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THE PROBABILISTIC ANALYSIS AND OPTIMAL DESIGN OF A BEVEL GEAR TRANSMISSION SYSTEM WITH FAILURE INTERACTION

PROBABILISTYCZNA ANALIZA I OPTYMALNE PROJEKTOWANIE UKŁADU PRZEKŁADNI STOŻKOWEJ Z UWZGLĘDNIENIEM INTERAKCJI MIĘDZY USZKODZENIAMI

The modelling of joint probability distributions of structural system with failure interaction and the reliability-based optimal design of system reliability under incomplete probability information remains a challenge that has not been studied extensively. This article aims to investigate the impacts of copulas for modelling dependence structures between each failure mode of a bevel gear transmission system under incomplete probability information. Firstly, a copula-based reliability method is proposed to evaluate the system reliability of the bevel gear transmission system with established performance functions for different failure modes. The joint probability of failure is estimated with selected copula functions based on the marginal distributions of each failure mode that are approximated by moment-based saddlepoint technology. Secondly, a reliability sensitivity problem is formulated and the formulas for calculating the reliability sensitivity with respect to the distribution parameters of the random variables are presented. Finally, the reliability-based robust optimal design problem is discussed and the optimal model is established. The robustness of the system reliability is ensured by involving the reliability sensitivity into the reliability-based design optimization model. A practical example of the bevel gear transmission system is given to verify the validity of the method. The proposed methods for joint failure probability estimation and robust design optimization are illustrated in the example. The failure probabilities of the system under different copulas can differ considerably. The Gaussian and Clayton copula produce the results that mostly close to the Monte Carlo simulation results. The reliability sensitivity-based robust design is performed based on the Clayton copula-based reliability model. The proposed method is based on the comparative analysis with selected copulas, the results obtained could be supplied as a reference for the optimal design of the gear transmission system.

Keywords: *copula; sensitivity analysis; system reliability; bevel gear; optimization.*

Modelowanie wspólnych rozkładów prawdopodobieństwa układu konstrukcyjnego, w którym zachodzą interakcje między uszkodzeniami, oraz oparte na niezawodności optymalne projektowanie takiego układu przy niekompletnych danych na temat prawdopodobieństwa pozostaje wyzwaniem dla badaczy tej tematyki. Niniejszy artykuł ma na celu zbadanie możliwości wykorzystania kopuł (funkcji powiązań) do modelowania struktury zależności pomiędzy poszczególnymi przyczynami uszkodzeń układu przekładni stożkowej w warunkach niepełnej informacji na temat prawdopodobieństwa. W pierwszej kolejności zaproponowano opartą na funkcjach kopułach metodę oceny niezawodności układu przekładni stożkowej z ustalonymi funkcjami stanu granicznego dla różnych przyczyn uszkodzeń. Wspólne prawdopodobieństwo uszkodzenia szacowano za pomocą wybranych kopuł w oparciu o rozkłady brzegowe poszczególnych przyczyn uszkodzeń aproksymowane opartą na znajomości momentów statystycznych metodą punktów siodłowych. Następnie sformułowano problem czułości niezawodności oraz przedstawiono wzory na obliczanie czułości niezawodności w odniesieniu do parametrów rozkładu badanych zmiennych losowych. Wreszcie, omówiono zagadnienie opartego na niezawodności optymalnego projektowania odpornego na działanie zakłóceń oraz opracowano odpowiedni optymalny model. Odporność i niezawodność systemu zapewniono poprzez wprowadzenie do opartego na niezawodności modelu optymalizacji projektowania, parametru czułości niezawodności. Poprawność metody zweryfikowano na przykładzie układu przekładni stożkowej. Proponowaną metodę wspólnej estymacji prawdopodobieństwa uszkodzenia oraz optymalizacji projektowania odpornego zilustrowano za pomocą przykładu Prawdopodobieństwo uszkodzenia systemu może się znacznie różnić w zależności od zastosowanej kopuły. Kopuły Gaussa i Claytona dają wyniki najbardziej zbliżone do wyników symulacji Monte Carlo. Projektowanie odporne oparte na czułości niezawodności wykonano na bazie modelu niezawodności opartego na kopule Claytona. Proponowana metoda opiera się na analizie porównawczej z użyciem wybranych kopuł. Otrzymane wyniki mogą stanowić punkt odniesienia dla optymalnego projektowania układu przekładni zębatej.

Słowa kluczowe: *kopuła; analiza czułości; niezawodność systemu; przekładnia stożkowa; optymalizacja.*

1. Introduction

Generally, multiple failure modes may occur in a gear transmission system. Due to the common-source of the input random variables of the system, such as the external loads, the geometric parameters and the material properties, the existing failure modes would be correlated to a certain extend [2, 13]. From another perspective, it means that a failure may cause another failure occurs more rapidly, or just the opposite. Actually, whether the dependence of different failure modes is considered during the reliability design process may affect the system reliability estimation greatly. The commonly used independent assumption between failure modes usually leads to defects in gear mechanism and the transmission accuracy accompanied with conservative and even unacceptable reliability results while the failure modes are highly dependent. Therefore, it is necessary to involve dependence modelling throughout the design process to provide a more accurate reliability estimation for the gear transmission system with failure interaction, and to enhance the optimization for the design.

Then the question arises. How to model the correlation property and to achieve a more accurate description of related patterns between failure modes? In the existing methods, the Pearson correlation coefficient is the most widely used method [15]. However, the method is just an approximation of the actual situation, which limits the use of occasions. When the limit state functions are nonlinear and the properties of the input random variables are unknown, that is common in the gear transmission system, the linear correlation coefficient cannot reflect the true dependency structure. Thus, a more reasonable and valid modelling approach should be put forward to realize the dependence modelling and the measure of the degree of relationship between failure modes in the gear transmission system. It is worth mentioning that the copula theory has found widespread application for constructing joint probability distribution of multivariate data, particularly bivariate data [9, 14, 16]. Copulas are functions that join multivariate distribution functions to their one-dimensional marginal distribution functions with undetermined coefficients [18]. It means that the joint modelling of distributions could be divided into two aspects, one is the approximation of the marginal distributions and the other is the connection of the marginal distributions with selected copulas. The copula method has been proved as an effective mathematical tool and greatly reduce the difficulty of joint probabilistic modelling. There are many copula families in the literature such as the Gaussian, t, Clayton, Gumbel, Frank and Farlie-Gumble-Morgensterm (FGM) copulas [23]. These copulas differ in the different dependence structures of their own.

