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Junxing LI
Zhihua WANG
Chengrui LIU
Ming QIU

ACCELERATED DEGRADATION ANALYSIS BASED ON A RANDOM-EFFECT WIENER PROCESS WITH ONE-ORDER AUTOREGRESSIVE ERRORS

PRZYSPIESZONA ANALIZA DEGRADACJI W OPARCIU O PROCES WIENERA Z EFEKTEM LOSOWYM Z BŁĘDAMI AUTOREGRESYJNYMI PIERWSZEGO RZĘDU

For highly reliable and long-life products, accelerated degradation test (ADT) is often an effective and attractive way to assess the reliability. To analyze the accelerated degradation data, it has been well recognized that it is necessary to incorporate three sources of variability including the temporal variability, the unit-to-unit variability and measurement errors into the ADT model. The temporal variability can be properly described by the Wiener process. However, the randomness of the initial degradation level, which is an important part of the unit-to-unit variability, has been often neglected. In addition, regarding the measurement errors, current ADT models often assumed them to follow a mutually independent normal distribution and ignored the autocorrelation that may probably exist in them. These problems lead to a poor accuracy for reliability evaluation in some situation. Thus, a random-effect Wiener process-based ADT model considering one-order autoregressive (AR(1)) errors is proposed. Then closed-form expressions for the failure time distribution (FTD) is derived based on the concept of first hitting time (FHT). A statistical inference method is adopted to estimate unknown parameters. Finally, a comprehensive simulation study and a practical application are given to demonstrate the rationality and effectiveness of the proposed model.

Keywords: *reliability evaluation, accelerated degradation modeling, Wiener process, unit-to-unit variability, measurement errors.*

W przypadku wysoce niezawodnych produktów o długim cyklu życia, przyspieszone badanie degradacji (ADT) często stanowi skuteczny i atrakcyjny sposób oceny niezawodności. Jak wiadomo, analiza danych z przyspieszonej degradacji wymaga włączenia do modelu ADT trzech źródeł zmienności, w tym zmienności czasowej, zmienności między jednostkami i błędów pomiarowych. Zmienność czasową można odpowiednio opisać za pomocą procesu Wienera. Jednak losowość początkowego poziomu degradacji, który stanowi ważną część zmienności między jednostkami, jest często w badaniach pomijana. Ponadto, w odniesieniu do błędów pomiaru, obecne modele ADT często zakładają, że mają one wzajemnie niezależne rozkłady normalne, ignorując możliwą autokorelację. Problemy te prowadzą w niektórych sytuacjach do niskiej trafności oceny niezawodności. W związku z powyższym, zaproponowano model ADT oparty na procesie Wienera z efektem losowym, w którym uwzględniono błędy autoregresyjne pierwszego rzędu (AR (1)). Następnie, w oparciu o pojęcie pierwszego czasu przejścia, wyprowadzono wyrażenia w postaci zamkniętej dla rozkładu czasu uszkodzenia (FTD). Do oszacowania nieznanych parametrów przyjęto metodę wnioskowania statystycznego. Na koniec przedstawiono kompleksowe studium symulacyjne i wskazano praktyczne zastosowanie modelu w celu wykazania jego racjonalności i skuteczności.

Słowa kluczowe: *ocena niezawodności, przyspieszone modelowanie degradacji, proces Wienera, zmienność między jednostkami, błędy pomiaru.*

1. Introduction

With increasing requirements from customers, more and more products are requested to have long life and high reliability. For products with these features, degradation data has been recognized as a valuable life information source and has been commonly adopted in reliability assessment [1, 26]. To guarantee the analysis accuracy, it is necessary to construct a reasonable degradation model. In the literature, many real applications suggest that degradation of a batch of products is usually affected by three types of variability including temporal variability (also defined as time-correlated structure), unit-to-unit variability and measurement errors. The temporal variability is referred to as the inherent variation of the degradation process over time. The unit-to-unit variability describes the heterogeneity among the degradation paths of multiple items. Measurement errors are usually created during the degradation investigation process due to imperfect measurements. For example, the imperfect measuring tool, the randomness of environmental factors and lower-skilled technicians may all result in imperfect measurements, especially when the data are obtained in an indirect way [19, 27, 28].

There is considerable interest on the part of the scientists and engineers in understanding and modeling the degradation process of products and components. Simultaneously considering the three types of variability, Peng and Tseng [19] proposed a general linear degradation model, Ye et al. [31] presented a well-adopted Wiener process degradation model and Li et al. [6] constructed a generalized nonlinear Wiener process-based degradation model. Meanwhile, Pan et al. [19] developed a reliability estimation approach based on EM algorithm for Wiener process degradation model by simultaneously

considering the variabilities. Moreover, degradation models adopted for the remaining useful life prediction that simultaneously considering the above three types of variability can be seen in [22, 23, 35].

In practical engineering, ADT has been recognized as an effective way to obtain degradation information quickly and efficiently within a reasonable time span and budget [31]. For accelerated degradation processes, temporal variability, unit-to-unit variability and measurement errors have also been recognized as three main uncertainty sources. Therefore, it is necessary to incorporate the three types of variability simultaneously for reasonable ADT modeling. Since the temporal variability of a degradation process can be properly described by a stochastic process via its the stochastic characteristics, stochastic process-based models including Wiener process, Gamma process and Inverse Gaussian Process have been in favor with many researchers in ADT analysis [11]. Among them, due to the attractive mathematical properties and physical interpretations, Wiener process and its various variants have been extensively developed and applied for accelerated degradation analysis of products [33-34]. Whitmore and Schenkelberg [30] adopted a Wiener process with one time scale transformation to model a constant stress ADT (CSADT).Liao and Tseng [10] provided an optimal step-stress ADT (SSADT) plan based on a Wiener process through a time transformation. Lim and Yum [11] developed an optimal ADT plan by assuming that the degradation characteristic follows a Wiener process.

Regarding unit-to-unit variability in ADT modeling, the degradation rate has been considered as a unit specific property. Then the drift coefficient (denoting the degradation rate) in Wiener process models is usually supposed to be random variable to describe the heterogeneity among test specimens. Tang et al. [25] proposed a nonlinear Wiener process to model ADT where the variability of an individual specimen was considered by assuming the drift coefficient as a random variable. Sun et al. [24] considered the individual variation by regarding the drift parameter as random in the Wiener process ADT model. Liu et al. [14] proposed a general Wiener process ADT model considering the unit-to-unit uncertainty. Meanwhile, measurement errors have been incorporated in Wiener process ADT modeling [4, 7].

According to the literature review, although a few Wiener process models have considered three types of variability into the ADT modeling, multiple problems have to be settled to enhance the model reasonability and the analysis accuracy. According to the best of our knowledge, it is a standard assumption for Wiener process-based ADT models that all measurement error terms are mutually independent and follow a normal distribution with zero mean and equal variance in the current literature. In practice, however, it is an oversimplification to suppose measurement errors are mutually independent. Degradation measurements (comprising the true degradation and measurement errors) are observed on a unit over time, and then it is reasonable to consider the degradation observation sequence and its measurement error dataset as time series [3, 17]. It is well known that time series datasets usually exhibit autocorrelation because of modeling errors or cyclic changes in ambient conditions (e.g., temperature)[5, 13]. Therefore, autocorrelation may probably exist in measurement errors, and it may be nonnegligible in many practical situations. It is further worth noticing that the autocorrelation may become stronger when the inspection time interval is relatively short. A Wiener process degradation model with AR(1) measurement errors for general degradation analysis was proposed in our previous work [9]. However, regarding accelerated degradation reliability analysis, the issue has not been reported in the literature. Thus, the first main objective of this paper is to extend the degradation model subject to autoregressive measurement errors in [9] to the case of accelerated degradation analysis.

On the other hand，in ADT modeling literature, the drift parameter has been usually supposed as a random variable to consider the unit-to-unit variability, while the initial degradation value has been always assumed or transformed as zero or a constant for most current Wiener process models [14, 25]. In real applications, however, the initial degradation level may probably not be a fixed value across all items, and may show unit-to-unit variabilities [4]. For example, the difference of the initial degradation level may be caused by the variant properties of material, the geometry differences of products, and so on. Therefore, it is necessary to incorporate unit-to-unit variabilities of both degradation rate and initial degradation level into the ADT modeling procedure to enhance its reasonability. That is the second objective of the current study.

