kuhn-Tucker theorem [20], we have:

$$\begin{aligned} \mathbf{c} \cdot \mathbf{x}^* &= \mathbf{b}^T \cdot \mathbf{y}_A^* = \mathbf{b}^T \cdot \mathbf{y}_B^* = \mathbf{b}^T \cdot \mathbf{y}_C^* = \\ &= \mathbf{b}^T \cdot [\alpha \cdot \mathbf{y}_A^* + \beta \cdot \mathbf{y}_B^* + \gamma \cdot \mathbf{y}_C^*] = \\ &= 1/3 \cdot \mathbf{b}^T \cdot [\alpha \cdot \tilde{\mathbf{y}}_A^* + \beta \cdot \tilde{\mathbf{y}}_B^* + \gamma \cdot \tilde{\mathbf{y}}_C^*] = \\ &= 1/3 \cdot \mathbf{b}^T \cdot [\tilde{\mathbf{y}}_B^* + \gamma \cdot (\tilde{\mathbf{y}}_C^* - \tilde{\mathbf{y}}_A^*) - (\alpha + \gamma) \cdot (\tilde{\mathbf{y}}_B^* - \tilde{\mathbf{y}}_A^*)] = \\ &= 1/3 \cdot \mathbf{b}^T \cdot [\tilde{\mathbf{y}}_B^* + \gamma \cdot \mathbf{w} + \lambda \cdot \mathbf{v}] = \mathbf{b}^T \cdot [\mathbf{y}^*(\gamma, \lambda)] = 666.6 \end{aligned} \quad (6)$$

where  $\lambda = \beta - 1 = -(\alpha + \gamma)$ ,  $\mathbf{w} = \tilde{\mathbf{y}}_C^* - \tilde{\mathbf{y}}_A^*$ ,  $\mathbf{v} = \tilde{\mathbf{y}}_B^* - \tilde{\mathbf{y}}_A^*$ , additionally  $\mathbf{b}^T \cdot \mathbf{w} = 0$  and  $\mathbf{b}^T \cdot \mathbf{v} = 0$  for  $\mathbf{w}^T = [0 \ 0 \ -2 \ 4 \ -1 \ 1]$  and  $\mathbf{v}^T = 2 \cdot [1 \ 0 \ 1 \ -2 \ 0 \ 0]$  is the orthogonality of vectors  $\mathbf{w}$  and  $\mathbf{v}$  to the vector  $\mathbf{b}$  of the right-hand side of the constraints in  $PM$ , which is simultaneously a vector of coefficients of the dual model objective function. Note that both vectors  $\mathbf{w}$  and  $\mathbf{v}$  orthogonal to the vector  $\mathbf{b}$  can be indicated as those rows of the matrix  $\mathbf{B}^{-1}$  inverse to the base matrix  $\mathbf{B}$ , which are orthogonal to the vector  $\mathbf{b}$  and thus in the product:

$$\mathbf{B}^{-1} \cdot \mathbf{b} = [x_3^* \ x_5^* \ x_1^* \ x_8^* \ x_4^* \ x_2^*]^T = [66.6(6) \ 100 \ 66.6(6) \ 0 \ 0 \ 66.6(6)]^T$$

they create zero values of dual variables remaining in the base in the last matrix of the simplex method. This means that the previous effort

to obtain one of the dual vectors  $\mathbf{y}_B^*$  and vectors  $1/3 \cdot \mathbf{w}$  and  $1/3 \cdot \mathbf{v}$  can be simplified using only the last matrix of the simplex method [58].