Many literatures have investigated the copulas in terms of multivariate modelling in the fields such as statistics [24], hydrology [25], engineering [12], and etc. However, the copula method has not yet received enough attention in the field of mechanical design, especially when it comes to the design of the mechanical components involve dependent failure modes. As an important transmission device, the bevel gear transmission system contains a variety of failure modes. Although research literatures on bevel gear pair addressing design optimization and reliability can be found [3, 19, 21], there is still a need for more in-depth research on the joint failure modelling of the bevel gear transmission system.

The dependency structure between different failure modes may be a positive correlation, negative correlation and etc. In order to effectively describe the potential dependency structure, the selected copulas are comparative analysed with the purpose to put forward a system reliability method for the bevel gear transmission system with failure interaction. The third-moment saddlepoint approximation technique are adopted so as to guarantee the system reliability estimation results and meet the needs of the engineering calculation [10]. The system reliability of the gear transmission system is estimated with the bound reliability theory based on the dependence model with different copulas. The work done in this paper aims to investigate the impacts of copula selection on the reliability of the bevel gear transmission system under incomplete probability information and failure interaction. This article is organized as follows. In Section 2, the performance functions of the bevel gear transmission system are established based on the tooth contact strength and the tooth bending strength. The system reliability estimation method based on the saddlepoint technology and the copula functions is proposed in Section 3. The moment-based saddlepoint approximation is used to obtain the marginal probabilistic distributions. The narrow bounds reliability method and the copula theory are combined to evaluate the system reliability. In the next Section, the reliability-based optimal design model is established based on the reliability sensitivity. The robustness of the system reliability is ensured by minimize the reliability sensitivity with respect to the highly dependent random variables. Summary and conclusions are given in the final Section.

2. The reliability modeling of the bevel gear transmission system

The schematic diagram of a bevel gear transmission is shown in Figure 1. As is shown in the figure, 1 and 2 refers to the pinion and the gear, respectively. According to the theory of machines and mechanisms, there are multiple failure modes exist when a bevel gear transmission system fails. To investigate the reliability of each failure mode and the system reliability considering the failure interaction, the reliability model for different failure modes of the gear transmission system is firstly defined.

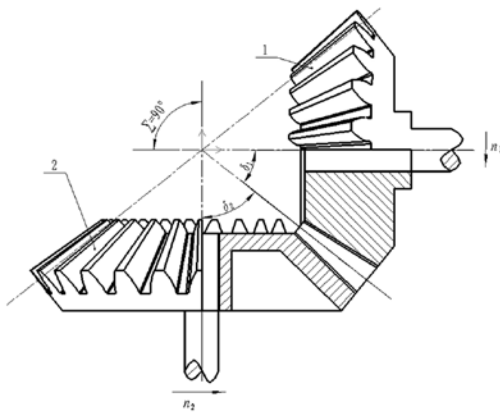


Fig. 1. The bevel gear transmission system

2.1. Reliability model based on the tooth contact strength

The contact stress of the bevel gears can be calculated with equation (1):

$$\sigma_H = Z_{M-B} Z_H Z_E Z_{LS} Z_\beta Z_K \sqrt{\frac{F_{mt}}{d_{v1} l_{bm}} \frac{u_v + 1}{u_v} K_A K_V K_{H\beta} K_{H\alpha}} \quad (1)$$

The contact fatigue strength of the bevel gears can be calculated with equation (2):

$$\sigma'_{H\lim} = \sigma_{H\lim} Z_{NT} Z_X Z_L Z_R Z_V Z_W \quad (2)$$

The limit state function is thus modeled as:

$$g_1(\boldsymbol{X}_1) = \sigma'_{H\lim} - \sigma_H \quad (3)$$

where $\boldsymbol{X}_1$ is the vector of the basic random variable and:

$$\boldsymbol{X}_1 = \left[Z_{M-B}, Z_H, Z_E, Z_{LS}, Z_\beta, Z_K, F_{mt}, d_{v1}, l_{bm}, K_A, K_V, K_{H\beta}, K_{H\alpha}, \sigma_{H\lim}, Z_{NT}, Z_X, Z_L, Z_R, Z_V, Z_W \right]^{\mathrm{T}}$$

2.2. Reliability model based on the tooth bending strength

The bending stress of the bevel gears can be calculated with equation (4):

$$\sigma_F = Y_{Fa} Y_{sa} Y_e Y_K Y_{LS} \frac{F_{mt}}{bm_{mn}} K_A K_V K_{F\beta} K_{F\alpha} \tag{4}$$

The bending fatigue strength of the bevel gears can be calculated with equation (5):

$$\sigma'_{F\lim} = \sigma_{F\lim} Y_{ST} Y_{NT} Y_{\delta relT} Y_{RrelT} Y_X \tag{5}$$

The limit state function of the pinion could be modeled as:

$$g_2(\boldsymbol{X}_2) = \sigma'_{F\lim} - \sigma_F \tag{6}$$

where:

$$\boldsymbol{X}_2 = [Y_{Fa1}, Y_{sa1}, Y_e, Y_K, Y_{LS}, F_{mt}, b, m_{mn}, K_A, K_V, K_{F\beta}, K_{F\alpha}, \sigma_{F1\lim}, Y_{ST}, Y_{NT}, Y_{\delta relT1}, Y_{RrelT1}, Y_{X1}]^T$$

.

Similarly, the limit state function of the gear could be modeled as:

$$g_3(\boldsymbol{X}_3) = \sigma'_{F\lim} - \sigma_F \tag{7}$$

where:

$$\boldsymbol{X}_3 = [Y_{Fa2}, Y_{sa2}, Y_e, Y_K, Y_{LS}, F_{mt}, b, m_{mn}, K_A, K_V, K_{F\beta}, K_{F\alpha}, \sigma_{F2\lim}, Y_{ST}, Y_{NT}, Y_{\delta relT2}, Y_{RrelT2}, Y_{X2}]^T$$

As described above, three failure modes are defined here. As can be seen from the vectors of random variables X_1, X_2, and X_3, some common random variables are shared in different failure. Thus, it is necessary to establish the joint probabilistic distribution between each failure modes.

3. Joint probabilistic modelling of correlated failure modes with copula function

The word “copula” was first employed in a mathematical or statistical sense by Abe Sklar in the theorem describing the functions that “join together” one-dimensional distribution functions to form multivariate distribution functions [18]. It means that a joint probabilistic distribution can be modelled with a certain copula when the marginal probability distribution is determined. In this work, the marginal probability distribution is approximated with the saddlepoint technology using the statistical moment information. Selected copula functions are used to describe the dependence structure and the system reliability of the bevel gear transmission system is comparatively studied.