In addition, it is well known that FHT of a standard Wiener process follows an inverse Gaussian distribution, and this is very useful for reliability analysis and maintenance decision-making [2]. For general degradation model and ADT model based on Wiener process, closed- form FTD expressions have been derived for situations when unit-to-unit variability regarding degradation rate is considered in the literature. As previously discussed, it is necessary for a reasonable ADT model to consider the unit specific variability caused by initial degradation level simultaneously. Considering the degradation models subject to measurement errors, most studies assume that a product is considered as been failed when its true degradation level first hits a predefined critical level [19, 35]. Therefore, the current study derives the closed-form ADT expressions along this line.

The remainder of the paper is organized as follows. Section 2 introduces an improved Wiener process ADT model which can consider the autoregressive measurement errors and the unit specific properties of both degradation rate and initial degradation level. The lifetime distribution is derived based on the FHT concept. Section 3 discusses MLEs for model parameters and an initial guesses method for optimization algorithm is given. In Section 4, the efficiency and reasonability of the established methodology is validated via a comprehensive Monte Carlo simulation study. In Section 5, the proposed approach is illustrated by a real application involving an electronic transistor ADT and comparative results are given. A summary and conclusion is given in Section 6.

2. ADT modeling

According to stress loading modes, there are mainly three ADT types including CSADT, SSADT and progressive stress ADT (PSADT). In real applications, comparing with SSADT and PSADT, CSADT has been recognized as the most commonly adopted type because of its simplicity and conveniences. Therefore, CSADT is focused in the current study.

2.1. Model formulation

A random-effect Wiener process-based ADT model considering autoregressive errors can be given by:

$$\begin{cases} Y(t) = X(t) + \varepsilon(t) \\ X(t) = X_0 + \upsilon(S,\boldsymbol{\beta})\Lambda(t,\boldsymbol{\theta}) + \sigma B(\Lambda(t,\boldsymbol{\theta})) \\ \varepsilon(t_i) = \sum_{k=1}^{p} \varphi_k \varepsilon(t_{i-k}) + e_i, \quad k \le i \end{cases} \tag{1}$$

where $Y(t)$ and $X(t)$ respectively denote the degradation inspection and the true performance degradation value at time t; $X_0 = X(0)$ is the true initial degradation level; $\upsilon(S,\beta)$ is the drift coefficient; S is the stress, and β is the unknown parameter vector in $\upsilon(S,\beta)$; $\Lambda(t,\theta)$ called transformed time scale is a positive non-decreasing function and θ is the unknown parameter vector in $\Lambda(t,\theta)$; For convenience, let $\Lambda = \Lambda(t,\theta)$; $B(\cdot)$ is a standard Wiener process and

$\sigma > 0$ denotes the diffusion coefficient; $\varepsilon(t)$ is the measurement error term at time t; p denotes the order of the autoregressive process; φ_k is the autocorrelation coefficient; $e_i \overset{iid}{\sim} N(0,\sigma_e^2)$ is the normally distributed and mutually independent. The improved ADT model expressed by Eq.(1) shows the following main properties:

1. The true performance degradation $X(t)$ under stress level S is assumed to be characterized by a Wiener process $B(\Lambda(t,\theta))$ and a drift coefficient $\upsilon(S,\beta)$ along with the diffusion coefficient σ and the true initial degradation level X_0. $\upsilon(S,\beta)$ is adopted to describe the accelerated relationship between the degradation rate and the stress S.
2. The accelerated model $\upsilon(S,\beta)$ can be constructed as $\upsilon(S,\beta)=\beta_0\lambda(S,\beta_1)$, where $\beta=(\beta_0,\beta_1)$ is unknown parameter vector; $\lambda(S,\beta_1)$ is a function that depends on stress S and can be obtained based on the acceleration relationship. Currently, commonly adopted acceleration models include the power rule model, Arrhenius model and Eyring model. For example, when Arrhenius model is considered as the acceleration model, the acceleration relationship can be supposed as $\lambda(S,\beta_1)=\exp(-\beta_1/S)$; While for the power rule model, one has $\lambda(S,\beta_1)=S^{\beta_1}$. In ADT analysis, to analyze the reliability at a use condition, the above accelerated models are usually utilized to describe the relation between the degradation rate and the stress [4, 7, 10-12, 14, 25].
3. The measurement error term $\varepsilon(t)$ in the accelerated degradation model is considered as a p-order autoregressive time series process $\varepsilon(t_i)=\sum_{k=1}^{p}\varphi_k\varepsilon(t_{i-k})+e_i$, where $|\varphi_k|<1$; i.e., AR(p). In practical engineering, it has been recognized that for longitudinal data or degradation data, an one-order autoregressive model can usually effectively describe the autocorrelation in the within-individual measurement errors [3, 9, 13]. Meanwhile, for the ADT model with p-order autoregressive error, the complex model increases the difficulties of the solving process and it is difficult to acquire the estimation of the unknown parameters. Thus, in this paper, the situation that the measurement error term $\varepsilon(t)$ is a one-order autoregressive time series process is focused, that is $\varepsilon(t_i)=\varphi\varepsilon(t_{i-1})+e_i$.
4. To consider the unit specific variant properties cause by the product-to-product differences, both the parameter β_0 in the drift coefficient $\upsilon(S,\beta)$ and the initial degradation level X_0 are assumed to be normally distributed random variables; i.e. $X_0 \sim N(\mu_0,\sigma_0^2)$, $\beta_0 \sim N(\mu_b,\sigma_b^2)$. Moreover, X_0, β_0, $B(\Lambda(t,\theta))$ and $\varepsilon(t)$ are assumed to be mutually independent of each other.

As described above, the proposed ADT model in Eq. (1) can depict the uncertainties from the temporal variability, the unit-to-unit variability ($\sigma_0 \neq 0$ or $\sigma_b \neq 0$) and measurement errors ($\sigma_e \neq 0$) incorporated in accelerated degradation processes, and is applicable for linear and nonlinear degradation processes. In addition, the proposed ADT model can cover several commonly Wiener process-based ADT models as its limiting cases, for example, if $\sigma_0 = 0$, $\sigma_e = 0$ and $\varphi_k = 0$, $k = 1,2,\ldots,p$, the proposed ADT model can be simplified to the existing widely used Wiener process-based ADT model [24, 25].

2.2. Derivation of lifetime distribution

To assess the product reliability at a use condition, it is necessary to derive FTD based on the proposed CSADT model. Without loss of generality, we first assume the degradation to be an increasing procedure over time, and a product is deemed to be failed when its true degradation first exceeds a predefined failure threshold. As discussed above, in some cases, it is necessary to incorporate the randomness of the initial degradation level X_0 into the ADT modeling procedure.

To derive the lifetime distribution, first let D_f^* denote the failure threshold when the initial true degradation level X_0 is considered as 0; i.e., $X_0 = 0$. According to the FHT concept, it is natural to define life T based on the true degradation path $X(t)$ as:

$$T_{|X_0=0} = \inf\{t : X(t) \geq D_f^* \mid X_0 = 0\} \tag{2}$$

In this situation, life T follows an inverse Gaussian distribution under the concept of FHT for a Wiener process according to the literature [2]. When the initial degradation level and the drift coefficient are supposed as $X_0 = 0$ and $\upsilon(S,\beta) \sim N(\mu_\upsilon,\sigma_\upsilon^2)$, the probability distribution function (PDF) and the cumulative distribution function (CDF) of life T for the Wiener process given in Eq. (1) can be expressed as [25]:

$$f_T(t|X_0=0) = \frac{D_f^*}{\sqrt{2\pi\Lambda^3(\sigma_\upsilon^2\Lambda+\sigma^2)}}\exp\left\{-\frac{(D_f^*-\mu_\upsilon\Lambda)^2}{2\Lambda(\sigma_\upsilon^2\Lambda+\sigma^2)}\right\}\frac{d\Lambda}{dt} \tag{3}$$

$$F_T(t|X_0=0) = \Phi\left(\frac{\mu_\upsilon\Lambda-D_f^*}{\sqrt{\sigma_\upsilon^2\Lambda^2+\sigma^2\Lambda}}\right)+\exp\left(\frac{2\mu_\upsilon D_f^*}{\sigma^2}+\frac{2\sigma_\upsilon^2(D_f^*)^2}{\sigma^4}\right)\Phi\left(-\frac{\sigma^2\mu_\upsilon\Lambda+(2\sigma_\upsilon^2\Lambda+\sigma^2)D_f^*}{\sigma^2\sqrt{\sigma_\upsilon^2\Lambda^2+\sigma^2\Lambda}}\right) \tag{4}$$

where $\Phi(\cdot)$ is the distribution function of a standard normal distribution. For the proposed CSADT model, $\mu_\upsilon = \mu_b\lambda(S,\beta_1)$ and $\sigma_\upsilon = \sigma_b\lambda(S,\beta_1)$.