Therefore, the optimal solution  $[66.6(6) \ 100 \ 66.6(6) \ 0 \ 0 \ 66.6(6)]^T$  in  $PM$  with the simplex method is connected by infinitely many optimal solutions  $\mathbf{y}^*(\gamma, \lambda)$  in  $DM$ , which in the space  $\mathbf{R}^6$  form a two-dimensional convex fragment of this space with a parametric form:

$$\mathbf{y}^*(\gamma, \lambda) = 1/3 \cdot (\tilde{\mathbf{y}}_B^* + \gamma \cdot \mathbf{w} + \lambda \cdot \mathbf{v}),$$

where for each pair of parameters  $(\gamma, \lambda)$  the vector  $\mathbf{y}^*(\gamma, \lambda)$  meets  $BR_{DM}$ . Therefore, by solving the system of 6 inequalities:

$$\frac{1}{3} \cdot \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \cdot \begin{pmatrix} 0 \\ 0 \\ -2 \\ 4 \\ -1 \\ 1 \end{pmatrix} + \lambda \cdot 2 \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \geq 0 \\ \leq 0 \\ \leq 0 \\ \geq 0 \\ \geq 0 \\ \geq 0 \end{matrix},$$

we determine the range for parameters  $(\gamma, \lambda)$  in the form of a convex figure  $T$ . And so  $T = \{(\gamma, \lambda) \in \mathbf{R}^2 : 0 \leq \gamma \leq 1, -1 \leq \lambda \leq \gamma\}$ . The range  $T$  has a trapezoidal shape (Fig. 2) with ABCD vertices, where point B coincides with the beginning of the  $O\gamma\lambda$  coordinate system. The trapezium  $T$  contains the previously determined range of parameters  $T_0 = \{(\gamma, \lambda) \in \mathbf{R}^2 : 0 \leq \gamma \leq 1, -1 \leq \lambda \leq -\gamma\}$  in the form of a triangle with vertices ABC, corresponding to a convex linear combination of only three vectors:

$$\mathbf{y}^*(0; -1) = \mathbf{y}_A^*, \quad \mathbf{y}^*(0; 0) = \mathbf{y}_B^*, \quad \mathbf{y}^*(1; -1) = \mathbf{y}_C^*.$$

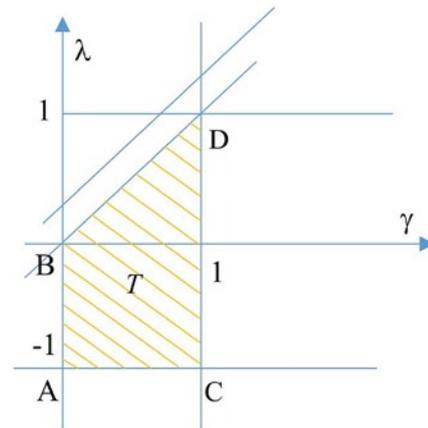


Fig. 2. The trapezium area  $T$  with ABCD vertices defines a range of parameters  $(\gamma, \lambda)$  for which the dual vector  $\mathbf{y}^*(\gamma, \lambda)$  is allowed in  $DM$

From here we obtain an additional nodal vector of dual prices:

$$\mathbf{y}^*(1; 1) = \mathbf{y}_D^* = [y_{D_1}^*, y_{D_2}^*, y_{D_3}^*, y_{D_4}^*, y_{D_5}^*, y_{D_6}^*]^T = [1.(3) \ 0 \ 0 \ 0.(3) \ 0 \ 0.(3)]^T = 1/3 \cdot \tilde{\mathbf{y}}_D^*$$

with unknown ranges of changes in the right-hand parameters of the  $CO$  corresponding to these prices.

For each  $(\gamma, \lambda) \in T$  in  $DM$ , the function value  $FC_{MD}(\gamma, \lambda)$  is constant and amounts to 666.67 and the vector  $\mathbf{y}^*(\gamma, \lambda)$  is an acceptable vector, i.e. all  $BR_{DM}$  conditions are simultaneously satisfied and the constraints of the dual model are then binding conditions in  $DM$ .

Moreover for  $j = 1, \dots, 6$  we obtain:

$$\mathbf{y}_j^*(\gamma, \lambda) \in \min \left( (y_{A_j}^*, y_{B_j}^*, y_{C_j}^*, y_{D_j}^*), \max (y_{A_j}^*, y_{B_j}^*, y_{C_j}^*, y_{D_j}^*) \right).$$

And from here and with tables 2 – 4 because  $y_2^*(\gamma, \lambda) \equiv 0$ , therefore, changing the upper limit of water demand by reducing the right-hand side in  $CO_2$  by 100 units or any increase therein will not change the optimal water abstraction plan  $\mathbf{x}^*$  or the total cost.

