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OPTIMISING EXTENDED WARRANTY POLICIES FOLLOWING THE TWO-DIMENSIONAL WARRANTY WITH REPAIR TIME THRESHOLD

OPTYMALIZACJA POLITYKI GWARANCYJNEJ W OKRESIE PO WYGAŚNIĘCIU DWUWYMIAROWEJ GWARANCJI Z USTALONĄ GÓRNĄ GRANICĄ CZASU NAPRAWY

This paper considers an optimal extended warranty policies after the expiration of base two-dimensional warranty with repair time threshold. During the base two-dimensional warranty period, each failure of the equipment can be either replaced or minimally repaired depending on a pre-specified repair time threshold. After the base warranty expires, the length of an extended warranty policy is available for selection. The equipment is minimally repaired on each failure during the extended warranty. In this study, the length of the extended warranty period is optimized by minimizing the expected cost rate incurred over the whole warranty coverage, from the views of customs and manufacturers respectively. For the purpose of illustration, we present and discuss some numerical examples. The effect of repair time threshold on the optimal strategy is also investigated numerically.

Keywords: *extended warranty, repair time threshold, two-dimensional warranty, renewing warranty.*

W niniejszej pracy rozważano optymalną politykę przedłużania gwarancji po wygaśnięciu podstawowej gwarancji dwuwymiarowej z ustaloną górną granicą czasu naprawy. W podstawowym okresie obowiązywania gwarancji dwuwymiarowej, po każdej awarii urządzenie zostaje poddane minimalnej naprawie, lub – jeśli naprawa nie może być wykonana we wcześniej ustalonym czasie naprawy – wymienione. Po wygaśnięciu gwarancji podstawowej, konieczne jest wybranie długości okresu obowiązywania gwarancji rozszerzonej. Podczas trwania okresu gwarancji przedłużonej, sprzęt naprawia się w sposób minimalny (naprawa minimalna) po każdorazowym uszkodzeniu. W niniejszej pracy, optymalizowano długość przedłużonego okresu gwarancyjnego poprzez minimalizację oczekiwanych kosztów poniesionych podczas całego okresu trwania gwarancji; optymalizację przeprowadzono z perspektywy klienta jak i producenta . Dla ilustracji, przedstawiono i omówiono wybrane przykłady numeryczne. Przeprowadzono także analizę numeryczną wpływu górnej granicy czasu naprawy na optymalną strategię gwarancyjną.

Słowa kluczowe: *przedłużona gwarancja, górna granica czasu naprawy, gwarancja dwuwymiarowa, odnowienie gwarancji.*

1. Introduction

Along with increasing the warranty period for complex systems, reducing the warranty servicing costs has become an issue of great importance to the manufacturers. One possible way to reduce the expected warranty servicing cost is by making sound decision on the product warranty and maintenance strategies [21]. A great number of warranty and maintenance models have been proposed in the literature. Blischke and Murthy [1, 2, 3] reviewed a variety of warranty policies and their cost models among them. Shafiee and Chukova [21] reviewed the literature published between 2001 and 2011. This paper is the first identifiable academic literature review to deal with warranty and maintenance and suggested a possible classification of the mathematical models employed in this area of research.

In recent years, warranties and maintenance models for various products such as operating in discrete time [7], products with different failure rate [5], products with competing failures [22], products from heterogeneous population [11] are discussed. Optimizations of warranty and maintenance policies for various system such as parallel systems [13], systems with multiple-failure-mode [14], systems with reliability thresholds [15] and so on are considered. Several two-phase warranty policies(Jung et al. [10], Wang and Su [26], Su and Wang [23]) are proposed.

