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Anna BORUCKA
Andrzej NIEWCZAS
Kamila HASILOVA

FORECASTING THE READINESS OF SPECIAL VEHICLES USING THE SEMI-MARKOV MODEL

PROGNOZOWANIE GOTOWOŚCI POJAZDÓW SPECJALNYCH NA PODSTAWIE MODELU SEMI-MARKOWA*

The vehicle exploitation system, consisting of statistically identical objects that perform intervention tasks, not subject to systematic changes, can be modelled as a stationary stochastic process. Such a model allows to determine the probabilistic indicators of current and boundary readiness of the system. This article presents the use of the semi-Markov process, based on three operating states: operation, ready-to-be-used and repair, to study a transport system consisting of special vehicles. On the example of a sample consisting of police patrol cars, experimental studies of the intensity of fleet utilization, time of failure-free operation of vehicles were carried out, and it was demonstrated that the examined transport system is characterized by a satisfactory, stationary readiness coefficient. The developmental possibilities of the presented modelling method were emphasized.

Keywords: *vehicle exploitation system, special vehicles, readiness, semi-Markov model.*

System eksploatacji samochodów, które realizują zadania interwencyjne, niepodlegający systematycznym zmianom może być modelowany jako stacjonarny proces stochastyczny. Taki model pozwala wyznaczyć probabilistyczne wskaźniki bieżącej i granicznej gotowości systemu. W niniejszym artykule, do modelowania systemu eksploatacji pojazdów specjalnych, wykorzystano proces semi-Markowa, oparty na trzech stanach eksploatacyjnych: użytkowania, postoju użytkowego i naprawy. Na przykładzie próby radiowozów policyjnych przeprowadzono doświadczalne badania intensywności użytkowania floty, czasu bezawaryjnej pracy pojazdów a także wykazano, że badany system transportowy charakteryzuje się zadowalającym, stacjonarnym współczynnikiem gotowości. Podkreślono rozwojowe możliwości przedstawionej metody modelowania.

Słowa kluczowe: *system eksploatacji samochodów, pojazdy specjalne, gotowość, model semi-Markowa.*

Introduction

The process of commercial vehicle exploitation can be analysed both in road transport companies, which operate in market conditions, as well as in rescue services and other services responsible for national security, such as the fire brigade, army, police, ambulance service. In the first group, the most important criterion for assessing the operational quality of a vehicle is efficiency, usually measured as a profit/cost ratio [1]. The second group, especially the Police, is identified above all with keeping the peace, protecting the lives and health of people and property, and preserving order. Therefore, most of the research in this thematic area is mainly related to security issues in the broad sense, and concerns, for example:

1. Estimates of the likelihood of a fatal accident when driving a police car [3] and the assessment of the risk of traffic incidents, including serious injuries resulting from participation in police operations [6, 25].
2. Possibilities of increasing the security level of police operations through the use of special methods or devices, such as the bulletproof panels mounted on police cars proposed by Michaelson [27] or the warning light systems described by Lyons [26].
3. Methods for planning and optimising patrol routes [8, 10], with particular emphasis on security issues [4] as well as the necessary number of patrol cars depending on the intensity of the activities carried out and the time of their occurrence [22].

On the other hand, the readiness and reliability of police vehicles is considered a kind of status quo. The studies presented in the literature on the assessment of readiness of complex intervention systems (not only police ones) are of a unitary nature. This is mainly due to the limitations associated with the confidential nature of the empirical data. Record and billing documentation is usually kept in paper form and the practice of creating electronic databases encounters organisational barriers.

Transport tasks are complex processes, which means that their modelling based on classical techniques of reliability theory can be complicated and may not produce satisfactory results [21]. Alternative methods are used in such a case, e.g. reliable phase diagrams proposed by Lu et al. [24] or Dong et al. [9], as well as Markov processes [11, 16, 34], which are particularly popular in the readiness assessment. In the literature one can find models describing single means of transport, e.g. a passenger car – as in the case of Girtler and Ślęzak [12], a bus in the case of Landowski et al. [23], or a helicopter in the case of Szawłowski [23]. Complex transport systems are also studied. Theoretical basis for such considerations are included in the papers [2, 13, 20]. The systems are analysed as a whole [7, 32, 35] or their individual components are considered independently, and each of them is described by a separate model. Often, the authors point to Markov processes as a tool to solve a particular exploitation problem [29, 30]. Unfortunately, transport systems models based on empirical data are few and far between. There are only single studies available,

(*) Tekst artykułu w polskiej wersji językowej dostępny w elektronicznym wydaniu kwartalnika na stronie ***www.ein.org.pl*** e.g. Migawa [28] studied the city bus exploitation system in this way, Żurek and Tomaszewska [39] analysed aircrafts and Restel [31] analysed urban rail transport systems.