Then, further considering the randomness of the initial degradation level X_0, let D_f denote the failure threshold, life T regarding the true degradation path $X(t)$ can be defined as:

$$T = \inf\{t : X(t) \geq D_f \mid X_0 \leq D_f\} \tag{5}$$

where the failure threshold $D_f = X_0 + D_f^*$. Then $D_f^* = D_f - X_0$ can be considered as a random variable with a normal distribution $D_f^* \sim N(D_f-\mu_0,\sigma_0^2)$. To this end, a proposition, which can significantly simplify the CDF and PDF derivation procedure, is given as follows:

Proposition 1: Let $X \sim N(\mu,\sigma^2)$ and a, b, c, $d \in \mathbf{R}$, then the following constructions hold:

$$E_X[\Phi(a+bX)] = \Phi\left((a+b\mu)/\sqrt{1+b^2\sigma^2}\right) \tag{6}$$

$$E_X\left[\exp\left(aX+bX^2\right)\Phi\left(c+dX\right)\right]=\frac{1}{\sqrt{1-2b\sigma^2}}\exp\left[\frac{2\mu\left(a+b\mu\right)+a^2\sigma^2}{2\left(1-2b\sigma^2\right)}\right]\times\Phi\left(\frac{c+d\mu+\left(ad-2bc\right)\sigma^2}{\sqrt{\left(1-2b\sigma^2\right)\left(1-2b\sigma^2+d^2\sigma^2\right)}}\right) \tag{7}$$

The proof is given in the **Appendix A**. Then based on Proposition 1, Eq. (6) and Eq. (7), CDF and PDF of life T for the proposed model can be obtained via the total probability law. It is shown in the following **Proposition 2**.

Proposition 2: When the initial degradation level is considered as a normal random variable $X_0 \sim N\left(\mu_0,\sigma_0^2\right)$, PDF and the CDF of life T for the proposed model can be expressed as:

$$f_T\left(t\right)=\frac{\left(\sigma_\upsilon^2\Lambda+\sigma^2\right)\left(D_f-\mu_0\right)+\mu_\upsilon\sigma_0^2}{\sqrt{2\pi\left(\sigma_\upsilon^2\Lambda^2+\sigma^2\Lambda+\sigma_0^2\right)^3}}\exp\left\{-\frac{\left(D_f-\mu_0-\mu_\upsilon\Lambda\right)^2}{2\left(\sigma_\upsilon^2\Lambda^2+\sigma^2\Lambda+\sigma_0^2\right)}\right\}\frac{d\Lambda}{dt} \tag{8}$$

$$F_T\left(t\right)=\Phi\left(\frac{\mu_0+\mu_\upsilon\Lambda-D_f}{\sqrt{\sigma_\upsilon^2\Lambda^2+\sigma^2\Lambda+\sigma_0^2}}\right)+\frac{\sigma^2}{\sqrt{\sigma^4-4\sigma_\upsilon^2\sigma_0^2}}\exp\left\{\frac{\left(D_f-\mu_0\right)\left[2\mu_\upsilon\sigma^2+2\sigma_\upsilon^2\left(D_f-\mu_0\right)\right]+2\mu_\upsilon^2\sigma_0^2}{\sigma^4-4\sigma_\upsilon^2\sigma_0^2}\right\}\times\Phi\left[-\frac{\mu_\upsilon\left(\sigma^2\Lambda+2\sigma_0^2\right)+\left(2\sigma_\upsilon^2\Lambda+\sigma^2\right)\left(D_f-\mu_0\right)}{\sqrt{\left(\sigma^4-4\sigma_\upsilon^2\sigma_0^2\right)\left(\sigma_\upsilon^2\Lambda^2+\sigma^2\Lambda+\sigma_0^2\right)}}\right] \tag{9}$$

The proof is given in the **Appendix B**. Then, the mean time to failure (MTTF) t_{MTTF} can be approximately obtained by:

$$t_{MTTF}=\Lambda^{-1}\left(\frac{D_f-\mu_0}{\mu_\upsilon}\right) \tag{10}$$

3. Parameter estimation

In a CSADT, let S_0 be a use stress level and $S_1<S_2<,\ldots,<S_L$ denote L higher stress levels. Suppose m_l units are tested under stress level S_l, and the corresponding performance degradation of the ith unit is measured at n_{li} test time points $t_{li1}<t_{li2}<,\ldots,<t_{lin_{li}}$, $l=1,2,\ldots,L$, $i=1,2,\ldots,m_l$.

3.1. MLE for unknown parameters

For simplicity, let $y_{lij}=Y\left(t_{lij}\right)$ denote the degradation inspection for unit i at time t_{lij} under stress level S_l, and suppose $\Lambda_{lij}=\Lambda\left(t_{lij}\right)$ and $\lambda_l=\lambda\left(S_l,\beta_1\right)$, $l=1,2,\ldots,L$, $i=1,2,\ldots,m_l$, $j=1,2,\ldots,n_{li}$. Meanwhile, further define $\Lambda_{li}=\left(\Lambda_{li1},\Lambda_{li2},\ldots,\Lambda_{lin_{li}}\right)'$, $\boldsymbol{y}_{li}=\left(y_{li1},y_{li2},\ldots,y_{lin_{li}}\right)'$, $\boldsymbol{y}_l=\left(\boldsymbol{y}_{l1},\boldsymbol{y}_{l2},\ldots,\boldsymbol{y}_{lm_l}\right)$ and $\boldsymbol{y}=\left(\boldsymbol{y}_1,\boldsymbol{y}_2,\ldots,\boldsymbol{y}_L\right)$. Then, $\boldsymbol{y}_{li}$ can be concluded to follow a multivariate normal distribution with mean $\mu_0 1_{li}+\mu_b\lambda_l\Lambda_{li}$ and covariance $\Sigma_{li}=\sigma_0^2 1_{li}1_{li}'+\sigma_b^2\lambda_l^2\Lambda_{li}\Lambda_{li}'+\sigma^2\boldsymbol{Q}_{li}+\gamma_0\Omega_{li}$, where $\gamma_0=\sigma_e^2/\left(1-\varphi^2\right)$, $1_{li}=\left(1,1,\ldots,1\right)'$ is an n_{li} dimensional column vector, and:

$$\boldsymbol{Q}_{li}=\begin{bmatrix}\Lambda_{li1} & \Lambda_{li1} & \cdots & \Lambda_{li1}\\ \Lambda_{li1} & \Lambda_{li2} & \cdots & \Lambda_{li2}\\ \vdots & \vdots & \ddots & \vdots\\ \Lambda_{li1} & \Lambda_{li2} & \cdots & \Lambda_{lin_{li}}\end{bmatrix}_{n_{li}\times n_{li}}$$ and

$$\Omega_{li}=\begin{bmatrix}1 & \varphi & \varphi^2 & \cdots & \varphi^{n_{li}-1}\\ \varphi & 1 & \varphi & \cdots & \varphi^{n_{li}-2}\\ \varphi^2 & \varphi & 1 & \cdots & \varphi^{n_{li}-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \varphi^{n_{li}-1} & \varphi^{n_{li}-2} & \varphi^{n_{li}-3} & \cdots & 1\end{bmatrix}_{n_{li}\times n_{li}}.$$