An increasing number of literatures focus on two-dimensional warranties and maintenance models recently(see Tong et al. [24], Huang et al. [8], Wang et al. [27], Cheng et al.[6], Wang and Su [26], Su and Wang [23], Peng et al. [20], Huang et al. [9], Wang et al. [25]). The traditional two-dimensional warranty policies are characterized by a region on the plane, where the axes represent the age and the usage of a system. These models under the traditional two-dimensional warranty policies have widely usefulness in various situations. It may be difficult for many products to obtain their usage information, but their failure and repair time information can be obtained conveniently by manufacturers. Furthermore many states in USA have acted a regulation named "lemon laws" to protect the customers who purchased defective vehicles, either new or used. The lemon law [4,12] regulates that, when the manufacturer or its authorized service agent cannot conform a vehicle to its warranty either by repairing or correcting any nonconformity within a repair time threshold or after a reasonable number of attempts, the manufacturer must replace the vehicle. The policy of warranty services for repair, replacement and refund in China has similar terms. Based on the aforementioned background, another two-dimensional warranty policy with repair threshold that is different from the classical one is introduced by Park et al. [16]. Under the two-dimensional warranty policy both failure time and repair time are considered simultaneously to determine the warranty action upon the equipment's failure. Manufacturers specify a repair time threshold in advance to set the limit on the repair time for the failed equipment. In the case that repair service can't be completed within the repair time threshold, the failed equipment is to be replaced by a new one carrying with the same original warranty terms instead of continuing the repair works. Otherwise, only minimal repair is conducted. Such a warranty policy is referred to as ''two-dimensional warranty policy with repair threshold'' throughout this paper. They obtained the optimal maintenance policy following the expiration of the warranty period by minimizing the expected cost rate per unit time during the life cycle of the system, from the customer's perspective.

The two-dimensional warranty policy with repair time threshold receives a great attention from many researchers due to its practical applications in automobile industry. From the view of manufacturers, Park et al. [17] determined the optimal warranty period to minimize the expected cost rate during the warranty. Park and Pham [18] proposed cost models for age replacement and block replacement policies under the two-dimensional warranty. Park et al. [19] proposed a periodic preventive maintenance policy after the expiration of the two-dimensional warranty.

Despite the fact manufacturers offer various types of warranty policies at the sale of an equipment, it is necessary for customers to purchase extended warranty services for safeguarding equipment maintenance. Furthermore the performing of extended warranty services is convenient and free from worry for customers. So many customers chose to purchase an extended warranty, instead of planning the maintenance activities after the expiration of a base warranty themselves. In this case the length of the extended warranty period is important from a customer's perspective. A long extended warranty period may be more costly, but it can be more cost-effective in the long run as any failures during the extended warranty will be served by the warranty provider. On the other hand, if an extended warranty service is scheduled for a period of time that is excessively long, the risk of potential costs for the manufacturer may increase. Therefore, establishing an optimal extended warranty policy would be crucial, particularly from the cost perspective.

In this study, we assume the pro-rata renewal two-dimensional warranty with repair time threshold is adopted during the base warranty. We also assume that the length of the extended warranty policies is available for selection. The length of the extended warranty period is optimized to minimize the expected cost per unit time over the whole warranty coverage.

The remainder of this paper is organized as follows. In Section 2, model formulation is given. Cost structures, from the view of customers and manufacturers respectively, are presented. In Section 3, the explicit expressions for the expected cost rates over the whole warranty coverage, from the views of customers and manufacturers respectively, are given. The optimal solution for the extended warranty period is obtained by minimizing the expected cost rate. In Section 4 numerical examples are presented for illustrative purpose, assuming the Weibull failure and repair times. Concluding remarks are given in Section 5.