The literature review shows that the Markov models are a good tool to assess the readiness of both whole systems and individual objects [5, 19]. However, they have their own requirements and limitations. These include, first of all, the form of available observations, the distribution of which should be exponential. It is an element that is often omitted in the presented analyses, which causes the use of Markov processes to be abused. It is more difficult to estimate parameters in the case of semi-Markov models, which is why they are less popular. They have less restrictive requirements concerning the form of distributions of studied variables (they can be arbitrary), therefore they are proposed in this article as a tool for police car fleet assessment. The aim of the presented study was to estimate the level of their readiness at the assumption of three operating states: operation, ready-to-be-used and repair (technical maintenance) and to present a method for a stochastic description of the exploitation process. Moreover, the intention of the authors was to emphasize that the three-state exploitation model may be a useful and sufficient tool to assess the readiness of special vehicles. The application of such a model does not require complicated calculations as it is the case with complex multi-state models and can be used in the current practice of fleet management.

2. Exploitation studies and preliminary analysis of the results

The subject of the study were police cars, performing patrol and intervention tasks in the capital city of Warsaw. A total of 20 Kia brand marked passenger cars were analysed. All vehicles came from a single production batch, which allowed the sample to be considered as homogeneous. The source database was the documentation of the use of police cars concerning police patrols and the records of technical services and repairs.

On the basis of the collected observations, a three-element set of operating states $S = \{S_1, S_2, S_3\}$ of the vehicles was singled out:

– operation (S_1),

– ready-to-be-used (S_2),

– repair (including technical maintenance) (S_3).

It was assumed that the time the vehicle remains in the state S_1 (state S_1 duration) falls within the range from the moment of departure in order to perform an intervention task (patrol) to the moment of returning to the depot. The time the vehicle remains in the state S_2 (state S_2 duration) falls within the range from the moment of starting a stop in the depot waiting for the instruction to perform the task until the moment of departure. The time the vehicle remains in the state S_3 (state S_3 duration) is determined by the time when the technical maintenance starts and ends.

Then, based on the actual interstate relations, permitted transitions were determined, which are presented in Fig. 1 in the form of a graph.

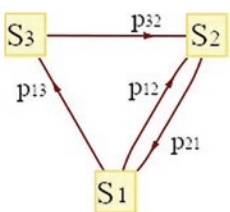

Fig. 1. Permitted transitions graph

The analysis of statistical time distributions (expressed in minutes) of individual operating states was also carried out. Matching of real observations to selected theoretical distributions (normal, log-normal, exponential, Gamma and Weibull) was examined. The parameters of these distributions were estimated using the Statistica program, applying the highest reliability method. The quality of matching was assessed by comparing the distribution of observed frequencies with the expected ones. The statistics of the Kolmogorov-Smirnov test and the Akaike Information Criterion were calculated. On the basis of the results obtained, a gamma distribution was selected as the most suitable one. An exemplary analysis was presented for the distribution of the operation state - S_1 (Fig. 2).

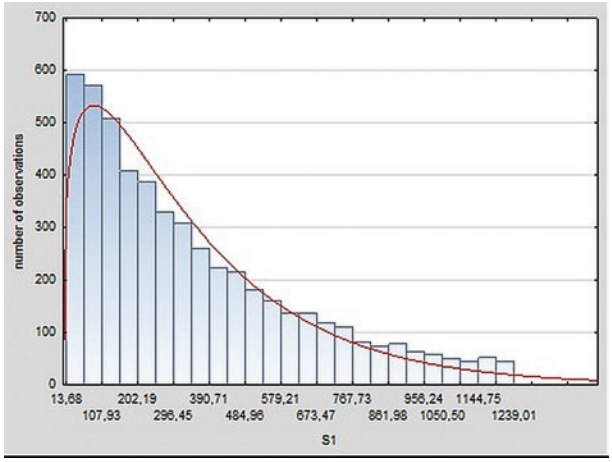

Fig. 2. Histogram of state duration times S_1

3. Estimation of parameters of semi-Markov model

3.1. Basic characteristics

The conclusion from the preliminary analyses was the lack of possibility to use the Markov model (it requires exponential form of distributions of variables) and the assumption to carry out analyses with the use of the semi-Markov model, for which the form of distributions may be arbitrary.