Moreover, to facilitate the estimation and inference, the parameters are re-parameterize as $\tilde{\sigma}_b^2=\sigma_b^2/\sigma_0^2$, $\tilde{\sigma}^2=\sigma^2/\sigma_0^2$, $\tilde{\gamma}_0=\gamma_0/\sigma_0^2$ and $\tilde{\Sigma}_{li}=\Sigma_{li}/\sigma_0^2$. $\Theta=\left(\mu_0,\sigma_0^2,\mu_b,\tilde{\sigma}_b^2,\beta_1,\theta,\tilde{\sigma}^2,\tilde{\gamma}_0,\varphi\right)$ is further defined as a vector involving all unknown parameters in the proposed model. Then the log-likelihood function (Log- LF) of Θ can be expressed as:

$$\ell\left(\Theta|\boldsymbol{y}\right)=-\frac{N}{2}\left[\ln\left(2\pi\right)+\ln\sigma_0^2\right]-\frac{1}{2}\sum_{l=1}^{L}\sum_{i=1}^{m_l}\ln\left|\tilde{\Sigma}_{li}\right|-\frac{1}{2\sigma_0^2}\sum_{l=1}^{L}\sum_{i=1}^{m_l}\left(\boldsymbol{y}_{li}-\mu_0 1_{li}-\mu_b\lambda_l\Lambda_{li}\right)'\tilde{\Sigma}_{li}^{-1}\left(\boldsymbol{y}_{li}-\mu_0 1_{li}-\mu_b\lambda_l\Lambda_{li}\right) \tag{11}$$

where $N=\sum_{l=1}^{L}\sum_{i=1}^{m_l}n_{li}$.

Taking the first derivative of $\ell\left(\Theta|\boldsymbol{y}\right)$ in Eq. (11) with respective to μ_0, μ_b and σ_0^2, one can obtain:

$$\frac{\partial\ell\left(\Theta|\boldsymbol{y}\right)}{\partial\mu_0}=\frac{1}{\sigma_0^2}\sum_{l=1}^{L}\sum_{i=1}^{m_l}1_{li}'\tilde{\Sigma}_{li}^{-1}\left(\boldsymbol{y}_{li}-\mu_0 1_{li}-\mu_b\lambda_l\Lambda_{li}\right) \tag{12}$$

$$\frac{\partial\ell\left(\Theta|\boldsymbol{y}\right)}{\partial\mu_b}=\frac{1}{\sigma_0^2}\sum_{l=1}^{L}\sum_{i=1}^{m_l}\lambda_l\Lambda_{li}'\tilde{\Sigma}_{li}^{-1}\left(\boldsymbol{y}_{li}-\mu_0 1_{li}-\mu_b\lambda_l\Lambda_{li}\right) \tag{13}$$

$$\frac{\partial\ell\left(\Theta|\boldsymbol{y}\right)}{\partial\sigma_0^2}=-\frac{N}{2\sigma_0^2}+\frac{1}{2\sigma_0^4}\sum_{l=1}^{L}\sum_{i=1}^{m_l}\left(\boldsymbol{y}_{li}-\mu_0 1_{li}-\mu_b\lambda_l\Lambda_{li}\right)'\tilde{\Sigma}_{li}^{-1}\left(\boldsymbol{y}_{li}-\mu_0 1_{li}-\mu_b\lambda_l\Lambda_{li}\right) \tag{14}$$

Then, by equating Eq. (12), Eq. (13) and Eq. (14) to zero respectively, the MLE of μ_0, μ_b and σ_0^2 can be obtained as:

$$\hat{\mu}_0=\left(A_3B_2-A_2B_3\right)/\left(A_1B_2-A_2B_1\right) \tag{15}$$

$$\hat{\mu}_b=\left(A_1B_3-A_3B_1\right)/\left(A_1B_2-A_2B_1\right) \tag{16}$$

$$\hat{\sigma}_0^2=\frac{1}{N}\sum_{l=1}^{L}\sum_{i=1}^{m_l}\left(\boldsymbol{y}_{li}-\hat{\mu}_0 1_{li}-\hat{\mu}_b\lambda_l\Lambda_{li}\right)'\tilde{\Sigma}_{li}^{-1}\left(\boldsymbol{y}_{li}-\hat{\mu}_0 1_{li}-\hat{\mu}_b\lambda_l\Lambda_{li}\right) \tag{17}$$

where $A_1=\sum_{l=1}^{L}\sum_{i=1}^{m_l}1_{li}^{'}\tilde{\Sigma}_{li}^{-1}1_{li}$, $A_2=\sum_{l=1}^{L}\sum_{i=1}^{m_l}\lambda_l 1_{li}^{'}\tilde{\Sigma}_{li}^{-1}\Lambda_{li}$,

$A_3=\sum_{l=1}^{L}\sum_{i=1}^{m_l}1_{li}^{'}\tilde{\Sigma}_{li}^{-1}\boldsymbol{y}_{li}$, $B_1=\sum_{l=1}^{L}\sum_{i=1}^{m_l}\lambda_l\Lambda_{li}^{'}\tilde{\Sigma}_{li}^{-1}1_{li}$, $B_2=\sum_{l=1}^{L}\sum_{i=1}^{m_l}\lambda_l^2\Lambda_{li}^{'}\tilde{\Sigma}_{li}^{-1}\Lambda_{li}$

and $B_3=\sum_{l=1}^{L}\sum_{i=1}^{m_l}\lambda_l\Lambda_{li}^{'}\tilde{\Sigma}_{li}^{-1}\boldsymbol{y}_{li}$.

Next, substituting Eq. (16) and Eq. (17) into Eq. (11), a profile log-likelihood function of unknown parameter $\tilde{\Theta}=\left(\tilde{\sigma}_b^2,\beta_1,\theta,\tilde{\sigma}^2,\tilde{\gamma}_0,\varphi\right)$ can be obtained as:

$$\ell\left(\tilde{\Theta}|\boldsymbol{y}\right)=C-\frac{N}{2}\ln\hat{\sigma}_0^2-\frac{1}{2}\sum_{l=1}^{L}\sum_{i=1}^{m_l}\ln\left|\tilde{\Sigma}_{li}\right| \tag{18}$$

where C is a constant. Based on a multiple-dimensional search optimization algorithm, MLE $\hat{\tilde{\Theta}}=\left(\hat{\tilde{\sigma}}_b^2,\hat{\beta}_1,\hat{\theta},\hat{\tilde{\sigma}}^2,\hat{\tilde{\gamma}}_0,\hat{\varphi}\right)$ of unknown parameters can be obtained by maximizing the profile Log-LF. Then MLE of μ_0, μ_b and σ_0^2 can be calculated by substituting $\hat{\tilde{\Theta}}=\left(\hat{\tilde{\sigma}}_b^2,\hat{\beta}_1,\hat{\theta},\hat{\tilde{\sigma}}^2,\hat{\tilde{\gamma}}_0,\hat{\varphi}\right)$ into Eq. (16) and Eq. (17). The other unknown parameters can further be determined via $\hat{\sigma}_b^2=\hat{\sigma}_0^2\hat{\tilde{\sigma}}_b^2$, $\hat{\sigma}^2=\hat{\sigma}_0^2\hat{\tilde{\sigma}}^2$, $\hat{\gamma}_0=\hat{\sigma}_0^2\hat{\tilde{\gamma}}_0$ and $\hat{\sigma}_e^2=\hat{\gamma}_0\left(1-\hat{\varphi}^2\right)$.