Nomenclature

pdf, *cdf*	probability density function, cumulative distribution function, respectively
i.i.d	independent, identically distributed
r. v.	random variable
ECR	expected cost rate
MRR	minimal repair-replacement
$RMRR$	renewable minimal repair-replace
T , Y	failure time and repair time, respectively
$f(t)$, $F(t)$, $\bar{F}(t)$	density function, distribution function, and reliability function of T , respectively
$g(y)$, $G(y)$, $\bar{G}(y)$	pdf, cdf and reliability function of Y , respectively
C_{U_r}	total replacement cost the customer is responsible for during the based warranty period
C_{Uf}	total failure stoppage cost the customer is responsible for during the warranty period
C_{Mr}	total replacement cost the manufacturer is responsible for during the warranty period
C_{Mm}	total minimal repair cost the manufacturer is responsible for during the warranty period
c_r	unit cost of replacement
c_{Uf_1}	unit cost of replacement stoppage during the based warranty period
c_{Uf_2}	unit cost of minimal repair stoppage during the based warranty period
c_{Uf_3}	unit cost of minimal repair stoppage during the extended warranty period
c_{Mf_2}	unit cost of minimal repair during the base warranty period
c_{Mf_3}	unit cost of minimal repair during the extended warranty period
w	the length of the original warranty period
c_e	unit cost for unit length of extended warranty period
L'	unit length of extended warranty period
r_0	repair time threshold

2. Model formulation

The warranty policy considered in this paper is a combination of base and extended warranties. The equipment is sold with an original warranty period w. The warranty term works as follows.

During the warranty period, a renewable minimal repair-replace (RMRR) warranty policy is performed. Under the RMRR policy, the manufacturer starts the minimal repair for the failed equipment immediately. The manufacturer sets the repair time threshold, r_0, in advance, which works as the time limit for the minimal repair service for the customer's satisfaction. If manufacturer successfully provides the minimal repair to fix the failed equipment within the repair time limit, the warranty would be effective only in the remaining warranty period and it would not be renewed. On the other hand the warranty policy is renewed for the replaced equipment with exactly the same warranty terms as the original one.

Once the base warranty expires, the customer purchases extended warranty service for a time period kL', where L' is a given time period and k $(k = 0,1,2,\cdots)$ can be optimized. During the extended warranty period, all failures (whose repair time can be within or not within the limit) are corrected with minimal repairs. The possible system state evolution path for such a warranty policy is shown in Fig. 1.

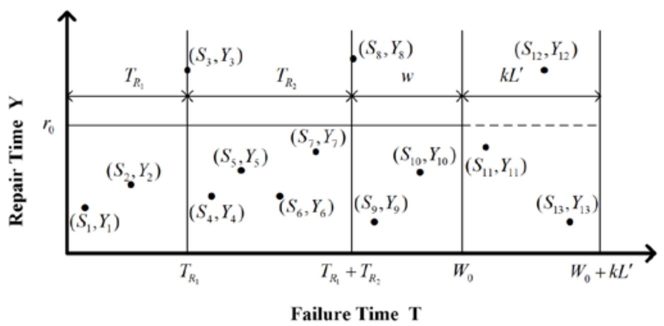

Fig. 1. A possible system state evolution path with two times of renewals

In Fig. 1, $(S_i, Y_i)(i = 1,2,\cdots,14)$ denote time instants of failures and times for repairs. The repair times for the 1th and 2th failures don't exceed the threshold r_0. While the 3th one exceeds it. Furthermore the total times for the three failures don't exceed the original warranty period w. So the equipment is replaced at time S_3 and the time for the 1th renewal of the warranty terms is T_{R_1}. Similarly, the equipment is replaced at the 8th failure and the inter-arrival time between the 1th and 2th renewal is T_{R_2}. During the following w times, the 9th and the 10th failures happen and their repair times don't exceed the threshold. Hence the base warranty period ends and the base warranty period is $T_{R_1} + T_{R_2} + w$. We denoted by W_0. Over the extended warranty period, the 11th, the 12th and the 13th failures happen and their repair times can exceed or not exceed the threshold. The warranty period is defined from the purchasing point of the equipment to the end of the extended warranty time at which the customer buy kL' extended warranty period after the based warranty expires, as shown in Fig.2.