Analysing tables 2 – 4, we infer that there are two alternative ways to reduce the total costs: a) by using value of the dual variable  $y_{B_1}^* = 0.(6)$  with only an apparently unacceptable decline of  $b_1$  or b) using the doubled value of  $y_{D_1}^* = 1.(3)$  with the unknown allowed drop range of  $b_1$ .

Although  $y_1^*(\gamma, \lambda) \in (0; 1.(3))$ , perhaps the largest decrease in the total costs in the amount of  $0.(6) \cdot 25 \approx 16.67$  monetary units by reducing the lower limit of water demand can be obtained by allowing declining of the right-hand side value 200 of the first constraint

condition  $CO_1$  to be reduced by as many as supposedly 25 units when using the dual value  $y_{B_1}^* = y_1^*(0,0) = 0.(6)$  read from the upper table 3 for  $DM$  when  $[\alpha \ \beta \ \gamma] = [0 \ 1 \ 0]$ , i.e. when  $[\gamma \ \lambda] = [0 \ 0]$ . But such a single variation of  $CO_1$  may, however, contribute nothing because of the remaining 4 binding constraint conditions. Two tables inform about it: 2 and 4. This means that a single change on the right-hand side of  $CO_1$ , as in a typical sensitivity analysis, additionally requires changing the right-hand sides of other constraint conditions. As it turns out, such a proper suggestion results from the fact of unnatural blocking in table 3 both the increases and decreases in the parameters of the right side of  $CO_1$  corresponding to a non-zero dual price. Similar objections concern the unnatural total blocking of changes in the right-hand parameters of constraints for several non-zero dual prices in tables 2 and 4.

Determining the change in the total cost of water abstraction  $\Delta OF_{DM}(\Delta \mathbf{b})$ , depending on the known vector  $\Delta \mathbf{b} = [\Delta b_1 \ \Delta b_2 \ \Delta b_3 \ \Delta b_4 \ \Delta b_5 \ \Delta b_6]^T$  of changes of the right-hand side of the constraint conditions in the  $PM$ , most simply requires re-launching appropriate software. But here an analyst faces the opposite problem, because he is looking for the whole vector of changes  $\Delta \mathbf{b}$ , the most favourable (most productive) for  $y_{B_1}^*$ . In addition,  $CO_1$  is not the only binding constraint condition but there are up to 5 binding constraints for the three decision variables. Therefore, having tables 2 - 4 a post-optimization question arises, not only a mathematical but also a managerial one: *Which changes  $\Delta b_j$  on the right-hand sides of the constraint conditions, which take into account various technical and economic problems of water abstraction and the principle of water flow in the river, must be accompanied by:*

- a)  $y_{B_1}^* = 0.(6)$  and a decrease in the demand for  $\Delta b_1 = 25$  water units, i.e. savings  $0.(6) \cdot 25 \approx 16.67$  of monetary units, and which
- b) accompany twice bigger  $y_{D_1}^* = 1.(3)$  with a decrease in demand unknown here  $\Delta b_1$ ?

To answer the above questions, note that  $OF = OF_{DM} = \mathbf{c} \cdot \mathbf{x}^* = \mathbf{b}^T \cdot [\mathbf{y}^*(\gamma, \lambda)]$ , i.e. costs are fixed for any allowed parameter values  $\gamma$  and  $\lambda$ . But only for the properly selected fixed change vector  $\Delta \mathbf{b}$  also new costs, i.e. the value  $(\mathbf{b} + \Delta \mathbf{b})^T \cdot \mathbf{y}^*(\gamma, \lambda)$  is constant for any of the above-accepted values of parameters  $\gamma$  and  $\lambda$ . Hence, a vector  $\Delta \mathbf{b}$  should be selected so that the cost change, i.e. the value  $\Delta OF_{DM}(\Delta \mathbf{b}; \gamma, \lambda) = (\Delta \mathbf{b})^T \cdot \mathbf{y}^*(\gamma, \lambda)$  would not depend on the choice of parameters  $\gamma$  and  $\lambda$ . The last equality indicates how the whole vector  $\mathbf{y}^*(\gamma, \lambda)$  should be selected for the whole vector  $\Delta \mathbf{b}$  used here. In particular, because also for any chosen parameters  $(\gamma, \lambda) \in T$  we have:

$$\Delta OF_{DM}(\Delta \mathbf{b}; \gamma, \lambda) = 1/3 (\Delta \mathbf{b})^T \cdot [\tilde{\mathbf{y}}_B^* + \gamma \cdot \mathbf{w} + \lambda \cdot \mathbf{v}] = 1/3 [(\Delta \mathbf{b})^T \cdot \tilde{\mathbf{y}}_B^*] = \Delta OF_{DM}(\Delta \mathbf{b}; 0, 0),$$

when two conditions are met:  $(\Delta \mathbf{b})^T \cdot \mathbf{w} = 0$  and  $(\Delta \mathbf{b})^T \cdot \mathbf{v} = 0$ . Therefore, the last orthogonality conditions describe this allowed proper way of selecting the constituents of the change vector  $\Delta \mathbf{b}$ , corresponding to operating the whole vector here  $\mathbf{y}_B^*$ .

Ad a) *What if, as in the classical sensitivity analysis, one should use only the one single component (read from table 3 of  $y_{B_1}^* = y_1^*(0;0) = 0.(6)$  the vector  $\mathbf{y}_B^*$ , i.e. when:*

$$\Delta OF_{DM}(\Delta \mathbf{b}; \gamma, \lambda) = \Delta b_1 \cdot y_{B_1}^* ?$$

Because the last equality is equivalent to the condition:

$$\sum_{j=1}^6 (\Delta b_j \cdot y_{B_j}^*) = \Delta b_1 \cdot y_{B_1}^*,$$

i.e. a requirement is created whereby in addition to  $y_{B_1}^*$  the remaining non-zero components  $y_{B_j}^*$  of the vector  $\mathbf{y}_B^*$  do not affect the determined value of the change of the dual objective function. For this purpose, firstly, we accept  $\Delta b_4 = \Delta b_5 = 0$ . Secondly, for any chosen parameters  $\gamma$  and  $\lambda$  the equation must follow:

$$\begin{aligned} \Delta OF_{DM}(\Delta \mathbf{b}; \gamma, \lambda) &= \frac{1}{3} \cdot (\Delta \mathbf{b})^T \cdot [\tilde{\mathbf{y}}_B^* + \gamma \cdot \mathbf{w} + \lambda \cdot \mathbf{v}] = \\ &= \frac{1}{3} \cdot \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \\ 0 \\ 0 \\ \Delta b_6 \end{bmatrix}^T \cdot \left( \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} 0 \\ -2 \\ 4 \\ -1 \\ 1 \end{bmatrix} + \lambda \cdot \begin{bmatrix} 2 \\ 0 \\ 2 \\ -4 \\ 0 \\ 0 \end{bmatrix} \right) = \Delta OF_{DM}(\Delta \mathbf{b}; 0, 0) = \frac{2}{3} \cdot \Delta b_1. \end{aligned}$$

Hence conditions:  $(\Delta \mathbf{b})^T \cdot \mathbf{w} = 0$  and  $(\Delta \mathbf{b})^T \cdot \mathbf{v} = 0$  take the form of  $-2\Delta b_3 + \Delta b_6 = 0$  and  $2\Delta b_1 + 2\Delta b_3 = 0$ . Then if  $-25 \leq \Delta b_1 \leq 28$  and  $-100 \leq \Delta b_2$ , the cost of water intake will change by  $2/3 \cdot \Delta b_1$  when:

$$\Delta \mathbf{b} = \Delta b_1 \cdot [1 \ 0 \ -1 \ 0 \ 0 \ -2]^T + \Delta b_2 \cdot [0 \ 1 \ 0 \ 0 \ 0 \ 0]^T.$$