3.1. Marginal probability of failure estimation by saddlepoint approximation

As stated above, multiple failure modes may occur in the bevel gear transmission system, which are involved in the reliability analysis with the consideration of correlation among them. Thus, the joint cumulative distribution function (CDF) or the joint probability density function (PDF) of these correlated failure modes has to be established in order to properly evaluate the system reliability. Based on the Sklar theory, the join CDF and the joint PDF is modelled as the combination of a copula function and marginal distributions. Thus, as the first step, the determination of the marginal distribution of each failure mode has to be made.

In this paper, the saddlepoint approximation (SPA) is designed to calculate the CDF and PDF of the performance function $Y=g(X)$ of each failure mode. The most fundamental SPA was proposed by Daniels [5]. The improvements and applications of the method can be found in [7, 8, 11]. The classcal saddlepoint approximation method is established with the most frequently used formulas that is expressed as:

$$F_Y(y) = P(Y \le y) = \Phi(w) + \varphi(w)\left(\frac{1}{w} - \frac{1}{v}\right) \tag{8}$$

Equation (8) is an exact approximation of the CDF of limit state function Y. Symbol $\Phi(\cdot)$ and $\varphi(\cdot)$ represents the CDF and PDF of a standard normal distribution function. Symbol w and v is expressed as:

$$w = sign(t_s)\left\{2\left[t_s y - K_Y(t_s)\right]\right\}^{0.5} \tag{9}$$

$$v = t_s\left[K''_Y(t_s)\right]^{0.5} \tag{10}$$

where $sign$ is the sign function with sign(t_s)= 1, -1, or 0 corresponding to the cases $t_s > 0$, $t_s < 0$, or $t_s = 0$, respectively.

This method requires an explicit cumulant generating function (CGF) which is not always exist for all distribution type. In situations that the CGFs do not exist, the SPA would be unable to use. To overcome such a disadvantage, a moment based saddlepoint approximation method is proposed which requires only the first three moment information of the performance function [10]. The failure probability of a failure mode with the limit-state function $Y=G(X)$ can be represented as:

$$P_f = P(Y \le 0) = P(Y_s \le -\beta_2) = \Phi\left[r + \frac{1}{r}\ln\left(\frac{q}{r}\right)\right] \tag{11}$$

in which:

$$Y_s = (Y - \mu_G)/\sigma_G,\ \beta_2 = \frac{\mu_G}{\sigma_G},$$

$$r = sign(\hat{t}_s)\left\{2\left[\hat{t}_s(-\beta_2) - K_{Ys}(\hat{t}_s)\right]\right\}^{1/2} \quad q = t_s\left[K''_{Ys}(\hat{t}_s)\right]^{1/2},$$

$$K_{Ys}(\hat{t}_s) = \begin{cases} -\dfrac{2}{\gamma_G}\hat{t}_s - \dfrac{2}{\gamma_G^2}\ln\left[\left(1 - \dfrac{\gamma_G}{2}t_s\right)^2\right], (\gamma_G \ne 0) \\ 0.5\hat{t}_s^{\,2}, (\gamma_G = 0) \end{cases}$$

$$K''_{Ys}(\hat{t}_s) = \frac{1}{(1 - 0.5\gamma_G \hat{t}_s)^2}$$

where K_{Ys} is the cumulant generating function (CGF) of the standardized variable Ys, $\gamma_G = \gamma'_G/\sigma_G^3$ is the skewness coefficient and the single saddlepoint is provided as $\hat{t}_s = 2\beta_2/(\gamma_G\beta_2 - 2)$.

The moment-based saddlepoint approximation provides an efficient and accurate result of the marginal failure probability. The method uses the first three moments of the performance function of the failure modes, different to the first order reliability method (FORM) and the fourth moment method, which is applicable in the engineering field commonly with incomplete probability information and not compromise on accuracy.

3.2. System probability of failure estimation with copulas and narrow bounds

Recent advances in mathematics show that the copula-based techniques [17, 20] could be provided as an efficient mathematical tool for the multivariate analysis problem. A copula is a multivariate distribution function of a random vector with uniform marginal distributions in (0,1), which decouples the marginal CDFs and the joint CDF. As thoroughly explained below, we shall use copulas to provide a two-dimension dependence characteristics for different failure modes.

Generally, let $F_i(G_i(X))$ and $F_j(G_j(X))$ denote the marginal distribution functions of the failure modes $G_i(X)$ and $G_j(X)$, respectively. Then, the joint distribution function of the two failure modes, $F_{ij}(G_i, G_j)$, can be expressed as equation (12) according to the Sklar theory [22]:

$$F_{ij}\left(G_i,G_j\right)=C\left[F_i\left(G_i\left(x\right)\right),F_j\left(G_j\left(x\right)\right)\right] \tag{12}$$

where $C(u,v)$ is the copula function.

According to the copula theory, any joint CDF or PDF can be modelled by a certain copula function with an unknown parameter to be estimated. However, the dependence characteristics between different failure modes would be differ considerably, which means that a specific copula function has to be carefully selected to describe the dependence exactly. Thus, it is necessary to compare the dependence characteristics underlying different copulas and a well-defined copula has to be chosen for each pair of failure modes. On the other hand, how to choose a suitable family of copulas is still an open problem in Statistics, and no clear procedures are available yet. However, our main goal here is to provide a procedure in order to model the dependence between the potential failure modes of the bevel gear transmission system. Thus, several well-known copulas are chosen herein to describe a wide range of dependence, including Gaussian, Clayton, Gumbel, and Frank copula, as displayed in Table 1. The Gaussian copula belongs to the Elliptical copulas family and is a link between a multivariate normal distribution and marginal distributions. The difference between the Gaussian copula and the well-known joint normal CDF is that the former allows non-normal and different marginal distributions while the latter does not. The selected copulas can describe both positive and negative dependence characteristics and the correlation coefficient to be estimated is limited within the range [-1,1].