3.2. Initial guesses

When a multiple-dimensional search optimization algorithm is applied to numerically maximize the log-likelihood function, a reasonable initial guess is necessary for numerical calculation. Thus, a simple method is given to obtain an educated guess for the initial interval. The detailed procedure is as follows:

1. Let X_{0li} and υ_{li} denote the initial degradation level and the drift coefficient parameter $\upsilon\left(S,\beta\right)$ for the ith unit under stress level S_l respectively, $l=1,2,\ldots,L$, $i=1,2,\ldots,m_l$. Based on the least square method, rough estimates of $X_{0l1},X_{0l2},\ldots,X_{0lm_l}$, $\upsilon_{l1},\upsilon_{l2},\ldots,\upsilon_{lm_l}$ and θ can be obtained by minimizing the mean squared error (MSE):

$$\mathrm{MSE}_1=\sum_{l=1}^{L}\sum_{i=1}^{m_l}\left(\boldsymbol{y}_{li}-X_{0li}1_{li}-\upsilon_{li}\Lambda_{li}\right)^{'}\left(\boldsymbol{y}_{li}-X_{0li}1_{li}-\upsilon_{li}\Lambda_{li}\right) \tag{19}$$

2. The rough estimates of μ_0 and σ_0^2 can be calculated by fitting the estimations $X_{0l1},X_{0l2},\ldots,X_{0lm_l}$, $l=1,2,\ldots,L$.
3. Based on the least square method, the rough estimates of $\beta_{0l1},\beta_{0l2},\ldots,\beta_{0lm_l}$, $l=1,2,\ldots,L$ and β_1 can be obtained by minimizing the following mean squared error (MSE):

$$\mathrm{MSE}_2=\sum_{l=1}^{L}\sum_{i=1}^{m_l}\left(\beta_{li}-\beta_{0li}\lambda\left(S_l,\beta_1\right)\right)^{'}\left(\beta_{li}-\beta_{0li}\lambda\left(S_l,\beta_1\right)\right) \tag{20}$$

4. The rough estimates of μ_b and σ_b^2 can be calculated by fitting the estimations $\beta_{0l1},\beta_{0l2},\ldots,\beta_{0lm_l}$, $l=1,2,\ldots,L$.
5. Based on the estimations of μ_0, σ_0^2, μ_b, σ_b^2, β_1 and θ, the rough estimates of σ^2, σ_e^2 and φ can be obtained by maximizing the profile log-likelihood function in Eq. (11).

Therefore, the starting intervals of the unknown parameters for maximizing the log likelihood function via a multiple-dimensional search optimization algorithm are determined.

4. Simulation study

In this section, to test the efficiency of the proposed method, a comprehensive simulation study has been conducted considering CSADT. For comparison, let M_0 denote the proposed method. To demonstrate the necessity of considering autoregressive measurement errors in the accelerated degradation modeling procedure, a Wiener process-based ADT model M_1, as a special case of model M_0 by setting $\varphi=0$, is considered as a reference method. Meanwhile, another reference model M_2 that setting $\sigma_0=0$ in the model M_0 is also adopted to show the benefits of incorporating the random effects of the initial degradation level. Therefore, both the model comparison and sensitivity analysis of the standard deviation $\sigma_0=0$ of initial degradation and the autocorrelation coefficient φ are conducted to test the efficiency and necessary of the proposed method.

4.1. Model Comparison

Without loss of generality, temperature is considered as the accelerated stress and the transformed time scale function is defined as $\Lambda=t^{\theta}$. The normal stress level is S_0=303.5 K (30 °C) and three accelerated levels are supposed to be S_1 =333.5 K (60 °C), S_2 = 343.5 K (70 °C) and S_3 =353.5 K (80 °C). And an Arrhenius model $\upsilon\left(S,\beta\right)=\beta_0\exp\left\{\beta_1/S\right\}$ is adopted to describe the accelerated relationship. The parameters in simulation model are preset as μ_0 =3, σ_0 =0.9, μ_b =7, σ_b =0.7, β_1 =1200, θ =1.5, σ =0.3, σ_e =0.5 and φ =0.8. A failure threshold value is predefined as D_f = 12. For convenience, suppose m units are tested under each stress level and all items are inspected at n time points with $t=i$, $i=1,2,\ldots,n$.

In order to examine the influence of the sample size on the analytical precision, different combinations of (m,n), which are chosen to be (5, 10), (10, 10), and (20, 10) are considered for the simulation study sequentially. For each combination of (m,n), the mean absolute relative errors (MREs) and the mean square errors (MSEs) of the medium life $t_{0.5}$ and the FTD percentile are calculated by Eq. (21) and Eq. (22) based on Monte Carlo replications.

$$MSE=\frac{1}{K}\sum_{k=1}^{K}\left(\hat{t}_{pk}-t_p\right)^2 \tag{21}$$

$$MRE=\frac{1}{K}\sum_{k=1}^{K}\frac{\left|\hat{t}_{pk}-t_p\right|}{t_p} \tag{22}$$

where K is the number of Monte Carlo replications, t_p denotes the true $100p$th FTD percentile value and $\hat{t}_{pk}$ is the corresponding estimated result under the kth simulation, $k=1,2,\ldots,K$. Comparative results are given in Table 1 and Table 2 based on K=5000 Monte Carlo replications.

From Table 1 and Table 2, it can be observed that for each combinations of (m,n), MSEs and MREs of $t_{0.5}$ and $t_{0.1}$ based on the proposed model M_0 are smaller than that from reference models M_1 and M_2. When sample size is small, MSEs and MREs from the proposed model M_0 are significantly lower than the results given by reference models M_1 and M_2. Meanwhile, when the sample size increases, although MSEs and MREs of $t_{0.5}$ and $t_{0.1}$ based on reference models

Table 1. MSEs and MREs of $t_{0.5}$ from different degradation models

Model	(5,10)		(10,10)		(20,10)	
	MSE	MRE	MSE	MRE	MSE	MRE
M_0	2.6371	0.0785	1.6664	0.0609	1.2013	0.0547
M_1	24.8524	0.1960	14.6416	0.1726	6.6128	0.1209
M_2	17.5616	0.1894	13.1900	0.1637	5.2263	0.1076

Table 2. MSEs and MREs of $t_{0.1}$ from different degradation models

Model	(5,10)		(10,10)		(20,10)	
	MSE	MRE	MSE	MRE	MSE	MRE
M_0	1.4047	0.0737	0.6919	0.0576	0.4719	0.0485
M_1	4.1695	0.1250	2.9078	0.1205	1.7061	0.0939
M_2	7.1508	0.1607	3.6679	0.1272	1.6009	0.0867

Table 3. Comparison of Log_LF and AIC results

Model	(5,10)		(10,10)		(20,10)	
	Log-LF	AIC	Log-LF	AIC	Log-LF	AIC
M_0	-166.14	350.29	-350.26	718.52	-701.94	1421.88
M_1	-171.56	359.11	-357.43	730.85	-715.94	1447.89
M_2	-169.91	355.82	-355.76	727.53	-712.75	1441.50

M_1 and M_2 becomes closer to those of the constructed model M_0, the proposed model M_0 still can yield smaller MSEs and MREs. In addition, one can see that MSEs and MREs of $t_{0.5}$ are larger than those of $t_{0.1}$. This is because that the true value $t_{0.1}$ is smaller comparing with the true value of $t_{0.5}$, which will consequently result in a smaller error.

To further illustrate the efficiency of model M_0, the log-likelihood function value (Log-LF) and the corresponding Akaike information criterion (AIC) value are calculated to compare the modeling reasonableness for each combination of (m,n), where AIC is defined as:

$$AIC = -2 \times \left\{\max\left[\log\left(likelihood\right)\right]\right\} + 2q \tag{23}$$

where q is the number of the unknown parameters in the adopted accelerated degradation model.

Table 3 lists the average results of Log-LF value and AIC value under different combinations of (m,n) based on K=5000 Monte Carlo replications. It is obvious that in both terms of Log-LF and AIC, the proposed model M_0 gives a better fitting and modeling property. Therefore, although the constructed model M_0 involves one more parameter than reference models M_1 and M_2, it is necessary and worthy to construct the complicated analysis procedure for reasonable and accurate analysis.

4.2. Sensitivity Analysis

In this section, to furthermore test the necessary of considering the autocorrelation among measurement errors and the randomness of initial degradation level into the unit-to-unit variability, the sensitivity of reliability estimation is analyzed by setting different values of the autocorrelation coefficient φ and the standard deviation σ_0 of initial degradation level for the simulation example.

To this end, we first set the autocorrelation coefficient φ =0.1 (0.1) 0.9, and keep all other parameters unchanged. After that, the absolute error and the relative error of reliability evaluation results at normal stress levels are calculated in such a case by comparing model M_0 with M_1. Then, we set the standard deviation σ_0 =0.25 (0.25) 2 and the errors are calculated by comparing model M_0 with M_2. If the autocorrelation coefficient φ and the standard deviation σ_0 are sensitivity, the errors of reliability evaluation results at normal stress levels should increase with the values of φ and σ_0.