This means that a decrease in total costs by  $0.(6) \cdot 25 \approx 16.67$  monetary units through acceptable reduction of the minimum water requirement from 200 to 175 units can be obtained by concurrent (acceptable) increase of the right-hand side of the third (technical) constraint condition of  $CO_3$  by 25 units and increase of the right-hand side of the sixth (also technical) condition restricting  $CO_6$  by 50 units, with the appropriate limitation of the right-hand side of the second constraint condition  $CO_2$  by 100 units. The correctness of the quoted analytical reasoning is confirmed by tables 5 and 6 for  $PM$  after changing 4 out of 6 parameters of the right-hand sides of the constraint conditions. This rightness exists despite the doubts that may have been raised in table 3 by zero values of acceptable growth for  $CO_3$  and for  $CO_6$ . But the change ranges  $\Delta b_j$  of right-hand side changes for  $CO_3$  and for  $CO_6$  indicated in table 3 refer to the change of this single parameter in combination with a properly identified set of several parameters described by the vector  $\Delta \mathbf{b}$ . The correctness of the simultaneous changes carried out in the set of parameters  $\Delta \mathbf{b}$  identified above was confirmed by conducting an analysis leading to table 5 and theoretical considerations in section 3.1.

Note that the reduction of the water supply minimums from the three sources, expressed as a reduction of the free expression in  $CO_1$ , will not result in decreasing the total water abstraction costs. It is only the collective action consisting in the simultaneous implementation of the three activities mentioned above that brings the intended effect of reducing the total cost of water consumption in the whole agglomeration from this system. The most effective measure is to maintain proportions in technical constraint conditions  $\Delta b_3 / \Delta b_1 = -1$ ,  $\Delta b_6 / \Delta b_1 = -2$  allowing for maximum reduction of water consumption  $\Delta b_1 = 25$ . The modification of the water abstraction system leads to the shut-off of the  $P_3$  intake point during the minimum water demand and at the same time requires the modernization of the  $P_2$  intake point to increase its ability to supply the agglomeration with water.

Table 5. Optimum water consumption for urban and industrial agglomeration from three water abstractions. PM after changing the 4 parameters of the right-hand sides of the constraint conditions. An answer report in the Excel application

Objective Cell (Min)					
Cell	Name	Original Value	Final Value		
\$E\$10	Total Cost	666.6666667	650		

Variable Cells					
Cell	Name	Original Value	Final Value	Integer	
\$A\$2	x1	66.66666666	50	Contin	
\$B\$2	x2	66.66666666	125	Contin	
\$C\$2	x3	66.66666666	3.55271E-14	Contin	

Constraints					
Cell	Name	Cell Value	Formula	Status	Slack
\$E\$4	Left Side Constraint 1	175	\$E\$4>=\$G\$4	Binding	0
\$E\$5	Left Side Constraint 2	175	\$E\$5<=\$G\$5	Not Binding	125
\$E\$6	Left Side Constraint 3	1825	\$E\$6<=\$G\$6	Binding	0
\$E\$7	Left Side Constraint 4	1000	\$E\$7>=\$G\$7	Binding	0
\$E\$8	Left Side Constraint 5	600	\$E\$8>=\$G\$8	Binding	0
\$E\$9	Left Side Constraint 6	250	\$E\$9>=\$G\$9	Binding	0

Table 6. Sensitivity report in the Excel application for PM environment after changing 4 parameters of the right-hand sides of the constraint conditions

Variable cells						
Cell	Name	Final Value	Reduced Cost	Coefficient Objective	Allowable Increase	Allowable Decrease
\$A\$2	x1	50	0	3	0.5	1
\$B\$2	x2	125	0	4	0.8	1
\$C\$2	x3	3.55271E-14	0	3	0.5	0.4