Table 1. Summary of the adopted bivariate copula functions

Copula	Copula function, $C(u,v;\theta)$	Range of θ
Gaussian	$\Phi_2\left(\Phi^{-1}(u),\Phi^{-1}(v);\theta\right)$	[-1,1]
Clayton	$C_{Cl}\left(u,v;\theta\right)=\left(u^{-\theta}+v^{-\theta}-1\right)^{-1/\theta}$	$(0,\infty)$
Gumbel	$C_G\left(u,v;\theta\right)=\exp\left(-\left[\left(-\ln u\right)^{\frac{1}{\theta}}+\left(-\ln v\right)^{\frac{1}{\theta}}\right]^{\theta}\right)$	$(0,1]$
Frank	$C_F\left(u,v;\theta\right)=-\frac{1}{\theta}\ln\left(1+\frac{\left(e^{-\theta u}-1\right)\left(e^{-\theta v}-1\right)}{e^{-\theta}-1}\right)$	$(-\infty,+\infty)\setminus\{0\}$

**u, v are the random variables of marginal distribution, θ is the copula parameter.*

The underlying marginal distributions could be arbitrary and approximated with the moment-based saddlepoint technique. The PDFs of these copulas are plotted in Figure 2, which illustrates the basic property of each copula function. It should be noted that the selected copulas only describe the dependence characteristics of pair variables, which is independent with the marginal distributions of failure modes. Although some high-dimensional copula functions are available, the undetermined parameters in high-dimension copulas are difficult to obtain [1].

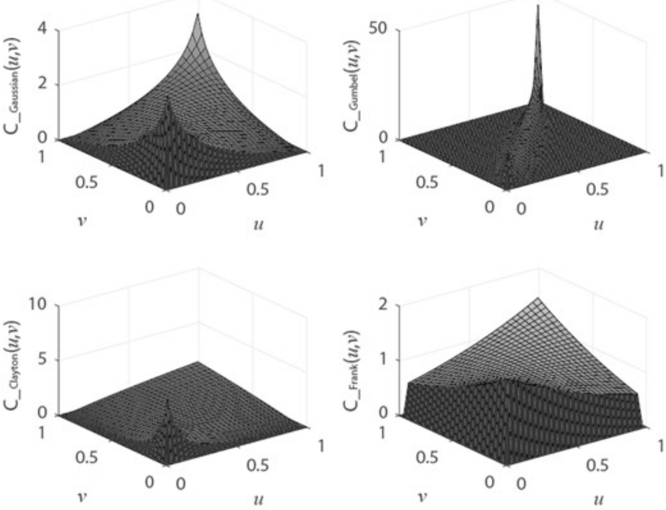


Fig. 2. The probability density distribution of adopted bivariate copulas

For each failure mode pair, the two-dimensional system failure probability could be expressed as equation (13):

$$\begin{aligned}P_{f\text{pair}}&=P_f\left[G_1\left(x\right)\le 0\bigcup G_2\left(x\right)\le 0\right]\\&=P_f\left[G_1\left(x\right)\le 0\right]+P_f\left[G_1\left(x\right)\le 0\right]-P_f\left[G_1\left(x\right)\le 0\bigcap G_2\left(x\right)\le 0\right]\\&=P_{f1}+P_{f2}-C\left(P_{f1},P_{f2}\right)\end{aligned} \tag{13}$$

in which the probability $C(P_{f1}, P_{f2})$ is estimated under different copulas.

System reliability bounds theory such as Cornell bounds [4] and Ditlevsen bounds [6] are commonly used to estimate system reliability. The Ditlevsen bounds, which is also known as the narrow reliability bounds, produce a narrower estimate of the system probability of failure by evaluating the joint failure probabilities of each pair of failure modes. For a series system with m failure modes, the narrow bounds for system reliability estimation is given by:

$$P_{f1}+\sum_{i=2}^{m}\max\left(P_{fi}-\sum_{j=1}^{i-1}P_{fij},0\right)\le P_{fs}\le\sum_{i=1}^{m}P_{fi}-\sum_{i=2}^{m}\max\left(P_{fij}\right) \tag{14}$$

where P_{fi} is the failure probability of the i-th failure mode, P_{fij} is the joint failure probabilities of each pair of failure modes that is evaluated with copulas. In this work, the narrow reliability bounds method is adopted to realize the system reliability estimation.

The bivariate copula functions shown in Table 1 are adopted in this work and will be comparatively analyzed. Since that the dependence structure and the degree of correlation between failure modes are previously unknown, the undetermined coefficient θ in each copula function is estimated using statistical approach. To facilitate the understanding of the procedure for system reliability calculation under different copulas, Figure 3 shows the flowchart of implementation for system reliability analysis. This procedure mainly includes three steps. Details of each step are summarized as follows:

(1) Draw n random samples of the basic random variables vector X_1, X_2, and X_3 of the bevel gear system that previously defined. These samples are then used to evaluate the performance functions and the samples of the responses (u) can be obtained. The kendall's tau, which is the index for correlation and used to estimate copula parameters, is calculated using MATLAB: corr(u_i, u_j,'type','kendall'). The joint probability of failure (P_{fij}) for each pair of failure modes can be calculated under different copulas with obtained correlation coefficient.

(2) Marginal probability of failure (P_{fi}) of each failure mode is estimated with the moment-based saddlepoint approximation. The high order moments of the performance function are estimated using the perturbation method, which is shown in Appendix. It should be noted that the accuracy of the adopted approximation depends on that of moments obtained.

(3) Based on the obtained marginal probability of failure P_{fi} and joint probability of failure P_{fij} , the system probability of failure $P_{f\mathrm{sys}}$ could be calculated using narrow bounds formula. The system probability of failure $P_{f\mathrm{sys}}$ under selected copulas are comparatively studied.

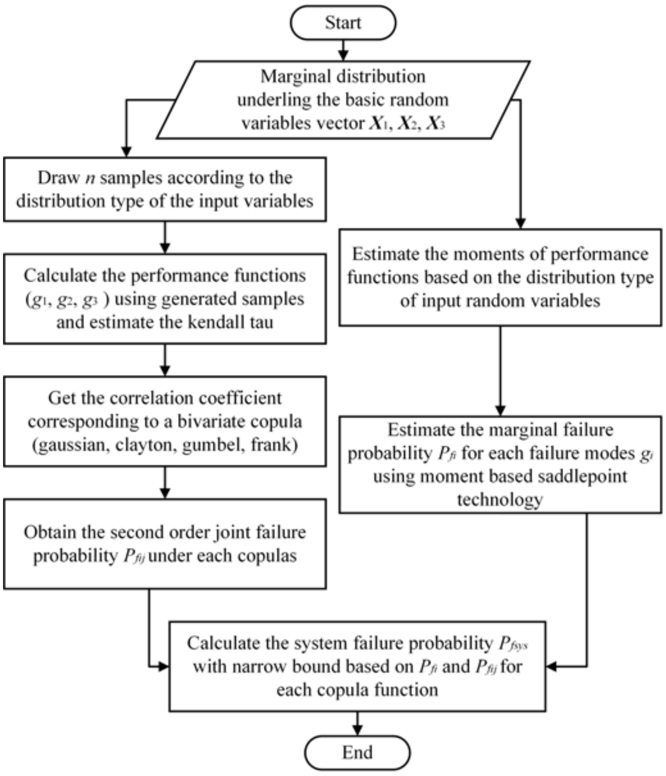


Fig. 3. Flowchart of system probability of failure using selected copulas

4. Reliability-based optimal design with copula function

To insure the robust design of the bevel gear transmission system, a reliability-based optimal design model is established here by introducing the reliability sensitivity information of design variables into the optimal design model. The reliability sensitivity reflects the impacts of the random variables on the system reliability. The robustness of the system reliability could be guaranteed by limiting the reliability sensitivity, which means that system reliability of the bevel gear transmission system would be stable despite the volatility of the random variables.