Herein, we repeated the simulation procedure of CSADT data for N_s = 100 times under the situation in Section 4.1. Then, the mean absolute error (MAE) and the mean absolute relative error (MARE) of reliability evaluation results for the autocorrelation coefficient φ and the standard deviation σ_0 can be given by:

$$\begin{cases} \mathrm{MAE}_i = \dfrac{1}{N_s}\displaystyle\sum_{k=1}^{N_s}\frac{1}{n}\sum_{j=1}^{n}\left|F_T^k\left(t_j|\mathrm{M}_0\right) - F_T^k\left(t_j|\mathrm{M}_i\right)\right| \\ \mathrm{MARE}_i = \dfrac{1}{N_s}\displaystyle\sum_{k=1}^{N_s}\frac{1}{n}\sum_{j=1}^{n}\frac{\left|F_T^k\left(t_j|\mathrm{M}_0\right) - F_T^k\left(t_j|\mathrm{M}_i\right)\right|}{F_T^k\left(t_j|\mathrm{M}_0\right)} \end{cases} \quad i = 1,2 \tag{24}$$

where $F_T^k\left(t_j|\mathrm{M}_i\right)$ is the CDF at time t_j under the normal stress level for the k th simulation under model M_i, $i = 0,1,2$, $k = 1,2,\ldots,N_s$.

The results of the sensitivity analysis of autocorrelation coefficient φ and standard deviation σ_0 are shown in Fig.1 and Fig.2 respectively. From Fig.1 and Fig.2, it can be observed that MAE and MARE of reliability evaluation results will increase with the autocorrelation coefficient φ and the standard deviation σ_0. Thus, it is clear that the effect of ignoring the autocorrelation among measurement errors and the randomness of initial degradation level on the reliability

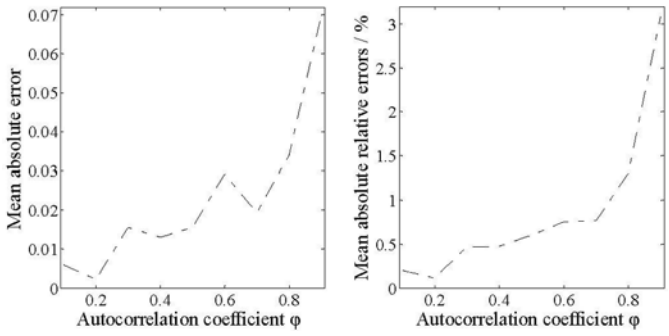

Fig. 1. Sensitivity analysis of autocorrelation coefficient φ

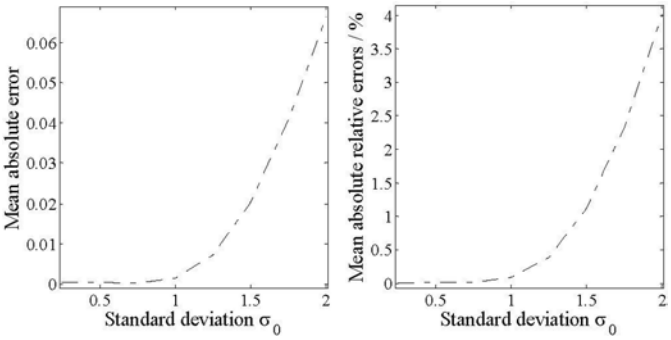

Fig. 2. Sensitivity analysis of standard deviation σ_0

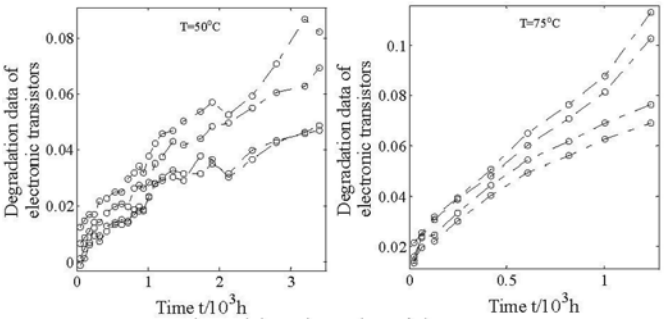

Fig. 3. Accelerated degradation data of electronic transistors

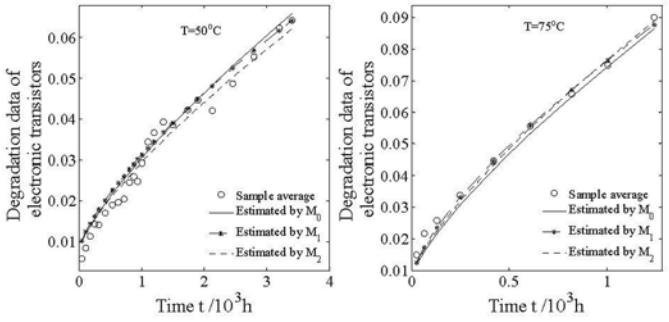

Fig. 4. Estimated mean degradation path based on three models

analysis is not critical under a small autocorrelation coefficient and the standard deviation σ_0 of initial degradation level situation. However, when the autocorrelation coefficient the standard deviation σ_0 are large, the effect is quite serious.

From the above analysis and comparison incorporating the reference models M_1 and M_2, one can conclude that it is necessary to consider the autoregressive measurement errors and the random effects of the initial degradation level into the Wiener process ADT modeling procedure, especially for limited sample size situations. In addition, if degradation model is mis-specified, unreliable results may probably be derived.

5. Illustrative example

In this section, a real application regarding a CSADT of electronic transistors is involved to further illustrate the necessary and validity of the proposed model M_0 for accelerated degradation analysis. An electronic transistor degrades over time and finally fails when its gain, a key performance of transistor, falls to a preset threshold level that makes it nonfunctional in the device where it is placed. To assess the electronic transistor reliability under a use stress level S_0=25ºC, a CSADT was conducted under two higher stress levels S_1=50ºC and S_2=75ºC. For each accelerated stress level, 4 electronic transistors are randomly selected for the degradation test. The original accelerated degradation data are given in [15], and is shown in Fig.3. The failure threshold is preset as D_f=0.15. Meanwhile, models M_1 and M_2 are also considered as reference for comparison.

According to [1, 18], Arrhenius acceleration model is a most common model and has been widely applied when the accelerated variable is temperature. Thus, without loss of generality, Arrhenius acceleration model is utilized to describe the relationship between the degradation rate of electronic transistors and the stress; i.e., $\upsilon(S,\beta)=\beta_0\lambda(S,\beta_1)$ and $\lambda(S,\beta_1)=\exp(-\beta_1/S)$. In addition, empirical studies have shown that $\Lambda(t,\theta)=t^{\theta}$ can be considered as a reasonable transformed time scale form [9, 15]. Consequently, this form is adopted in the current study.

To test the fitting goodness, models M_0, M_1 and M_2 are adopted to fit the accelerated degradation dataset. Unknown parameters of different models are estimated according to the inference procedure given in Section 4. Meanwhile, both Log-LF and AIC values are calculated. Results are summarized in Table 4. From Table 4, on can see that compared with models M_1 and M_2, model M_0 displays a best fitting with a largest Log-LF value and a smallest AIC value.

The one-order autocorrelation coefficient values φ estimated by models M_0 and M_2 are larger than or equal to 0.9, which indicates that the autocorrelation among measurement errors is non-ignorable for this practical problem. Additionally, it is obvious that the estimated value of σ_0 in models M_0 and M_1 are also relatively larger, and one can conclude that the random effect of the initial degradation level is necessary to be considered in ADT modeling.

Based on the results in Table 4, estimated mean degradation paths for the above three models are obtained and shown in Fig.4. From Fig. 4, it shows that for both stress levels, the estimated mean degradation paths based on the three models are all in conjunction with the sample average.