For the examined process of car exploitation, a semi-Markov model with a finite set of states was determined by means of the Markov renewal process, based on [12, 13, 20]:

For N denoting a set of non-negative integers, S – a certain finite set, R_+ – a set of non-negative real numbers, while $\Omega, \mathcal{F}, \mathrm{P})$ – a probabilistic space in which for each $n \in \mathrm{N}$ random variables are specified:

$$\xi_n : \Omega \to S \tag{1}$$

$$\vartheta_n : \Theta \to R_+ \tag{2}$$

Two-dimensional sequence of random variables $\{\xi_n, \vartheta_n : n \in N\}$ is referred to as the Markov renewal process if for each $n \in N$, $i, j \in S$, $t \in R_+$:

$$P\{\xi_{n+1} = j, \vartheta_{n+1} < t | \xi_n = i, \xi_{n-1}, \ldots, \xi_0, \vartheta_n, \ldots \vartheta_0\} = P\{\xi_{n+1} = j, \vartheta_{n+1} < t | \xi_n = i\} \tag{3}$$

and

$$P\{\xi_0 = i, \vartheta_0 = 0\} = P\{\xi_0 = i\} \tag{4}$$

This definition shows that Markov renewal process is a specific case of the two-dimensional Markov process [14]. Transition probabilities of this process depend solely on the discrete value of the coordinate. Markov renewal process $\{\xi_n, \vartheta_n : n \in N\}$ is called homogeneous if probabilities:

$$P\{\xi_{n+1} = j, \vartheta_{n+1} < t | \xi_n = i\} = Q_{ij}(t) \tag{5}$$

do not depend on n.

Functional matrix:

$$Q(t) = [Q_{ij}(t)], i, j \in S \tag{6}$$

is called the renewal kernel. Semi-Markov process is defined basing on the homogeneous Markov renewal process 14.

Let:

$$M(t) = \sup\{m \geq 0 : \tau_m \leq t\} \tag{7}$$

where:

$$\tau_m = \vartheta_0 + \vartheta_1 + \ldots + \vartheta_m \tag{8}$$

The stochastic process $\{M(t) : t \in R_+\}$ is constant within the range $[\tau_m, \tau_{m+1})$. The stochastic process $\{X(t) : t \in R_+\}$ is determined by the formula:

$$X(t) = \xi_{M(t)} \tag{9}$$

is a semi-Markov model,

Defining a model semi-Markov process requires defining, in addition to the kernel of the process, its initial distribution [13, 17, 38]. The process of vehicle exploitation was divided into three phases of random duration. In this case, the renewal kernel of the semi-Markov process, according to the graph of permitted transitions (Fig. 1), takes the form:

$$Q(t) = \begin{bmatrix} 0 & Q_{12}(t) & Q_{13}(t) \\ Q_{21}(t) & 0 & 0 \\ 0 & Q_{31}(t) & 0 \end{bmatrix} \tag{10}$$

This matrix constitutes a model of changes in the distinguished states of the process. The non-zero elements $Q_{ij}(t)$ of the matrix $Q(t)$ are the conditional probabilities of the process of transition from the state S_i to the state S_j, within a time period of not more than t, specified according to the formula (11). They depend on the distribution of random variables, namely the process durations in the distinguished states:

$$Q_{ij}(t) = P\left(X(\tau_{m+1}) = j,\ \tau_{m+1} - \tau_m \leq t | X(\tau_m) = i\right) \text{ for } t \geq 0 \tag{11}$$

where a random variable τ_m means the moment of m-th change of state.