4.1. Reliability-based sensitivity analysis

Based on the moment-based saddlepoint approximation, the reliability sensitivity for each failure modes with respect to the distribution parameters of input random variables could be derived as:

$$\frac{\mathrm{D}P_{fi}}{\mathrm{D}\xi}=\frac{\partial P_{fi}}{\partial r}\frac{\partial r}{\partial \xi}+\frac{\partial P_{fi}}{\partial q}\frac{\partial q}{\partial \xi} \tag{15}$$

in which:

$$\frac{\partial P_{fi}}{\partial r}=\varphi\left[r+\frac{1}{r}\ln\left(\frac{q}{r}\right)\right]\left(1-\frac{1+\ln\left(q/r\right)}{r^2}\right)$$

$$\frac{\partial r}{\partial \xi}=\frac{\partial r}{\partial \hat{t}_s}\frac{\partial \hat{t}_s}{\partial \xi}+\frac{\partial r}{\partial \beta_2}\frac{\partial \beta_2}{\partial \xi}+\frac{\partial r}{\partial K_{Zs}}\frac{\partial K_{Zs}}{\partial \xi}$$

$$\frac{\partial r}{\partial \hat{t}_s}=sign\left(\hat{t}_s\right)\left\{2\left[\hat{t}_s\left(-\beta_2\right)-K_{Zs}\left(\hat{t}_s\right)\right]\right\}^{-1/2}\left[-\beta_2-\frac{\partial K_{Zs}}{\partial \hat{t}_s}\right]$$

$$\frac{\partial \hat{t}_s}{\partial \xi}=-\frac{4}{\left(\gamma_G\beta_2-2\right)^2}\left(\frac{1}{\sigma_G}\frac{\partial \mu_G}{\partial \xi}-\frac{\mu_G}{\sigma_G^2}\frac{\partial \sigma_G}{\partial \xi}\right)-\frac{2\beta_2^2}{\left(\gamma_G\beta_2-2\right)^2}\frac{\partial \gamma_G}{\partial \xi}$$

$$\frac{\partial r}{\partial \beta_2}=-\hat{t}_s sign\left(\hat{t}_s\right)\left\{2\left[\hat{t}_s\left(-\beta_2\right)-K_{Zs}\left(\hat{t}_s\right)\right]\right\}^{-1/2}$$

$$\frac{\partial r}{\partial K_{Zs}}=-sign\left(\hat{t}_s\right)\left\{2\left[\hat{t}_s\left(-\beta_2\right)-K_{Zs}\left(\hat{t}_s\right)\right]\right\}^{-1/2}$$

$$\frac{\partial K_{Zs}}{\partial \xi}=\left(-\frac{2}{\gamma_G}+\frac{2}{\gamma_G\left(1-0.5\gamma_G\hat{t}_s\right)}\right)\frac{\partial \hat{t}_s}{\partial \xi}+\left(\frac{2\hat{t}_s}{\gamma_G^2}+\frac{4\ln\left(1-0.5\gamma_G\hat{t}_s\right)^2}{\gamma_G^3}+\frac{2\hat{t}_s}{\gamma_G^2\left(1-0.5\gamma_G\hat{t}_s\right)}\right)\frac{\partial \lambda_\mathrm{G}}{\partial \xi}$$

$$\frac{\partial P_{fi}}{\partial q}=\frac{1}{qr}\varphi\left[r+\frac{1}{r}\ln\left(\frac{q}{r}\right)\right]$$

$$\frac{\partial q}{\partial \xi}=\sqrt{K''_{Zs}}\frac{\partial \hat{t}_s}{\partial \xi}+\frac{\hat{t}_s}{2\sqrt{K''_{Zs}}}\frac{\partial K''}{\partial \xi}$$

$$\frac{\partial K''_{Zs}}{\partial \xi}=\frac{\partial K''_{Zs}}{\partial \hat{t}_s}\frac{\partial \hat{t}_s}{\partial \xi}+\frac{\partial K''_{Zs}}{\partial \lambda_\mathrm{G}}\frac{\partial \lambda_\mathrm{G}}{\partial \xi}$$

$$\frac{\partial K''_{Zs}}{\partial \hat{t}_s}=\frac{\gamma_G}{\left(1\text{-}0.5\gamma_G\hat{t}_s\right)^3}$$

$$\frac{\partial K''_{Zs}}{\partial \gamma_G}=0.5\beta_2\left(\gamma_G\beta_2-2\right)$$

In equation (15), symbol ξ represents the distribution parameters of the random variables, such as the mean value and the standard deviation.

4.2. Reliability-based robust design optimization

According to the definition of the robust optimal design involving reliability sensitivity, the reliability-based robust design optimal model of the bevel gear transmission system can be established as follows:

$$\begin{aligned} \min \quad & f(\boldsymbol{X})=\sum_{k=1}^{M}\omega_k f_k(\boldsymbol{X}) \\ s.t.\quad & P\left[g_i(\boldsymbol{X})>0\right]\le P_{F_i}^{tar}, i=1,\ldots,N \\ & \boldsymbol{d}^L \le \boldsymbol{d} \le \boldsymbol{d}^U,\ \ \boldsymbol{d}\in \mathrm{R}^{ndv} \end{aligned} \tag{16}$$