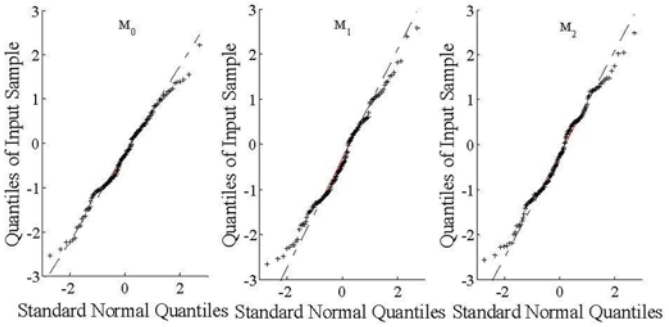

Fig. 5. Comparison of normal probability plots

For further illustration, the normal probability plot is adopted to assess the fitting goodness of models M_0, M_1 and M_2. Fig.5 gives the normal probability plots of the three adopted models. It is well known that if a normal probability plot approximates a straight line, a good fitting can be concluded. Otherwise, more proper degradation models should be considered when a poor fitting is derived. From Fig.5, it is

Table 4. Comparison fitting goodness for different degradation models

Model	Estimated parameters									Log -LF	AIC
	μ_0	σ_0	$\mu_b/10^4$	σ_b	β_1	θ	σ	σ_e	φ		
M_0	0.0077	0.0012	7.6756	0.0117	4.86	0.7707	0.0097	0.0002	0.9995	566.7	-1115.4
M_1	0.0074	0.0028	6.5938	0.0276	4.79	0.7129	0.0087	0.0014	---	563.4	-1110.8
M_2	0.0079	---	24.4991	0.0021	5.25	0.7592	0.0082	0.0018	0.9000	564.3	-1112.5

Table 5. Comparative results of different predicting life

Model	Estimated percentiles of the FTD	
	$t_{0.5}/10^3$h	$t_{0.1}/10^3$h
M_0	55.91	32.62
M_1	70.97	43.00
M_2	73.02	44.40

clear that the proposed model M_0 shows a best fitting compared with the reference models.

In addition, the medium life $t_{0.5}$ and the 10th FTD percentile $t_{0.1}$ for the three models are given in Table 5. It can be observed that results of $t_{0.5}$ and $t_{0.1}$ by reference models M_1 and M_2 are significantly larger than those from the constructed model M_0. It is well known that q-percentile life is commonly considered as an important evidence for making effective maintenance schedule. In practical engineering, a conservative q-percentile life estimation may lead to hysteretic maintenance and increase the failure risk at an early time.

From the above analysis, one can conclude that it is necessary to consider the autocorrelation among measurement errors for reasonable results when modeling accelerated degradation processes. Meanwhile, it is also necessary to incorporate the random effect of the initial degradation level into Wiener process ADT modeling. Although the proposed method may illustrate a more complicated modeling procedure because of the one more parameter, reasonable and reliable results can be governed.

6. Conclusions

Motivated by real applications, this paper proposed a Wiener process accelerated degradation model, which simultaneously considers the temporal variability, the unit-to-unit variability, and measurement errors. In the ADT modeling process, a one-order autoregressive model is utilized to reasonably describe the autocorrelation that may exist among measurement errors. Moreover, the random effects of both the initial degradation level and the degradation rate are incorporated regarding unit specific properties. Then, explicit form of lifetime distribution is derived based on the FHT concept, and a statistical inference method is given for unknown parameter estimation.

A comprehensive simulation study has demonstrated the necessity and efficiency of the proposed model with respect to CSADT analysis via an enhanced accuracy. Finally, a real application about CSADT of electronic transistors has verified the effectiveness and superiority of the constructed method comparing with the commonly used Wiener process models.

In this paper, accelerated degradation analysis for CSADT in focused. However, in practical engineering, SSADT is another effective way to evaluate the reliability of highly reliable products. Thus, the future research will focus on SSADT modeling. In addition, it may be of interest to predict the remaining useful life based on the proposed model. Meawhile, the failure threshold of many degradation process may be unknown and has uncertainty, which can be worth studying. We will work on these problems and hope to have useful findings.

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Appendix A

Proof of Proposition 1:

Let $Z \sim N(0,1)$ and $\gamma, \kappa \in \mathbf{R}$, one can derive [22]:

$$E_Z\left[\Phi(\gamma+\kappa Z)\right]=\Phi\left(\gamma/\sqrt{1+\kappa^2}\right) \tag{25}$$

Then, let $X=\mu+\sigma Z$. It can further be obtained that:

$$E_X\left[\Phi(a+bX)\right]=E_Z\left[\Phi(a+b\mu+b\sigma Z)\right]=\Phi\left((a+b\mu)/\sqrt{1+b^2\sigma^2}\right) \tag{26}$$

$$\begin{aligned}E_X\left[\exp\left(aX+bX^2\right)\Phi(c+dX)\right]&=\frac{1}{\sigma}\int_{\mathbf{R}}\exp\left(aX+bX^2\right)\Phi(c+dX)\phi\left(\frac{X-\mu}{\sigma}\right)dX\\&=\exp\left[\mu(a+b\mu)\right]\int_{\boldsymbol{R}}\exp\left[(a+2b\mu)\sigma Z+b\sigma^2Z^2\right]\Phi(c+d\mu+d\sigma Z)\phi(Z)dZ\\&=\frac{1}{\sqrt{1-2b\sigma^2}}\exp\left[\frac{2\mu(a+b\mu)+a^2\sigma^2}{2\left(1-2b\sigma^2\right)}\right]E_Z\left[\Phi\left(c+d\mu+\frac{(a+2b\mu)d\sigma^2}{\left(1-2b\sigma^2\right)}+\frac{d\sigma}{\sqrt{1-2b\sigma^2}}Z\right)\right]\end{aligned} \tag{27}$$

According to Eq. (26), we can derive:

$$E_Z\left[\Phi\left(c+d\mu+\frac{(a+2b\mu)d\sigma^2}{\left(1-2b\sigma^2\right)}+\frac{d\sigma}{\sqrt{1-2b\sigma^2}}Z\right)\right]=\Phi\left(\frac{c+d\mu+(ad-2bc)\sigma^2}{\sqrt{\left(1-2b\sigma^2\right)\left(1-2b\sigma^2+d^2\sigma^2\right)}}\right) \tag{28}$$

Thus:

$$E_X\left[\exp\left(aX+bX^2\right)\Phi(c+dX)\right]=\frac{1}{\sqrt{1-2b\sigma^2}}\exp\left[\frac{2\mu(a+b\mu)+a^2\sigma^2}{2\left(1-2b\sigma^2\right)}\right]$$

$$\times\Phi\left(\frac{c+d\mu+(ad-2bc)\sigma^2}{\sqrt{\left(1-2b\sigma^2\right)\left(1-2b\sigma^2+d^2\sigma^2\right)}}\right) \quad (29)$$

This completes the proof of **Proposition 1**.

Appendix B

Proof of Proposition2:

Considering the randomness of the initial degradation level X_0, the failure threshold $D_f^* = D_f - X_0$ can be considered as a random variable with the normal distribution; i.e., $D_f^* \sim N\left(D_f-\mu_0,\sigma_0^2\right)$. Thus, according to Eq. (4) and the law of total probability, CDF of life T for the proposed model can be expressed as:

$$F_T(t)=E_{D_f}\left\{\Phi\left(\frac{\mu_\upsilon\Lambda-D_f}{\sqrt{\sigma_\upsilon^2\Lambda^2+\sigma^2\Lambda}}\right)\right\}$$
$$+E_{D_f}\left\{\exp\left(\frac{2\mu_\upsilon D_f}{\sigma^2}+\frac{2\sigma_\upsilon^2 D_f^2}{\sigma^4}\right)\Phi\left(-\frac{\sigma^2\mu_\upsilon\Lambda+\left(2\sigma_\upsilon^2\Lambda+\sigma^2\right)D_f}{\sigma^2\sqrt{\sigma_\upsilon^2\Lambda^2+\sigma^2\Lambda}}\right)\right\} \quad (30)$$

For the first term in Eq. (30), let:

$$a=\frac{\mu_\upsilon\Lambda}{\sqrt{\sigma_\upsilon^2\Lambda^2+\sigma^2\Lambda}},\quad b=-\frac{1}{\sqrt{\sigma_\upsilon^2\Lambda^2+\sigma^2\Lambda}} \quad (31)$$

Meanwhile, for the second term in Eq. (30), let:

$$a=\frac{2\mu_\upsilon}{\sigma^2},\ b=\frac{2\sigma_\upsilon^2}{\sigma^4},\ c=-\frac{\sigma^2\mu_\upsilon\Lambda}{\sigma^2\sqrt{\sigma_\upsilon^2\Lambda^2+\sigma^2\Lambda}},\ d=-\frac{\left(2\sigma_\upsilon^2\Lambda+\sigma^2\right)}{\sigma^2\sqrt{\sigma_\upsilon^2\Lambda^2+\sigma^2\Lambda}} \quad (32)$$

Then, according to **Proposition 1**, CDF of life T can be derived for the proposed model. Furthermore, PDF of life can be accordingly obtained by taking the derivative of CDF with respect to t.