Constraints						
Cell	Name	Final Value	Price dual	Constraint R. H. Side	Allowable Increase	Allowable Decrease
\$E\$4	Left Side Constraint 1	175	1.333333333	200	75	7.10543E-15
\$E\$5	Left Side Constraint 2	175	0	300	1E+30	125
\$E\$6	Left Side Constraint 3	1825	0	1800	1E+30	2.66454E-14
\$E\$7	Left Side Constraint 4	1000	0.333333333	1000	1.33227E-14	150
\$E\$8	Left Side Constraint 5	600	0	600	1.42109E-14	1E+30
\$E\$9	Left Side Constraint 6	250	0.333333333	200	75	1.42109E-14

Ad b) What does the analogous adjusted sensitivity analysis procedure look like if only the highest first component  $y_{D1}^* = y_1^*(1;1) = 1.3$  of the vector  $y_D^*$ , expressed as an amount of money, should be used, i.e. when:  $\Delta OF_{DM}(\Delta b; \gamma, \lambda) = \Delta b_1 \cdot y_{D1}^*$ ?

Then the two-dimensional set of dual vectors should be parameterized to the form:

$$y^*(\gamma, \lambda) = y^*(1 + \gamma_1, 1 + \lambda_1) = 1/3 \cdot (y_D^* + \gamma_1 \cdot w + \lambda_1 \cdot v)$$

constructed on the basis of vectors  $y_D^*$ ,  $w$ ,  $v$ , where now D point of the trapezoid ABCD (Fig. 2) coincides with the beginning of the  $O\gamma_1\lambda_1$  coordinate system. Then, as before, we accept  $\Delta b_4 = \Delta b_6 = 0$  and solve the last of the following equations:

$$\Delta OF_{DM}(\Delta b; \gamma_1, \lambda_1) = \frac{1}{3} \cdot (\Delta b)^T \cdot [y_D^* + \gamma_1 \cdot w + \lambda_1 \cdot v] = \frac{1}{3} \cdot \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \\ \Delta b_3 \\ 0 \\ \Delta b_5 \\ 0 \end{bmatrix}^T \cdot \left( \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \gamma_1 \cdot \begin{bmatrix} 0 \\ 0 \\ -2 \\ 4 \\ -1 \\ 1 \end{bmatrix} + \lambda_1 \cdot \begin{bmatrix} 2 \\ 0 \\ 2 \\ -4 \\ 0 \\ 0 \end{bmatrix} \right) = \frac{4}{3} \cdot \Delta b_1$$

where conditions  $(\Delta b)^T \cdot w = 0$  and  $(\Delta b)^T \cdot v = 0$  take the form of  $-2\Delta b_3 - \Delta b_5 = 0$  and  $\Delta b_1 + \Delta b_3 = 0$ . Hence:

$$\Delta b = \Delta b_1 \cdot [1 \ 0 \ -1 \ 0 \ 2 \ 0]^T + \Delta b_2 \cdot [0 \ 1 \ 0 \ 0 \ 0 \ 0]^T$$

This means that the decrease in the total costs by the amount of  $1.3 \cdot 20 \approx 26.67$  monetary units through reduction of the minimum water demand by only  $\Delta b_1 = 20$  units from 200 to 180 units allowed by simultaneous (acceptable) increase of the right-hand side of the third (technical) condition limiting  $CO_3$  by 20 units and decrease on the right of the fifth constraint condition  $CO_5$  by 40 units, with the permissible drop of right-hand side of the second constraint condition  $CO_2$  by 100 units. This time one should keep the proportions of changes  $\Delta b_3 / \Delta b_1 = -1$  and  $\Delta b_5 / \Delta b_1 = 2$ .

The choice between the two solutions a) and b) requires additional information about the post-optimized water supply system.

### 3. Discussion of alternative scenarios

#### 3.1. Two scenarios

We notice that like in a) zero values for dual prices  $y_{B3}^*$  and  $y_{B6}^*$  contained in tab. 3 indicate that only the technical binding constraint conditions of  $CO_3$  and  $CO_6$  together with the binding  $CO_1$  with the exclusive exploitation of a non-zero value  $y_{B1}^* = 0.6$ . And in b) the use of only  $y_{D1}^* = 1.3$  means a greater financial benefit with a smaller decrease in the minimum water demand, but at the same time requires a slightly different scope of intervention related to the technical binding constraint condition  $CO_3$  and limitation of the parameter related to the right-hand side of another constraint condition  $CO_5$ , since  $y_{D3}^* = y_{D5}^* = 0$ . Therefore, the choice between these two solutions requires full consideration of information about the post-optimized water supply system, both included in the model and additional information from outside the model.