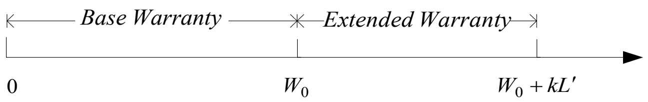

Fig. 2. The warranty span with k extended warranty periods

Length of time necessary to repair/replace the failed system is negligible and not included within the warranty period. Under the renewing PMRR, the customer is responsible for the pro-rated replacement cost. Failure cost, which incurs for each failure due to the stoppage of system operation, is charged to the customer during the based and extended warranty. Minimal repair is performed at no charge to the customer during the based and extended warranty period. We assume that all the warranty claims are valid and accepted.

From the view of customers and manufacturers respectively, the number of the extended warranty periods k is optimized by minimizing the expected cost rate (ECR) per unit time over the whole warranty coverage in this paper.

3. Formulation of the expected cost rate

3.1. Expected length of the warranty period

Let T_R be the time interval between two adjacent replacements. According assumptions in Section 2, the warranty terms would be renewed if $T_R < w$. Otherwise the warranty policy would not be renewed and the based warranty period will end. Let W_0 denote the length of the base warranty. It is clear that W_0 is a r. v. depending on the total number of warranty renewals, the inter-arrival times between two successive renewals of warranty terms and the length of original warranty period. Then, the based warranty period W_0 can be expressed as:

$$W_0 = T_{R_1} + T_{R_2} + \cdots T_{R_{N_R}} + w, \quad (1)$$

where N_R is the number of renewals during the base warranty period, $T_{R_i}(i = 1,2,\cdots,N_R)$ are the time intervals between two subsequent warranty renewals and they are independent and identically distributed non-negative random variables.

After the based warranty expires, the customer purchases k units of extended warranty periods, then the warranty period $L(k)$ can be expressed as:

$$L(k) = W_0 + kL'. \quad (2)$$

From Eq. (1), Eq. (2) can be rewritten as:

$$L(k) = T_{R_1} + T_{R_2} + \cdots + T_{R_{N_R}} + w + kL'. \quad (3)$$

Let T be the r. v. denoting the failure time of the equipment having $f(t)$, $h(t)$, $F(t)$, $\bar{F}(t)$ as its probability density function, hazard function, cumulative distribution function and reliability function, respectively. Let $T_i(i = 1,2,\cdots)$ denote the time intervals between the $i-1$ th and i th failures of the equipment, then $T_i(i = 1,2,\cdots)$ are i.i.d. continuous with the same distribution to T and $T_i = S_i - S_{i-1}(i = 1,2,\cdots,S_0 = 0)$, where S_i is the time that the i th failure occurs. We assume that $Y_i(i = 1,2,\cdots)$ is the repair time for the i th failure. They are assumed to be i.i.d. continuous r.v.'s having $G(y)$ and $\bar{G}(y)$ as its cdf and reliability function, respectively. According to assumptions in Section 2, upon failures, the system is replaced with probability $\bar{G}(r_0)$ and corrected minimally with probability $G(r_0)$.

According to Block et al. [4], the cumulative distribution function of the time to minimal repairs is given by:

$$F_M(t) = 1 - \exp[-\int_0^t G(r_0) h(u) du]. \quad (4)$$

The cdf of the time interval between two subsequent replacements (T_R), is:

$$F_R(t) = 1 - \exp[-\int_0^t \overline{G}(r_0)h(u)du]. \quad (5)$$

Deriving on Eq. (5), the pdf of T_R , can be given by:

$$f_R(t) = \overline{G}(r_0)h(t)\exp[-\int_0^t \overline{G}(r_0)h(u)du]. \quad (6)$$

Since the warranty terms would be renewed when $T_R < w$, the time interval between two subsequent warranty renewals $T_{R_i}(i = 1,2,\cdots)$ is the truncated distribution of T_R over the interval $(0,w)$ and its pdf is $f_R(t)/F_R(w)(0 < t < w)$. Note that this expression is different from Eq.(2) of Park et al. [16].