This completes the proof of **Proposition 2**.

References

1. Chen D G, Lio Y, Ng H K T, Tsai T R. Statistical Modeling for Degradation Data. Springer, 2017, https://doi.org/10.1007/978-981-10-5194-4.
2. Chhikara R S, Folks J L. The inverse Gaussian distribution: theory, methodology, and applications, Marcel Dekker, Inc.1989.
3. Chi E M, Reinsel G C. Models for longitudinal data with random effects and AR (1) errors, Publications of the American Statistical Association 1989; 84: 452-459, https://doi.org/10.1080/01621459.1989.10478790.
4. Hao S, Yang J, Berenguer C. Nonlinear step-stress accelerated degradation modelling considering three sources of variability, Reliability Engineering & System Safety 2017; 172.
5. Josephlu C, Meeker W. Using degradation measures to estimate a Time-to-Failure distribution. Technometrics 1993: 161-174.
6. Lei Y, Li N, Jia F, Lin J, Xing S. A nonlinear degradation model based method for remaining useful life prediction of rolling element bearings, Prognostics and System Health Management Conference 2016: 1-8.
7. Li J, Wang Z, Liu X, Zhang Y, Fu H, Liu C. A Wiener process model for accelerated degradation analysis considering measurement errors, Microelectronics Reliability 2016; 65: 8-15,https://doi.org/10.1016/j.microrel.2016.08.004.
8. Li J, Wang Z, Zhang Y, Fu H, Liu C, Krishnaswamy S. Degradation data analysis based on a generalized Wiener process subject to measurement error, Mechanical Systems & Signal Processing 2017; 94: 57-72, https://doi.org/10.1016/j.ymssp.2017.02.031.
9. Li J, Wang Z, Zhang Y, Liu C, Fu H. A nonlinear Wiener process degradation model with autoregressive errors, Reliability Engineering & System Safety 2018: 173, https://doi.org/10.1016/j.ress.2017.11.003.
10. Liao C M, Tseng S T. Optimal design for step-stress accelerated degradation tests, IEEE Transactions on Reliability 2006; 55(1):59-66, https://doi.org/10.1109/TR.2005.863811.
11. Lim H, Yum B J. Optimal design of accelerated degradation tests based on Wiener process models, Journal of Applied Statistics 2011; 38(2):17-27, https://doi.org/10.1080/02664760903406488.
12. Limon S, Yadav O P, Liao H. A literature review on planning and analysis of accelerated testing for reliability assessment. Quality and Reliability Engineering International 2017; 33: 2361-2383, https://doi.org/10.1002/qre.2195.
13. Lin J, Wei B. Testing for heteroscedasticity and/or autocorrelation in longitudinal mixed effect nonlinear models with AR(1) errors. Commun Stat –Theory Methods 2007; 36: 67–86, https://doi.org/10.1080/03610920601001816.
14. Liu L, Li X, Sun F, Wang N. A General Accelerated Degradation Model Based on the Wiener Process, Materials 2016; 9, https://doi.org/10.3390/ma9120981.
15. Lu J. Degradation processes and related reliability models, Ph.D. thesis. Montreal, Canada: McGill University,1995.
16. Lu C J, Meeker W Q, Escobar L A. A comparison of degradation and failure-time analysis methods for estimating a time-to-failure distribution. Stat. Sin. 1996; 6: 531–546.
17. Madsen H, Time Series Analysis, Chapman & Hall/crc Boca Raton, 2007.
18. Meeker W Q, Escobar L A. Statistical methods for reliability data, John Wiley & Sons, 2014.
19. Pan D, Wei Y, Fang H, Yang W. A reliability estimation approach via Wiener degradation model with measurement errors. Applied Mathematics & Computation, 2018; 320,https://doi.org/10.1016/j.amc.2017.09.020.
20. Peng C Y, Tseng S T. Mis-Specification Analysis of Linear Degradation Models, IEEE Transactions on Reliability 2009; 58: 444-455, https://

doi.org/10.1109/TR.2009.2026784.
21. Shahraki A F, Yadav O P, Liao H. A Review on Degradation Modelling and Its Engineering Applications, International Journal of Performability Engineering 2017;13: 299-314, https://doi.org/10.23940/ijpe.17.03.p6.299314.
22. Si X S, Wang W, Hu C H, Chen M Y, Zhou D H. A Wiener-process-based degradation model with a recursive filter algorithm for remaining useful life estimation, Mechanical Systems & Signal Processing 2013; 35: 219-237, https://doi.org/10.1016/j.ymssp.2012.08.016.
23. Si X S, Wang W, Hu C H, Zhou D H. Estimating remaining useful life with three-source variability in degradation modeling. IEEE Transactions on Reliability 2014; 63:167-190, https://doi.org/10.1109/TR.2014.2299151.
24. Sun L, Gu X, Song P. Accelerated Degradation Process Analysis Based on the Nonlinear Wiener Process with Covariates and Random Effects, Mathematical Problems in Engineering 2016; 1-13, https://doi.org/10.1155/2016/5246108.
25. Tang S, Guo X, Yu C, Xue H, Zhou Z. Accelerated Degradation Tests Modeling Based on the Nonlinear Wiener Process with Random Effects, Mathematical Problems in Engineering 2014: 1-11,https://doi.org/10.1155/2014/560726.
26. Vališ D, Žák L, Pokora O. Perspective approach in using anti-oxidation and anti-wear particles from oil to estimate residual technical life of a system, Tribology International 2018; 118: 46-59, https://doi.org/10.1016/j.triboint.2017.09.017.
27. Vališ D, Žák L, Pokora O, Lánský P. Perspective analysis outcomes of selected tribodiagnostic data used as input for condition based maintenance, Reliability Engineering & System Safety 2016; 145: 231-242,https://doi.org/10.1016/j.ress.2015.07.026.
28. Vališ D, Nováček O, Hasilová K, Leuchter J. Modelling of degradation and a soft failure moment during the operation of a supercapacitor applying selected diffusion processes, Engineering Failure Analysis 2017; 82: 566-582, https://doi.org/10.1016/j.engfailanal.2017.04.019.
29. Whitmore G A. Estimating degradation by a wiener diffusion process subject to measurement error, Lifetime Data Analysis 1995; 1: 307-319, https://doi.org/10.1007/BF00985762.
30. Whitmore G A, Schenkelberg F. Modelling accelerated degradation data using Wiener diffusion with a time scale transformation, Lifetime Data Analysis 1997; 3: 27-45, https://doi.org/10.1023/A:1009664101413.
31. Yang G. Life cycle reliability engineering. Wiley, 2008.
32. Ye Z S, Wang Y, Tsui K L, Pecht M. Degradation Data Analysis Using Wiener Processes With Measurement Errors, IEEE Transactions on Reliability 2013; 62: 772-780, https://doi.org/10.1109/TR.2013.2284733.
33. Ye Z S, Xie M. Stochastic modelling and analysis of degradation for highly reliable products, Applied Stochastic Models in Business & Industry 2015; 31: 16-32, https://doi.org/10.1002/asmb.2063.
34. Zhang Z, Si X, Hu C, Lei Y. Degradation Data Analysis and Remaining Useful Life Estimation: A Review on Wiener-Process-Based Methods, European Journal of Operational Research 2018,https://doi.org/10.1016/j.ejor.2018.02.033.
35. Zheng J F, Si X S, Hu C H, Zhang Z X, Jiang W. A Nonlinear Prognostic Model for Degrading Systems With Three-Source Variability, IEEE Transactions on Reliability 2016; 65: 736-750, https://doi.org/10.1109/TR.2015.2513044.

Junxing LI
School of Mechatronical Engineering
Henan University of Science and Technology
Luoyang 471003, China

Zhihua WANG
School of Aeronautic Science and Engineering
Beihang University
Beijing 100191, China

Chengrui LIU
Beijing Institute of Control Engineering
Beijing 100080, China

Ming QIU
School of Mechatronical Engineering
Henan University of Science and Technology
Luoyang 471003, China

E-mails: lijun-xing2008@163.com, wangzhihua@buaa.edu.cn, liuchengrui_502@163.com, qiuming69@126.com