The fact of the above-mentioned necessity of simultaneous proportional change of several parameters of the water supply system results from the broader note below.

### 3.2. Interpretation of zero dual prices

Removal of the constraint condition in  $PM$  results in the deletion of the corresponding dual variable from  $DM$ . And leaving this dual variable in  $DM_1$  equivalent to  $DM$  requires the recognition that in the optimal solution for  $DM_1$  the value of this dual variable is zero. Also the non-binding constraint conditions in  $PM$  correspond to zero dual variable values in  $DM$ . Thus, the zeroing of a certain dual variable results from the already existing relaxation in the non-binding condition or indicates that to use the remaining non-zero values in the entire dual vector there is a possible necessity to make non-zero changes to the right-hand side of the constraint condition where the dual vector is zeroed. In other words, the zero optimal value of the dual variable corresponds to 1) the non-binding condition in  $PM$  or 2) the lack of this dual variable in  $DM$ . In turn, the lack of a dual variable in  $DM$  results from the removal of the effects of the corresponding constraint condition in  $PM$  which corresponds to the full relaxation of the constraints resulting from this constraint condition or due to the change in the value of the right-hand side of this constraint condition. This means that the zero value of the dual variable with a non-binding condition in  $PM$  corresponds to the possibility of independent change of the right-hand side of this constraint condition. And the zero value of the dual variable under the binding condition in  $PM$  corresponds to the possible necessity of the total change of the right-hand side of this constraint condition in the case of an attempt to change the right-hand side of another binding condition in  $PM$  with a corresponding non-zero value of the dual price.

### 3.3. Additional technical scenarios

Based on the last statement, we note that the decrease in water consumption (i.e. decrease in the right-hand side  $CO_1$ ) may also be a consequence of six further scenarios changing: c) right-hand sides of binding conditions  $CO_3, CO_4, CO_5$  in  $PM$  using the corresponding non-zero values  $y_{A_3}^*, y_{A_4}^*, y_{A_5}^*$  of the dual price vector  $y_A^*$ ; d) the right-hand sides of the binding conditions  $CO_3, CO_4, CO_6$  in  $PM$  using the corresponding non-zero values  $y_{C_3}^*, y_{C_4}^*, y_{C_6}^*$  of the  $y_C^*$  dual price vector. The construction of these scenarios requires, among others analogous reparametrizing that shift the origin of the coordinate system to the vertices A and C of the trapezoid respectively. Each of these eight scenarios of reducing water consumption brings separate changes in the total cost of water consumed. These scenarios are the consequences of using, individually, eight of the 12 non-zero dual price values occurring for the nodes of the ABCD set. The four remaining non-zero values do not bring scenarios with the effect of reducing water consumption. The selection of the best scenario  $\Delta b$  out of 8 or a linear combination of the two best ones will be determined by a detailed analysis of, among others, financial, technical, biological, ecological and social consequences related to these scenarios. Among the considered constraint conditions, those that result from the principle of the integrity of water flow in the river either consistently do not change or should also be modified by legal, economic or additional technical steps.

For the vectors  $\Delta b$  located on a unitary sphere in  $\mathbf{R}^6$  i.e. when:

$$|\Delta b| = \sqrt{\sum_{j=1}^6 (\Delta b_j)^2} = 1,$$

changing the value of the objective function as a scalar product:

$$\Delta OF_{DM}(\Delta b; \gamma, \lambda) = (\Delta b)^T \cdot y^*(\gamma, \lambda)$$

reaches the value of: i) maximum when vectors  $\Delta b$  and  $y^*(\gamma, \lambda)$  have identical direction and sense; (ii) minimum when vectors  $\Delta b$  and  $y^*(\gamma, \lambda)$  have the same direction and opposite sense; (iii) zero when vectors  $\Delta b$  and  $y^*(\gamma, \lambda)$  are orthogonal. These facts determine the choice of scenarios with extreme or zero changes in the total costs of water consumed and require separate analysis, as well as the other 7 types of post-optimization analysis mentioned in chapter 1.5.