Given $N_R = n$, the conditional expected length of based warranty period can be represented as:

$$E(W_0 \mid N_R = n) = \sum_{j=1}^{n} E(T_{R_j}) + w, \quad (7)$$

where $T_{R_i}(i = 1,2,\cdots,n)$ are i.i.d. non-negative random variables. Eq. (7) can be written as:

$$E(W_0 \mid N_R = n) = n\frac{\int_0^w tf_R(t)dt}{F_R(w)} + w. \quad (8)$$

According to assumptions in Section 2, the warranty terms would be renewed if the time interval between two adjacent replacements $T_R < w$. Thus, if $N_R = n$, then the event $T_R < w$ occurs in the first n trials and the event $T_R > w$ occurs in the following $n+1$ th trial. Since these events are independent of each other,

$$P(N_R = n) = (F_R(w))^n(1 - F_R(w)), n = 0,1,2,\cdots. \quad (9)$$

Note that this expression is also different from Eq. (4) in Park et al. [16].

By taking the expectation for the conditional expectation of Eq. (8) with respect to N_R , we obtain the following expected length of the based warranty period:

$$\begin{aligned} E(W_0) &= E(E(W_0 \mid N_R)) \\ &= \sum_{n=0}^{\infty} P(N_R = n)\, E(W_0 \mid N_R = n) \\ &= \sum_{n=0}^{\infty} (n\int_0^w tf_R(t)dt \,/\, F_R(w) + w)(F_R(w))^n(1 - F_R(w)) \\ &= \frac{1}{1 - F_R(w)}\int_0^w tf_R(t)dt + w. \end{aligned} \quad (10)$$

From Eq.(10), the expected length of whole warranty period is:

$$E(L(k)) = E(W_0) + kL' = \frac{1}{1 - F_R(w)}\int_0^w tf_R(t)dt + w + kL'. \quad (11)$$

3.2. Expected cost rate from the view of customers

In this section, we will derive the expected cost rate per unit time during the warranty period of the equipment from the view of customers. Let C_{Ur} denote total replacement cost that the customer is responsible for during the base warranty period. Assume that C_{Uf} represents the total failure stoppage cost during the base and extended warranty that charged to the customer. Further, let c_e be the fixed unit cost for unit length of extended warranty period. Then, the user would be charged the total amount of cost equaling $C_{Ur} + C_{Uf} + kc_e$.

Under the renewing PMRR, the customer is responsible for prorated replacement cost during the based warranty period and thus, the replacement cost can be expressed as a function of T_{R_i} 's as (see Park et al. [16]),

$$C_{Ur} = \sum_{j=1}^{N_R} c_r \frac{T_{R_j}}{w}, \quad (12)$$

where c_r is the unit cost of replacement.

Given $N_R = n$, the conditional replacement cost can be represented as:

$$E(C_{Ur} \mid N_R = n) = \frac{c_r}{w} n\frac{\int_0^w tf_R(t)dt}{F_R(w)}. \quad (13)$$

Consequently, the total expected replacement cost during the based warranty period:

$$\begin{aligned} E(C_{Ur}) &= E(E(C_{Ur} \mid N_R)) \\ &= \sum_{n=0}^{\infty} P(N_R = n)E(C_{Ur} \mid N_R=n) \\ &= \sum_{n=0}^{\infty} P(N_R = n)\sum_{j=1}^{n} c_r \frac{E(T_{R_j})}{w} \\ &= \frac{c_r}{w}\sum_{n=0}^{\infty}(F_R(w))^n(1 - F_R(w))n\frac{\int_0^w tf_R(t)dt}{F_R(w)} \\ &= \frac{c_r}{w(1 - F_R(w))}\int_0^w tf_R(t)dt. \end{aligned} \quad (14)$$

Under the renewable MRR warranty policy, the expected total number of replacements can be given by:

$$\begin{aligned} E(N_R) &= \sum_{n=0}^{\infty} nP(N_R = n) \\ &= \sum_{n=0}^{\infty} n(F_R(w))^n(1 - F_R(w)) \\ &= \frac{F_R(w)}{(1 - F_R(w))}. \end{aligned} \quad (15)$$