Initial distribution: $p_i(0),\ i \in S = \{1, 2, 3\}$ was adopted in the following form:

$$p_i(0) = \begin{cases} 1, \text{ if } i = 1 \\ 0, \text{ if } i \neq 1 \end{cases} \tag{12}$$

where:

$$p_i(0) = P\{X(0) = i\}, \quad i = 1, 2, 3 \tag{13}$$

These elements make it possible to determine the probabilistic parameters of the exploitation process that are being searched for. For the semi-Markov model, the transition probabilities, defined as conditional probabilities [15], are important:

$$P_{ij}(t) = P\{X(t) = j | X(0) = i\}, i, j \in S \tag{14}$$

$P_{ij}(t)$ are the probabilities of transition from the state S_i to the state S_j at the moment t. They were calculated on the basis of real interstate relations, according to the formula (15).

$$p_{ij} = \frac{n_{ij}}{\sum_{k \in S} n_{ik}} \tag{15}$$

where:

n_{ij} – number of transitions from the state S_i to the state S_j,

$\sum\limits_{k \in S} n_{ik}$ - number of all transitions (exits) from the state S_i,

The distribution of probability of changes of the distinguished operating states (in one step), assuming that each graph arch of the exploitation process representation (Fig. 1), connecting two states of the process, corresponds to the value of probability p_{ij}, is presented in Table 1.

Table 1. Transitions probabilities matrix p_{ij}

p_{ij}	S_1	S_2	S_3
S_1	0	0.8	0.2
S_2	1	0	0
S_3	0	1	0

The calculated values of probabilities of transitions refer to sets of states, not time period. For example, $p_{13} = 0.2$ means that among all the exits from the state S_1, transitions from the state S_1 to S_2 constitute 20%.

3.2. Boundary properties

An important role in the study of the process of exploitation of cars modelled by the Markov chain is played by its boundary properties [13, 20], and especially by the boundaries of probabilities $p_j(n)$ and $p_{ij}(n)$ at $n \to \infty$, , which describe the behaviour of the process after a long time [13, 36]. An important concept in this respect is the stationary distribution of homogeneous Markov chain, described by the vector Π [14]:

$$\Pi = [\pi_1, \pi_2, \pi_3] \tag{16}$$

so as:

$$\Pi = \Pi \mathrm{P} \tag{17}$$

where:

$$\mathrm{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \tag{18}$$

and:

$$\sum_{j=1}^{3} \pi_j = 1 \tag{19}$$

this means that if the chain at a certain point in time m reaches the stationary distribution, then for each subsequent moment n greater than m the unconditional distribution will remain the same.

In the case of the examined process, some limits exist:

$$\lim_{n\to\infty} p_{ij}(n) = \pi_j \quad i, j = 1,2,3 \tag{20}$$

where:

$p_{ij}(n)$ – probability of transition from the state S_i to the state S_j in n steps.

Calculated probability matrix of changes in operating states inserted into the Markov chain process (Table 1) made it possible to determine the stationary probabilities π_j, according to a system of equations (17).

For the examined process, for the 3-state model, the estimation of the stationary probabilities π_j required the solution of the matrix equation:

$$\begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}^T \cdot \begin{bmatrix} 0 & p_{12} & p_{13} \\ p_{21} & 0 & 0 \\ 0 & p_{32} & 0 \end{bmatrix} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix}^T \tag{21}$$

with the normalization condition:

$$\pi_1 + \pi_2 + \pi_3 = 1 \tag{22}$$

which is equivalent to the following system of equations:

$$\begin{cases} \pi_2 \cdot p_{21} = \pi_1 \\ \pi_1 \cdot p_{12} + \pi_3 \cdot p_{32} = \pi_2 \\ \pi_1 \cdot p_{13} = \pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \tag{23}$$

After substituting the value of probability of transitions (Table 1), we get:

$$\begin{cases} \pi_2 = \pi_1 \\ 0.8\pi_1 + \pi_3 = \pi_2 \\ 0.2\pi_1 = \pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \tag{24}$$

The solution of the system of equations is presented in Table 2.

Table 2. Stationary probabilities π_j of the distinguished operating states

	S_1	S_2	S_3
π_i	0.455	0.455	0.09
$\pi_i\,[\%]$	45.5	45.5	9

Next, on the basis of the directed graph (Fig. 1) determining the probability of transitions of Markov chain states (Table 1), and on the basis of empirical times t_{ij} of duration of individual states, conditional estimations of expected $E(T_{ij})$ times of duration of process $X(t)$ states were made on the basis of the estimator defined by the formula (24):

$$\widehat{E(T_{ij})} = \overline{T}_{ij} = \frac{t_{ij}}{\sum_{j\in S} t_{ij}} \tag{25}$$

The matrix $\overline{T} = \left[\overline{T}_{ij}\right]$, $i, j = 1,2,3$ of estimated conditional values of expected times T_{ij} is presented in Table 3.