where $\boldsymbol{d}=\{d_i\}^{\mathrm{T}}=\mu(\boldsymbol{X})$ is the mean value of design vector of the bevel gear transmission system and $X=[X_j]^{\mathrm{T}}{}_{j=1,..,N}$. $P_{F_j}^{tar}$ is the target failure probability of the *i*th constraint of reliability. *M*, *N*, and *ndv* represent the number of sub-objective functions, probabilistic constraints and design variables, respectively. d^L and d^U is the lower and upper bound of the design variables. The optimal function is built as a weighted sum of two objective functions, *i.e.*, the reliability sensitivity with respect to $\mu(\boldsymbol{X})$, and the minimum volume of the gear transmission system. symbol ω_k represents the weighting coefficients, the value of which can be determined by considering the importance degree of each objective functions. Herein, ω_k is given as follows:

$$\begin{aligned} \omega_k =& \left[f_1(\boldsymbol{X}^{*k})-f_1(\boldsymbol{X}^{*1})\right] \Big/ \Big\{\left[f_1(\boldsymbol{X}^{*k})-f_1(\boldsymbol{X}^{*1})\right] \\ &+\left[f_2(\boldsymbol{X}^{*(k-1)})-f_2(\boldsymbol{X}^{*2})\right] \\ &+\cdots+\left[f_k(\boldsymbol{X}^{*1})-f_k(\boldsymbol{X}^{*k})\right]\Big\} \end{aligned} \tag{17}$$

while the weighting coefficients should satisfy the following conditions:

$$\sum_{k=1}^{n}\omega_k = 1 \ \ (\omega_k \ge 0) \tag{18}$$

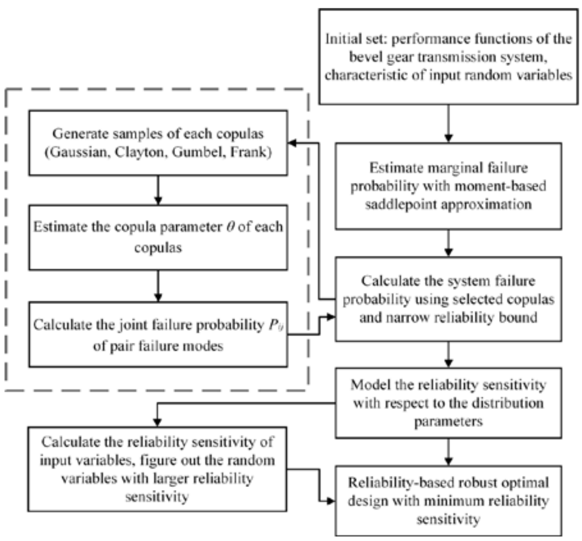


Fig. 4. The flowchart of the proposed copula-based reliability optimal method

The optimal design of the bevel gear transmission system could be carried out according to the established optimization model based on the results of the reliability analysis. The whole process of the method is shown in Figure 4.

5. Illustrative example

In this section, the probabilistic analysis and optimal design based on the reliability sensitivity of a bevel gear transmission system is used to illustrate the application of the proposed method. The stochastic structural parameters are listed in Table 2 and the variables are normally distributed.

Table 2. Random variables in the bevel gear transmission system

Variable	Mean	Std	Units	Variable	Mean	Std	Units
F_{mt}	4709.078	70.636	N	Z_X	1	0.033	-
K_A	1	0.033	-	Z_L	0.9658	0.0318714	-
K_V	1.034147	0.034127	-	Z_V	0.968213	0.031951	-
$K_{H\beta}$	1.65	0.05445	-	Z_R	0.936408	0.030901	-
$K_{F\beta}$	1.65	0.05445	-	Z_W	1	0.033	-
K_{Ha}	1	0.033	-	Y_{Fa1}	2.243285	0.074028	-
K_{Fa}	1	0.033	-	Y_{Fa2}	2.249644	0.074238	-
l_{bm}	27.138	0.13569	mm	Y_{sa1}	1.880157	0.06204	-
b	28	0.14	mm	Y_{sa2}	1.889086	0.06234	-
d_{v1}	50.886	0.25433	mm	Y_e	0.717334	0.0035867	-
m_{mn}	2.54827	0.01274	mm	Y_K	1.000244	0.005	
$Z_{M\text{-}B}$	0.9874	0.00494	-	$\sigma_{F\lim}$	380	76	N/mm^2
Z_H	2.49457	0.01247	-	Y_{ST}	2.0	0.066	-
Z_E	189.8117	9.4905	(N/mm^2)$^{1/2}$	Y_{NT1}	0.912	0.030096	-
F_{mt}	4709.078	70.636	N	Y_{NT2}	0.933	0.030789	-
Z_{LS}	1	0.033	-	Y_{relT1}	1.004782	0.033158	-
Z_β	1	0.033	-	Y_{relT2}	1.003269	0.033108	-
Z_K	0.8	0.004	-	Y_{RrelT1}	1.024202	0.033809	-
$\sigma_{H\lim}$	1370	164.4	N/mm^2	Y_{RrelT2}	1.024202	0.033809	-
Z_{NT}	1.07	0.03531	-	Y_{X1}	1	0.033	-

5.1. System failure probability estimation with selected copulas

The equations (1) and (4) are used to calculate the contact stress and the bending stress of the bevel gears with the mean value of the input random variables. The reliability of the three failure modes is then estimated with the equations (3), (6) and (7) considering the stochastic structure and design parameters for the system. The reliability of each failure modes and the system probability of failure with failure interaction is therefore estimated. As a matter of convenience, the mean of the reliability bounds that obtained under each copula function is used for comparative analysis, which is displayed in Table 3. Herein, the Monte Carlo simulation is used to provide the benchmark of the system reliability. And the system reliability under the independent assumption is also estimated.

From the above Table it can be observed that the probability of failure for the system differ a lot under different copulas. The commonly used Gaussian copula produces a probability with the relative error of 0.97%, which denotes that the most commonly used Gaussian copula is applicable in the bevel gear pair design when the true

Table 3. The system probability of failure under selected copulas

Copula	System failure probability	Reliability index	Relative error
Independent	0.012763	2.2333	5.8%
Gaussian	0.009440	2.3479	0.97%
Gumbel	0.010323	2.3144	2.38%
Clayton	0.008368	2.3925	0.92%
Frank	0.012329	2.2467	5.23%
MCS	0.008876	2.3708	-

dependence structure is unknown. The relative error with the Clayton copula is the least among the selected copulas. By comparing the scatter plots of Clayton functions and system random variables samples for g_2 and g_3, both of the two figures show a lower tail correlation as shown in Figure 5, which can be understand as that the reduction of the reliability of g_2 leads to a more likelihood of reduction of the reliability of g_3.