Conditioning on the possible time between two subsequent warranty renewals, the expected total number of minimal repairs between subsequent warranty renewals can be given by:

$$E(N_{M_r}) = \int_0^w \int_0^t G(r_0)h(u)f_R(t) \,/\, F_R(w)dudt. \quad (16)$$

From Eq. (16), the expected total number of minimal repairs during the N_R times renewals,

$$\begin{aligned} E(N_{M_R}) &= E(E(N_{M_R} \mid N_R)) \\ &= \sum_{n=0}^{\infty} P(N_R = n) E(N_{M_R} \mid N_R = n) \\ &= \sum_{n=0}^{\infty} P(N_R = n) n E(N_{M_r}) \\ &= E(N_{M_r}) \sum_{n=0}^{\infty} P(N_R = n) n \\ &= E(N_{M_r}) E(N_R) \\ &= \frac{1}{1-F_R(w)} \int_0^w \int_0^t G(r_0) h(u) f_R(t) du dt. \end{aligned} \quad (17)$$

During the last base warranty period w, minor repairs are performed with intensity function $G(r_0)h(t)(0<t<w)$, consequently, the expected total number of minor repairs is $\int_0^w G(r_0)h(t)dt$.

In sum, the expected total number of minor repairs over the based warranty period is:

$$E(N_M) = \frac{1}{(1-F_R(w))} \int_0^w \int_0^t G(r_0) h(u) f_R(t) du dt + \int_0^w G(r_0) h(u) du \quad (18)$$

During the extended warranty period kL', minimal repair service is conducted on each failure. Consequently, the expected total number of minor repairs $E(N_e) = \int_w^{w+kL'} h(u)du$.

The expected total failure cost due to the stoppage of system operation during the based and extended warranty period is:

$$E(C_{Uf}) = c_{Uf_1} E(N_R) + c_{Uf_2} E(N_M) + c_{Uf_3} E(N_e) \quad (19)$$

where c_{Uf_1} and c_{Uf_2} are unit cost of replacement and minimal repair stoppage time during the based warranty period, respectively. c_{Uf_3} $(c_{Uf_2} < c_{Uf_3} < c_{Uf_1})$ is the unit cost of minimal repair stoppage time during the extended warranty period.

From Eq.(14), Eq.(15), Eq.(17), Eq.(18), Eq.(19), the expected cost rate over the whole warranty period can be formulated as:

$$ECR_U(k) = \frac{E(C_{Ur}) + c_{Uf_1} E(N_R) + c_{Uf_2} E(N_M) + c_{Uf_3} E(N_e) + kc_e}{\frac{1}{1-F_R(w)} \int_0^w t f_R(t) dt + w + kL'}. \quad (20)$$

3.3. Expected cost rate from the view of manufacturers

Let C_{Mr}, C_{Mm} be the r.v.'s representing total replacement cost, total minimal repair cost incurred during the warranty period that the manufacturer is responsible for, respectively. Then (see Park et al. [16]):

$$C_{Mr} = \sum_{j=1}^{N_R} c_r \frac{w - T_{R_j}}{w} \quad (21)$$

According to Eq. (13) and Eq. (15):

$$E(C_{Mr}) = \frac{c_r (w F_R(w) - \int_0^w t f_R(t) dt)}{w(1-F_R(w))}. \quad (22)$$

From Eq.(15) and Eq.(16), the expected total minor repair cost over the whole warranty period can be given as:

$$\begin{aligned} E(C_{Mm}) = c_{Mf2} \frac{1}{1-F_R(w)} \int_0^w \int_0^t G(r_0) h(u) f_R(t) du dt + \\ c_{Mf2} \int_0^w G(r_0) h(u) du + c_{Mf3} \int_w^{w+kL'} h(u) du. \end{aligned} \quad (23)$$

where c_{Mf_2} and c_{Mf_3} ($c_{Mf_2} < c_{Mf_3} < c_r$) are unit costs of minimal repair during the base and the extended warranty period, respectively. The first term in Eq. (23) corresponds to the expected cost incurred during the renewal based warranty period. The second term is those during the non-renewal based warranty period.