The significance of the result obtained here can be attested by the fact that so far some researchers, due to the analytical and interpretative difficulties, have generally discouraged the development of a similar analysis [8, p. 44]. Apart from  $PM$ , the authors support  $DM$  and software diversity to avoid the need to build a permutation algorithm supporting linear programming to generate all node solutions in the ABCD trapeze. The complexity of analytical calculations can be demonstrated by the lack of reproducibility on the basis of tables 5 and 6 return steps to table 3 allowing the return to the initial  $PM$ . And the post-optimization analysis made available in table 6, indicates the possibility of further recursive modification of the considered water abstraction system. But this requires repeating analytical steps similar to those described above. Of course, any changes to the right-hand constraint conditions in the model should reasonably reflect the operational needs of the agglomeration and the environmental potential of the hydrological system.

## 4. Conclusions

1. In its classic form, linear programming models for the exploitation of complex technical systems most often in the sensitivity reports contain ambiguous responses. Therefore, in section 1.4, some results referring to the dual price of water presented in the literature were critically evaluated. Because this study is devoted to eliminating the unreliability of standard idealistic sensitivity analysis procedures, which combine the value of each dual price only with the change of a single parameter of the right-hand constraint conditions, the proposed post-optimization methodology can be widely used by engineers, not only in the management of operational processes.
2. Inference of the market price of water based on an ambiguous dual price (and such situations almost always occur while examining technical issues [29]) leads to:
  - a) lack of recognition of many scenarios for modification of the water supply system, and in turn, to a selection of a random scenario, which may require carrying out a difficult or even unachievable modernization of the water supply system.
  - b) achieving lower savings in the functioning of the water supply network, than it is feasible in another way.

Presenting the foundations of technical scenarios for a simplified example, the authors present the methodology of:

- ad. a) identification, through constraint conditions, of parts of the water supply system which are subject to change when implementing various technical scenarios,
  - ad. b) finding various ways to obtain savings associated with the implementation of alternative scenarios, which allows for selecting the best system modification.
3. The methodology developed in the study allows for generating eight alternative technical water saving scenarios, of which two were outlined.
  4. The ratios of  $\Delta b$  components determined in the work are used to simplify and improve the procedures of single and multi-parametric linear programming [54]. Included among those

- procedures are: post-optimization analysis with a functional dependence of changes in the parameters of the right-hand sides of constraint conditions, or changes in the coefficients of the objective function, which are analogous to the case of type 6 post-optimization in section 1.5. These problems are not only without descriptions available in the literature, but also become more complicated with a larger number of decision variables.
5. The study examines the use of dual vectors  $\mathbf{y}^*$  and  $\mathbf{w}$ ,  $\mathbf{v}$  vectors orthogonal to  $\mathbf{b}$  in determining effective proportional changes  $\Delta \mathbf{b}$ . Thus, it generalises the importance of vectors balancing in the transport model and orthogonal to  $\mathbf{b}$  and allows for generalization of the dual price matrix method [24, 55-58] for models with inequality constraint conditions.
  6. The methodology presented above can be directly adapted to the post-optimization of various operational processes of complex technical systems, e.g.
    - a) in the water abstraction system for an urban agglomeration from several sources under various operating conditions [9-10, 57];
    - b) to operating costs of other multi-source systems, e.g. supplying gas through the municipal gas network [34, 50];
    - c) to operating costs of medical equipment and technical infrastructure [51];
    - d) in the problem of improving the regionalization of emergency medical stations [39, 52];
    - e) to the average system residence times in subsets of reliability states [31-32].
  7. Currently, the authors are working on a full algorithmizing of the methodology presented here.

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