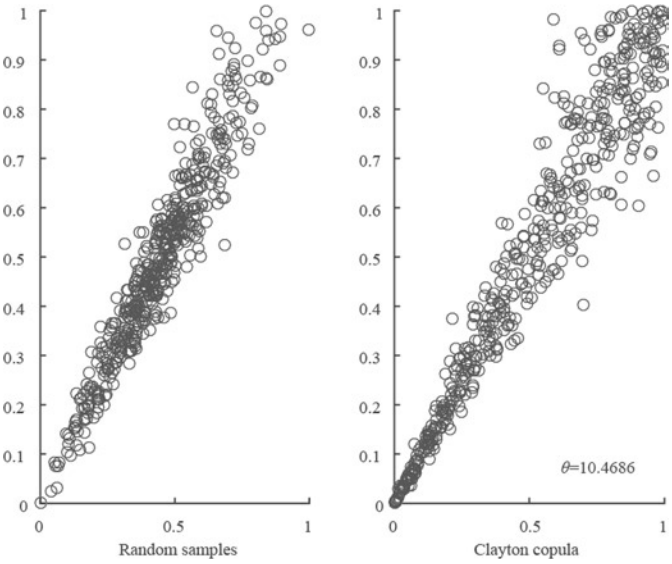


Fig. 5. The scatter plots for g_2 and g_3 under random samples and Clayton copula

The impacts of the performance degradation on the system probability of failure that estimated under different copulas is also studied. Figure 6 shows the variation curve of the system probability of failure when the reliability of each failure mode decreases respectively. It can be clearly seen that the system probability of failure increases with the increase of P_{fi}. On the other hand, the system probability of failure under Frank copula always produce the biggest value, and then is the Gumbel and Gaussian copula. The Clayton copula helps to provide the smallest value of the system probability of failure, which is also the most accurate results as shown in Table 3. This relations and trends appear to be consistency in three figures (a), (b) and (c).

5.2. Reliability-based optimal design with minimum reliability sensitivity

Due to that the reliability obtained by the Clayton copula is the closest one to the Monte Carlo simulation result, the reliability sensitivity analysis is furthermore conducted based on the Clayton copula-based reliability results. According to the equations deduced in section "reliability-based sensitivity analysis", the reliability sensitivity with respect to the mean of the input variables of the bevel gear transmission system can be obtained.

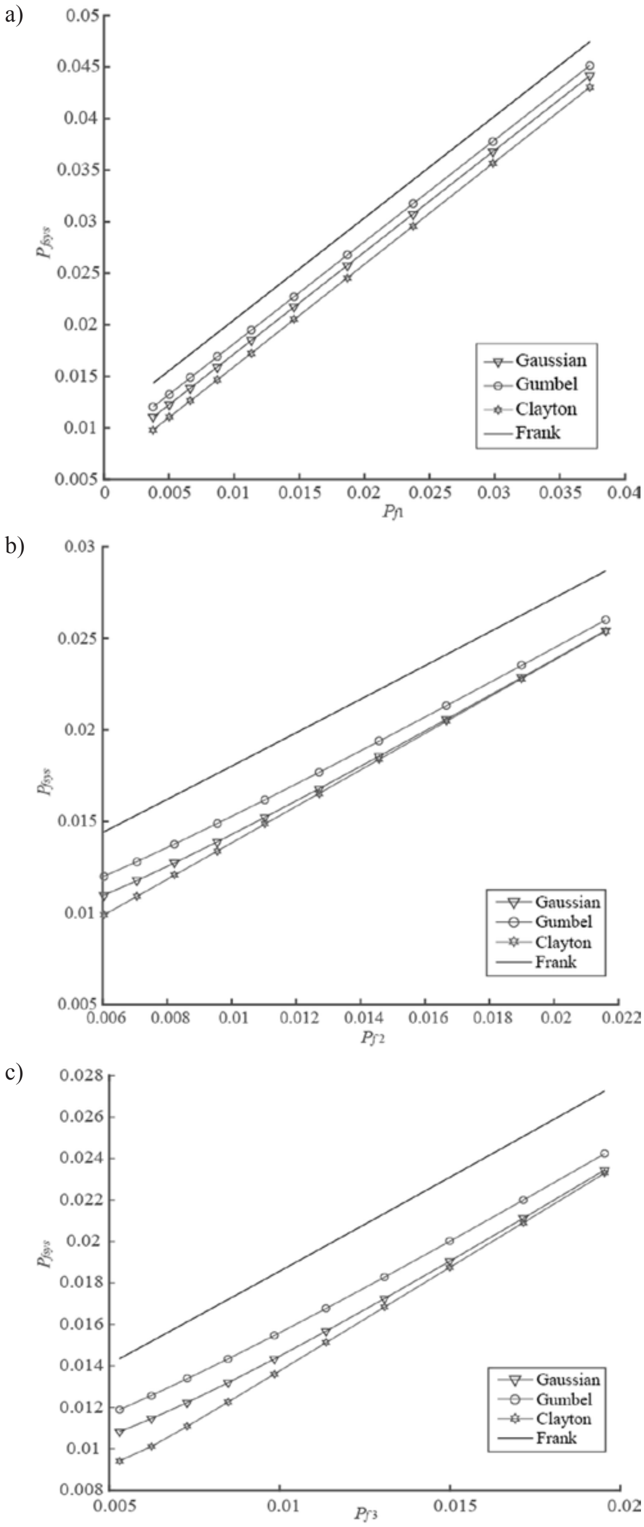
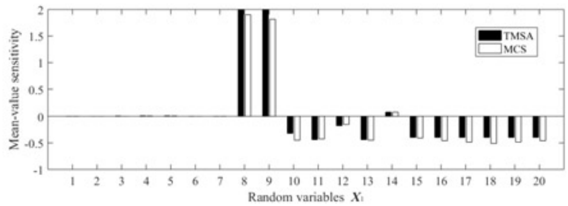
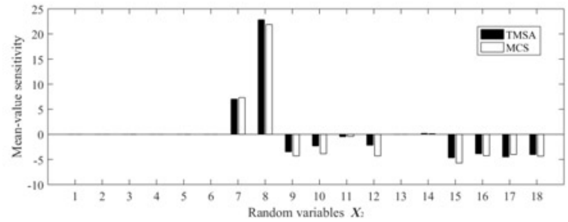


Fig. 6. The comparison of system probability of failure under selected copulas with p_{f1}, p_{f2} andp_{f3} increase