Table 3. Estimated expected values of conditional times T_{ij}

$\overline{T}_{ij}\,[minutes]$	S_1	S_2	S_3
S_1		844	845
S_2	479		
S_3		388	

When the elements of the matrix P and $\overline{T}$ are known, the expected values $ET_i, i = 1,2,3$ of the unconditional duration times of individual states of the process can be estimated according to the dependency:

$$\widehat{ET_i} = \overline{T}_i = \sum_{j=1}^{3} p_{ij} \cdot \overline{T}_{ij} \tag{26}$$

For the examined 3-state process of vehicle exploitation, the problem of estimating the values of expected unconditional duration of individual states of the process boiled down to the solution of the following system of equations:

$$\begin{cases} \overline{T}_1 = p_{12} \cdot \overline{T}_{12} + p_{13} \cdot \overline{T}_{13} \\ \overline{T}_2 = p_{21} \cdot \overline{T}_{21} \\ \overline{T}_3 = p_{32} \cdot \overline{T}_{32} \end{cases} \tag{27}$$

The estimated values of unconditional times $\overline{T}_i$ are shown in Table 4.

The random variables $T_i, i = 1,2,3$ have finite positive expected values. This makes it possible to determine the boundary distribution of the semi-Markov process. Based on the stationary distribution of the inserted Markov chain (Table 2) and the estimated expected values

Table 4. Unconditional times $\overline{T}_i$ [minutes] of process duration in 3 operating states

state	$\overline{T}_i$ [minutes]
1	844.2
2	479
3	388

of the process duration times (Table 4), boundary probabilities were estimated according to the formula (28) [20]:

$$P_i = \frac{\pi_i \cdot \overline{T}_i}{\sum_{k \in S} \pi_k \cdot \overline{T}_k}, \; i = 1, 2, 3 \tag{28}$$

The calculated boundary distribution of probability of semi-Markov process states is presented in Table 5.

Table 5. Boundary probabilities distribution P_i

Percentage	P_1	P_2	P_3
probabilities	0.6026	0.3419	0.0555
distribution	60	34	6

The values P_i constitute boundary probabilities determining that in a long period of operation ($t \to \infty$) the vehicle will remain in a given operating state.

The highest values were achieved for the state of operation (60%), which is a very good result. Ready-to-be-used reaches the boundary value of 34%, which is also a satisfactory result and shows, on the one hand, a high level of readiness of the examined vehicles and, on the other hand, a significant reserve which, however, in the case of structures operating in an unforeseen, intervention-based manner, seems rational. In boundary terms, there is only 5.5% probability of vehicles being a repair state.

The technical readiness factor K is the sum of appropriate probabilities of reliability states. For the proposed model of vehicle operation, the states S_1 and S_2 are roadworthy, while the state S_3 is the state of unfitness. Hence, the readiness of the examined vehicles can be calculated as the sum of the boundary probabilities of the states S_1 and S_2:

$$K = P_1 + P_2 \tag{29}$$

The calculated readiness factor is $K = 94.45$ and means that the vehicles from the examined group for almost 95% of the time remain in the technical readiness state.

3.3. Time of first transition of the vehicle exploitation process to a subset of states (time of failure-free operation)

Another important characteristic describing the processes of vehicle exploitation is the time of the first transition of the process to a separate state or a set of states $\{A\}$ [18]. Based on the distribution of this time and its parameters, the probability of vehicles being in a particular state or set of states may be determined [20, 37]. Function in a form:

$$\Phi_{iA}(t) = P\left(\Theta_A \le t \,|\, X(0) = i\right), t \ge 0 \tag{30}$$

is a distribution function of the distribution of a random variable $\Theta_A = \tau_{\Delta_A}$, which means the time elapsing from the moment when the semi-Markov process takes the value $i \in A'$ until the moment when the process takes any value from the subset of states A, where $A \subset S$ and $A' = S - A$.
while:

$$\Delta_A = \min\left\{n \in N : X(\tau_n) \in A\right\} \tag{31}$$

For regular semi-Markov processes, in which the subset A is strongly achievable from any state belonging to A', random variables T_{ij} have finite and positive expected values $E(T_{ij})$, there are expected values $E(\Theta_{A'})$ and they are the only solutions of the system of equations [13, 20]:

$$\left(I - P_{A'}\right)\overline{\Theta}_{A'} = T_{A'} \tag{32}$$

where:

$P_{A'}$ – probability matrix of transitions within the set A'

$\overline{\mathrm{E}}_{A'}$ – process kernel specified in the set A'

$T_{A'}$ – random variables of unconditional duration times of the process in the set of states A'

Since in the process under consideration the transport task will be performed if there is no failure of the means of transport, the distribution of time of task execution (failure-free operation of the system) can be found by reducing the original model by the state S_3 - repair. In such a case the subset of states $A' = \{S_1, S_2\}$, while the subset of states $A = \{S_3\}$, and the elements of the equation (32) take the form:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \overline{\Theta}_{A'} = \begin{bmatrix} \overline{\Theta}_{13} \\ \overline{\Theta}_{23} \end{bmatrix}, \quad \overline{T}_{A'} = \begin{bmatrix} E(T_1) \\ E(T_2) \end{bmatrix}, \tag{33}$$

$$P_{A'}(s) = \begin{bmatrix} 0 & p_{12} \\ p_{21} & 0 \end{bmatrix} \tag{34}$$

where:

$$P_{A'} = \left[p_{ik}\right] \quad i,k \in A' \tag{35}$$

is a sub matrix of matrix P_{ij} (Table 1). Random variable Θ_{ij} means the time elapsed between the initial time and the time when the repair condition is first reached, provided that one of the conditions of set A' has commenced at the time considered initial. Hence, it means the time of system failure-free operation. For the analysed semi-Markov model, the matrix equation (32) takes the following form:

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & p_{12} \\ p_{21} & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} \overline{\Theta}_{13} \\ \overline{\Theta}_{23} \end{bmatrix} = \begin{bmatrix} E(T_1) \\ E(T_2) \end{bmatrix} \tag{36}$$

After substituting appropriate values from Table 1 and Table 4 we get:

$$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0.8 \\ 1 & 0 \end{bmatrix}\right) \cdot \begin{bmatrix} \overline{\Theta}_{13} \\ \overline{\Theta}_{23} \end{bmatrix} = \begin{bmatrix} 844.2 \\ 479 \end{bmatrix} \tag{37}$$

which comes down to solving the system of equations:

$$\begin{cases} \bar{\Theta}_{13} - 0.8\bar{\Theta}_{23} = 844.2 \\ -\bar{\Theta}_{13} + \bar{\Theta}_{23} = 479 \end{cases} \tag{38}$$

The results of calculations of the above equations are presented in Table 6.

Table 6. Values of the elements of the matrix $\bar{\Theta}$ of the time of first transition for all vehicles

$\bar{\Theta}$	$[min]$	$[h]$
$\bar{\Theta}_{13}$	6137	102.3
$\bar{\Theta}_{13}$	6616	110.3

If the initial decomposition of the exploitation process is a vector:

$$p = [p_1, p_2, p_3] \tag{39}$$

which in the examined process, according to the original assumption (12), takes the form:

$$p = [1,0,0] \tag{40}$$

then the first row of the single-column matrix solving this equation is the expected value of the task execution time, which in this case is more than 102 hours.

It is also possible to determine the time distribution for the correct operation of the object. Using the information that the probability of the transition $P_{ij}(t)$, defined as conditional probabilities [20]:

$$P_{ij}(t) = P\{X(t) = j \mid X(0) = i\},\ i,j \in S \tag{41}$$

fulfil the Feller's equations:

$$P_{ij}(t) = \delta_{ij}[1 - G_i(t)] + \sum_{k \in S} \int_0^t P_{kj}(t-x)\,dQ_{ik}(x),\ i,j \in S \tag{42}$$

it is possible to find the solution to this system using Laplace – Stieltjes transformation:

$$P_{ij}(t) = \delta_{ij}[1 - G_i(t)] + \sum_{k \in S} \int_0^t P_{kj}(t-x)\,dQ_{ik}(x),\ i,j \in S \tag{43}$$

$$\tilde{p}_{ij}(s) = \int_0^\infty e^{-st}\,dP_{ij}(t) \tag{44}$$

$$\tilde{q}_{ik}(s) = \int_0^\infty e^{-st}\,dQ_{ik}(t) \tag{45}$$

where $Q_{ik}(t)$ is the kernel of the process of renewal of a subset of states A' while $G_i(t)$ denotes the distribution function of a random variable T_i of the duration of the *i-th* state of the semi-Markov process, regardless of the state to which the transition occurs at the moment τ_{n+1} [13]:

$$G_i(t) = P\{T_i < t\} = P\{\tau_{n+1} - \tau_n < t / X(\tau_n) = 1\}, i \in S \tag{46}$$

In this case, the above system of integral equations is correspondent to the system of algebraic equations with unknown transforms $p_{ij}(s),\ i,j \in S$:

$$\tilde{p}_{ij}(s) = \delta_{ij}\left[\frac{1 - \tilde{g}_i(s)}{s}\right] + \sum_{k \in S} \tilde{q}_{ik}(s)\tilde{p}_{kj}(s),\ i,j \in S \tag{47}$$

the system in the matrix notation takes the form:

$$\tilde{P}(s) = \frac{1}{s}[I - \tilde{q}(s)]^{-1}[1 - \tilde{g}(s)] \tag{48}$$

When solved, a transforms matrix is obtained. Since the initial state is the state S_1, the first line is simultaneously a one-dimensional distribution of the process.

For of the examined system:

$$Q(t) = \begin{bmatrix} 0 & Q_{12}(t) \\ Q_{21}(t) & 0 \end{bmatrix} \tag{48}$$

where $Q_{12}(t)$ and $Q_{21}(t)$ are distribution functions of estimated Gamma decompositions:

$$Q_{12}(t) = \frac{e^{-\frac{t}{\beta_1}} t^{-1+\alpha_1} \beta^{-\alpha_1}}{\Gamma[\alpha_1]},\quad t > 0 \tag{50}$$

$$Q_{21}(t) = \frac{e^{-\frac{t}{\beta_2}} t^{-1+\alpha_2} \beta^{-\alpha_2}}{\Gamma[\alpha_2]}\quad t > 0 \tag{51}$$

and:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \tag{52}$$

since for the Gamma decomposition the Laplace-Stieltejes transform takes the form:

$$\tilde{f}(s) = \left(\frac{\beta}{\beta + s}\right)^\alpha \tag{53}$$

the elements of equation (48) take the form:

$$\frac{1}{s}[I - \tilde{q}(s)]^{-1} = \begin{bmatrix} \frac{0.1}{s\left(1 - \beta_1^{\alpha_1}(s+\beta_1)^{-\alpha_1}\beta_2^{\alpha_2}(s+\beta_2)^{-\alpha_1}\right)} & \frac{0.1\beta_1^{\alpha_1}(s+\beta_1)^{-\alpha_1}}{s\left(1 - \beta_1^{\alpha_1}(s+\beta_1)^{-\alpha_1}\beta_2^{\alpha_2}(s+\beta_2)^{-\alpha_1}\right)} \\ \frac{0.1\beta_2^{\alpha_2}(s+\beta_2)^{-\alpha_1}}{s\left(1 - \beta_1^{\alpha_1}(s+\beta_1)^{-\alpha_1}\beta_2^{\alpha_2}(s+\beta_2)^{-\alpha_1}\right)} & \frac{0.1}{s\left(1 - \beta_1^{\alpha_1}(s+\beta_1)^{-\alpha_1}\beta_2^{\alpha_2}(s+\beta_2)^{-\alpha_1}\right)} \end{bmatrix} \tag{54}$$

and:

$$[1 - \tilde{g}(s)] = \begin{bmatrix} 1 - \beta_1^{\alpha_1}(s+\beta_1)^{-\alpha_1} & 1 \\ 1 & 1 - \beta_2^{\alpha_2}(s+\beta_2)^{-\alpha_1} \end{bmatrix} \tag{55}$$