The expected cost rate over the whole warranty period can be formulated as:

$$ECR_M(k) = \frac{E(C_{Mr}) + E(C_{Mm})}{\frac{1}{1-F_R(w)} \int_0^w t f_R(t) dt + w + kL'}. \quad (24)$$

Substituting Eq's (22) and (23) into Eq.(24), we can obtain the expression of $ECR_M(k)$.

4. Numerical experiments

4.1. Optimisation of the length of the extended warranty

This section presents numerical examples to illustrate the optimal warranty policy model derived in Section 3. The failure time and the repair time of the equipment are assumed to follow Weibull distributions with the failure rate function $h(t) = (\beta/\eta)(t/\eta)^{\beta-1} (\beta, \eta > 0)$ and cdf $G(t) = 1 - e^{-(t/\lambda)^{\nu}} (\lambda, \nu > 0)$ respectively. The parameter values we set for this particular numerical example are listed in Table 1.

For this particular numerical example, we set $w = 6$ (unit: year) and $r_0 = 1/12$ (unit: year). This implies that if the failed equipment requires the repair time of more than one month, it is rather replaced by a new one instead of being repaired. In practice, the extended warranty of an equipment might has a limit number of possibility. Hence, we consider situations where $L' = 0.25$, $k = \{0, 1, \cdots, 12\}$. Under such a situation, the customer purchases k extended warranties, each with the length of 0.25 years and the longest extended warranty is 3 years.

From the view of customers, the cost rate is decided by unit cost for unit length of extended warranty period (c_e), unit cost of minimal repair stoppage time(c_{Uf_3}) and the length of the extended warranty period after the based warranty expires. So we determine optimal extended warranties for various c_e and c_{Uf_3} by using Matlab software. See Table 2 for detail. The optimal length of extended warranty period

Table 1. Parameter values assumed for our numerical example

w	L'	c_r	c_{Uf_1}	c_{Uf_2}	c_{Mf_2}	r_0	β	η	ν	λ
6	0.25	5.5	0.05	0.02	0.85	1/12	1.097	0.241	0.787	0.033

is denoted by k^* and its corresponding expected cost rate is denoted by $ECR_U(k^*)$.

Table 2 . Optimal extended warranties for various c_e and c_{Uf_3} (from the view of customers)

c_e	c_{Uf3}	k^*	$ECR_U(k^*)$
0.01	0.035	8	0.988291
	0.034	12	0.987665
	0.033	12	0.98696
	0.032	12	0.986256
	0.031	12	0.985551
0.02	0.035	3	0.988809
	0.034	7	0.988537
	0.033	11	0.988019
	0.032	12	0.987319
	0.031	12	0.986615
0.03	0.035	0	0.988945
	0.034	2	0.988895
	0.033	5	0.988707
	0.032	9	0.98832
	0.031	12	0.987679
0.04	0.035	0	0.988945
	0.034	0	0.988945
	0.033	1	0.98894
	0.032	3	0.988835
	0.031	7	0.988561

Table 3. Optimal extended warranties for various c_e and c_{Mf_3} (from the view of manufacturers)

c_e	c_{Mf_3}	k^{**}	$ECR_M(k^{**})$
0.01	1	10	6.427558
	1.01	7	6.428713
	1.02	4	6.429429
	1.03	1	6.429752
	1.04	0	6.429782
	1.05	0	6.429782
0.02	1	12	6.471456
	1.01	9	6.472935
	1.02	6	6.473955
	1.03	3	6.474555
	1.04	0	6.474781
	1.05	0	6.474781
0.03	1	12	6.515258
	1.01	11	6.516934
	1.02	8	6.518271
	1.03	5	6.519163
	1.04	2	6.519652
	1.05	0	6.519779
0.04	1	12	6.559059
	1.01	12	6.560748
	1.02	10	6.562367
	1.03	7	6.563567
	1.04	4	6.564338
	1.05	1	6.564725