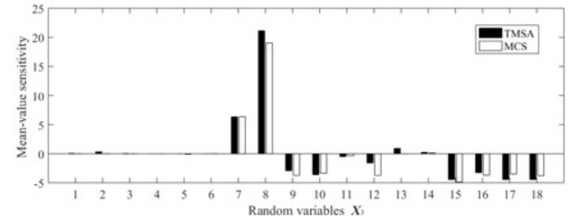
As illustrated in Figure 7, the random variables d_{v1}, l_{bm}, b and m_{mn} has relatively larger values of reliability sensitivity, which means that the failure of the system is highly dependent on these random variables. The volatility of these random variables may lead to significant changes of the failure probability. In order to obtain more robust reliability results of the system, the reliability sensitivity of these variables is involved in the reliability-based optimal design model. The robust design model is thus established as follows:



(a) *Sensitivity index of the mean value for g_1 mean*



(b) *Sensitivity index of the mean value for g*



(c) *Sensitivity index of the mean value for g_3*

Fig. 7. The reliability sensitivity results for each failure mode with respect to mean

$$\min \quad f_1(\boldsymbol{d}) = \frac{\pi}{8} u(u+1)(m_{mn} L/L_m)^3 z_1^3 \frac{b}{L}\left(1 - \frac{b}{L} + \frac{1}{3}\left(\frac{b}{L}\right)^2\right)$$

$$f_2(\boldsymbol{h}) = \sqrt{\left(\frac{\partial p_{f1}}{\partial \boldsymbol{h}_1}\right)^2 + \left(\frac{\partial p_{f2}}{\partial \boldsymbol{h}_2}\right)^2 + \left(\frac{\partial p_{f3}}{\partial \boldsymbol{h}_2}\right)^2}$$

$$\begin{aligned} s.t.\ \ & P\left[g_i(\boldsymbol{X}) > 0\right] \le P_{g_j}^{tar}, j = 1, \ldots, N \\ & m_{mn} L / L_m - 1.5 \ge 0; \\ & 8 - m_{mn} L / L_m \ge 0 \\ & z_1 - 17 \ge 0 \\ & 36 - z_1 \ge 0 \\ & b - 0.2L \ge 0 \\ & 0.35L - b \ge 0 \end{aligned}$$

The sub-objective function f1(d) represents the volume of the gear and the design variable vector is d=$[m_{mn}, z_1, b]^T$. The sub-objective function $f_2(h)$ minimize the reliability sensitivity with respect to the vector $h_1=[d_{v1}, l_{bm},]^T$ and $h_2= [b, m_{mn}]^T$, which represents the random variables that have larger reliability sensitivity values. The initial value is set as $d_0=[2.55, 19, 28]^T$. The optimal results are obtained and shown in Table 4. It clearly shows that the reliability sensitivity with respect to h_1 and h_2 is reduced, and the volume of the gear is also greatly reduced. Besides, the system probability of failure is reduced, that means the gear transmission system could become less sensitive to the volatility of the random variables with the system reliability increases.

Table 4. The summary of the optimal results under Clayton copulas

	Design variables	P_{fsys}	V	$f_2(h)$
Initial	d_0=[2.55, 19, 28]^T	0.009036	2.0702e05	30.6945
Optimal	d^*=[2.23 17 32.54]^T	0.007655	1.3292e05	17.1299

6. Conclusions

A numerical procedure has been proposed for evaluating the system probability of failure of the bevel gear transmission system with dependent failure modes. The proposed procedure utilizes selected copula functions to efficiently describe the dependence structure between failure modes and the most suitable copula is determined by comparative analysis. The probability of failure and the system reliability index were determined by the moment-based saddlepoint technique and the narrow bounds theory. Based on the reliability results obtained, some conclusions could be made.

(1) Different copula function can be used to describe the dependence structure between each failure mode, but result in different joint failure probability. What can be seen from the example is that the system reliability is underestimated using Gumbel and Frank copula, while the ones obtained with the Gaussian and Clayton copula get results that close to the simulation results.

(2) By comparing the obtained reliability index, the one under Clayton copula produce the most accurate result. According to the properties of the Clayton copula, the copula describes the tail correlation between random variables, which denotes that the reliability of one failure modes reduces, the reliability of the other one is more likely to reduce.

(3) The reliability-based optimal design is performed and the corresponding optimal variables are obtained. The reliability sensitivity is obtained based on marginal failure probability, and the value of which is decreased to a certain extend to ensure the robust of system reliability.

(4) The failure probability analysis is performed under the assumption that the dependence structures among different failure modes of the gear system are the same based on the engineering experience. If the number of failure modes is increased, the dependence structures between different response variables would be more complex. The combination of copulas for the multivariate joint modelling will serve as the main research content of the future.

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Appendix

Herein, the statistical perturbation method is adopted to estimate the first three moments of each failure mode. Given the vector of random variable X_i (i=1,2,3) and the limit state function for each failure mode $g_i(X_i)$, the moments could then be estimated by:

$$\mathrm{E}\left[g_i(\boldsymbol{X}_i)\right] = \mathrm{E}\left[g_{id}(\boldsymbol{X}_i)\right] + \varepsilon \mathrm{E}\left[g_{ip}(\boldsymbol{X}_i)\right] = g_{id}(\boldsymbol{X}_i) = \bar{g}_i(\boldsymbol{X}_i) \tag{A.1}$$

$$\sigma_{gi}^2 = \mathrm{Var}\left[g_i(\boldsymbol{X}_i)\right] = \varepsilon^2 \mathrm{E}\left\{\left(\frac{\partial g_{id}(\boldsymbol{X})}{\partial(\boldsymbol{X}_i)^{\mathrm{T}}}\right)^{[2]} \boldsymbol{X}_{ip}^{[2]}\right\} = \left[\frac{\partial g_{id}(\boldsymbol{X})}{\partial(\boldsymbol{X}_i)^{\mathrm{T}}}\right]^{[2]} \mathrm{Var}(\boldsymbol{X}_i) \tag{A.2}$$

$$\gamma_{gi} = \mathrm{C}_3\left[g_i(\boldsymbol{X}_i)\right] = \varepsilon^3 \mathrm{E}\left\{\left(\frac{\partial g_{id}(\boldsymbol{X}_i)}{\partial(\boldsymbol{X}_i)^{\mathrm{T}}}\right)^{[3]} \boldsymbol{X}_{ip}^{[3]}\right\} = \left[\frac{\partial g_{id}(\boldsymbol{X}_i)}{\partial(\boldsymbol{X}_i)^{\mathrm{T}}}\right]^{[3]} C_3(\boldsymbol{X}_i) \tag{A.3}$$

where σ_{gi}^2 and γ_{gi} are the variance and the third central moments of $g_i(X_i)$ respectively.

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