The solution is a matrix whose elements of the first line are as follows:

$$\tilde{P}_1(s) = \frac{0.1\beta_1^{\alpha_1}(s+\beta_1)^{-\alpha_1}}{s\left(1 - \beta_1^{\alpha_1}(s+\beta_1)^{-\alpha_1}\beta_2^{\alpha_2}(s+\beta_2)^{-\alpha_1}\right)} + \frac{0.1\left(1 - \beta_1^{\alpha_1}(s+\beta_1)^{-\alpha_1}\right)}{s\left(1 - \beta_1^{\alpha_1}(s+\beta_1)^{-\alpha_1}\beta_2^{\alpha_2}(s+\beta_2)^{-\alpha_1}\right)} \tag{56}$$

$$\tilde{P}_2(s) = \frac{0.1}{s\left(1-\beta_1^{\alpha_1}(s+\beta_1)^{-\alpha_1}\beta_2^{\alpha_2}(s+\beta_2)^{-\alpha_1}\right)} + \frac{0.1\beta_1^{\alpha_1}(s+\beta_1)^{-\alpha_1}\left(1-\beta_2^{\alpha_2}(s+\beta_2)^{-\alpha_1}\right)}{s\left(1-\beta_1^{\alpha_1}(s+\beta_1)^{-\alpha_1}\beta_2^{\alpha_2}(s+\beta_2)^{-\alpha_1}\right)} \tag{57}$$

After calculating the reverse transforms, the boundary distribution of the intensity of use of the object is obtained. For the state S_1 we get the function in the form:

$$P_1(t) = 0.0026857\,e^{-0.66956t} - 0.006151\,e^{-0.23043t} - 0.129546\,e^{-0.0446957t} + 0.6$$

The graph of this function is shown in Fig. 3

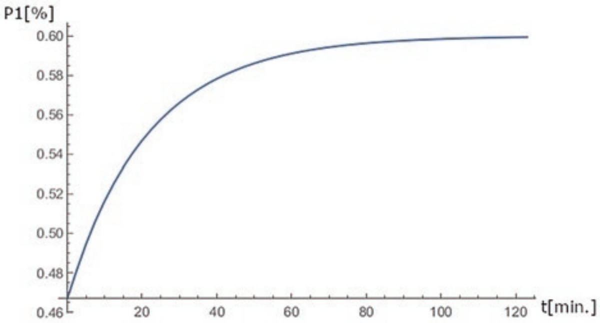

Fig. 3. Function graph $P_1(t)$

The function stabilizes in about 120 minutes, and within the boundary, for $t \to \infty$, it aims towards the previously calculated boundary value of the semi-Markov process of $P_1 = 60\%$.

4. Conclusions

The use of semi-Markov processes allows to determine the boundary readiness factor and to carry out the analysis of the duration times of distinguished operating states of special vehicles. It also enables an objective assessment of the intensity of vehicle operation and the time of its failure-free operation. Analysing readiness factors, it is possible to search for optimal algorithms of vehicle operation and maintenance, as well as to analyse the quality of vehicle fleet selection.

The validity of the above assumptions was confirmed by the conducted research. The proposed semi-Markov model made it possible to diagnose the system of exploitation of police cars indicating that it is characterized by a satisfactory level of probability of vehicles being in the state of operation ($P_1 = 0.6$) and ready-to-be-used ($P_2 = 0.34$). The forecast technical readiness factor amounted to $K = 95\%$.

Therefore, the effectiveness of the application of semi-Markov processes to model the readiness of special vehicle exploitation systems has been demonstrated. The three-state model distinguishing the state of operation of the vehicle, the state of ready-to-be-used and the state of repair (technical maintenance) proved to be justified. In this case, it was not necessary to create complex, multi-state structures of the exploitation process model requiring advanced computational programs. The presented, three-state model is expandable in a situation where a deeper analysis of selected aspects of the system readiness would be necessary.

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Anna BORUCKA
Military University of Technology
ul. Kaliskiego 2, 01-476, Warsaw, Poland

Andrzej NIEWCZAS
Motor Transport Institute,
ul. Jagiellońska 80, 03-301, Warsaw, Poland

Kamila HASILOVA
University of Defence,
Kounicova 65, 662 10 Brno, Czech Republic

E-mails: anna.borucka@wat.edu.pl, andrzej.niewczas@its.waw.pl, kamila.hasilova@unob.cz