For instance, $k^* = 2$ and $ECR_U(k^*)$=0.988876 when c_e=0.01 and c_{Uf_3}=0.035 from Table 2. This indicates that the expected cost rate is minimized when the customer buys 8 units extended warranty, and the customer's cost rate becomes 0.988291 per year over the whole warranty period of the equipment in this situation. In some cases, such as c_e=0.01 and c_{Uf_3}=0.034, c_e=0.02 and c_{Uf_3}=0.032, c_e=0.03 and c_{Uf_3}=0.031 and so on, the cost rate is minimized by buying the longest extended warranty periods. However, in some cases, such as c_e=0.01 and c_{Uf_3}=0.034, c_e=0.02 and c_{Uf_3}=0.032, c_e=0.03 and c_{Uf_3}=0.031 and so on, the cost rate is minimized when customers don't buy extended warranty services.

Similarly, we can obtain the optimal extended warranties for various c_e and c_{Mf_3}, from the view of manufacturers. See Table 3 for detail. The optimal length of extended warranty period and its corresponding expected cost rate are denoted by k^{**} and $ECR_M(k^{**})$ respectively.

4.2. The effect of the repair time threshold on the optimal strategy

To analysis the effect of the repair time threshold r_0 on the optimal strategy, we change it from 1/11.4 to 1/12.6 and set w=6, $L' = 0.25$, $c_r = 5.5$, $c_e = 0.04$, $c_{Uf_1} = 0.05$, $c_{Uf_2} = 0.02$, $c_{Uf_3} = 0.031$, c_{Mf_2}=0.85, c_{Mf_3}=1.02. The optimal strategies and its corresponding expected cost rates are presented in Tables 4 and 5. The tables indicate that the value of r_0 has some effect on k^*, k^{**}, $ECR_U(k^*)$ and $ECR_M(k^{**})$. As r_0 decreases, the length of the optimal extended warranty and the corresponding expected cost rate increase from both the views of customers and manufacturers. It is due to the fact that as

Table 4. Optimal extended warranties for various r_0 (from the view of customers)

r_0	k^*	$ECR_U(k^*)$
1/11.4	4	0.971906
1/11.6	5	0.977926
1/11.8	6	0.983465
1/12	7	0.988561
1/12.2	7	0.993246
1/12.4	8	0.997554
1/12.6	9	1.001518

Table 5. Optimal extended warranties for various r_0 (from the view of manufacturers)

r_0	k^{**}	$ECR_M(k^{**})$
1/11.4	0	6.336242
1/11.6	2	6.39698
1/11.8	5	6.457649
1/12	8	6.518271
1/12.2	11	6.578851
1/12.4	12	6.639496
1/12.6	12	6.700443

r_0 decreases, the chance of the failed equipment is replaced increases and so has a higher cost.

5. Conclusions

An extended warranty decision model considering the "lemon law" is built in this paper. It is of interest and practical especially for the products including automobiles, etc. Under the two-dimensional renewal repair-replacement warranty policy, each failure of the system can be either replaced or minimally repaired depending on a repair time threshold. We obtained the optimal extended warranty period, from the views of customers and manufacturers respectively, so that the expected cost rate over the whole warranty coverage, is optimized. By assuming Weibull distributions for failure and repair times, numerical examples are discussed to demonstrate the applicability of the methodology derived in the paper. Numerical example indicate the larger the repair time thresholds are, the smaller are the expected cost rates.

The repair time the threshold in this study is a constant. However in practice it may be changed with the degradation of the equipment. For customer's satisfaction, the time limits of repair services are shorter at the early stage of the warranty period. So the extended warranty models with different repair time thresholds would be an interesting future research.

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