The reliability of technical systems is one of the most important research subjects in the point reached by modern science. In many recent studies, this problem is solved by determining the operation performance of determined one or more components operating under stress.\cite{1} At this point, \( R = P(X < Y) \) is taken as a basis. Here, \( X \) is the stress applied on the operating component and \( Y \) is the strength of the component. In this study we aim to propose a new method by using Kullback-Leibler divergence for computing the reliability of the component under stress-strength model. The superiority of the proposed method is that when the component durability is equal to applied stress Kullback-Leibler divergence is equal to zero. In addition to that when more than one stresses exists at the same time the formed function can include all stresses at the same time. When \( R \) is used, this is not possible because of stresses are evaluated separately. As Kullback-Leibler divergence is calculated depending on time, the strength of the component is evaluated within a dynamic structure.

**Keywords:** reliability, stress-strength model, multistate system model, Kullback-Leibler divergence.

Multi-state systems have been found to be more flexible tool than binary systems for modeling engineering systems. In literature, much attention has been paid to multi-state system modeling.\cite{14} El-Neweihi et al.\cite{14} proposed axioms extending the standard notion of a coherent system to the new notion of a multistate coherent system. For such systems they obtained deterministic and probabilistic properties for system performance which are analogous to well-known results for coherent system reliability.\cite{19} Hudson and Kapur\cite{19} presented several models and their applications, in terms of reliability analyses, to situations where the system and all its components have a multiple states.

Ebrahimi\cite{11} proposed two types of multistate coherent system and presented various properties related to them. Brunella and Kapur\cite{7} studied a series of reliability measures and expanded their definitions to be consistent with binary, multistate and continuum models. Kuo and Zuo\cite{22} focused on multistate system reliability models and introduced several special multistate system reliability models. Eryilmaz\cite{15} studied mean residual and mean past lifetime concepts for multistate systems. Also, for more details about multi-state system model one can see Andrzejczak\cite{2} and \cite{3}.

For reliability analysis, stress-strength models are of special importance. In the simplest terms, stress-strength model can be described as an assessment of the reliability of the component in terms of \( X \) and \( Y \) random variables where \( X \) is the random “stress” experienced by the component and \( Y \) is the random “strength” of the component. Therefore, the components failure can lead to the degradation of the entire multi-state system performance. The performance rates of the components can range from perfect functioning up to complete failure. The quality of the system is completely determined by components. In some cases, the status of the system depends on the effect of several stresses which cause degradation. The system may not fail fully, but can degrade and there may exist several states of the system. This situation corresponds to multistate systems. For an excellent review of multistate system we refer to Andrzejczak\cite{1}.

1. **Introduction**

All technical systems have been designed to perform their intended tasks in a specific ambient. Some systems can perform their tasks in a variety of distinctive levels. A system that can have a finite performance rates is called a multi-state system. Generally, such systems are referred to as multi-state system. The performance rates of components can also vary as a number of performance rates is called a multi-state system. Generally, such systems are referred to as multi-state system. The performance rates of components can also vary as a number of performance rates is called a multi-state system.
component and $Y$ is the random “strength” of the component available to overcome the stress. From this simplified explanation, the reliability of the component is the probability that the component is strong enough to overcome the stress applied on it. Then the reliability of the system is defined as:

$$P(X < Y) = \int_0^\infty F(x) dG(x),$$

(1)

where $F(x)$ and $G(x)$ are distribution functions of $X$ and $Y$, respectively. Also, for $x<0, F(x)=G(x)=0$.

Extensive works have been done for the reliability of the component and its estimation under different choices for stress and strength distributions. Chandra and Owen [8] studied the estimation of the reliability of a component where component is subject to several stresses whereas its strength is a single random variable. Awad and Gharrafi[4] used a simulation study which compares minimum variance unbiased estimator, the maximum likelihood estimator and bayes estimator for $R$ when $X$ and $Y$ are two independent but not identically distributed Burr random variables. Kotz et al. [20] presented comprehensive information about all methods and results on the stress-strength model. Nadarajah and Kotz [24] calculated the marginal distribution function

In this section, we introduce a new approach for determining the component operation performance where component is subject to $X_1(t), X_2(t), \ldots, X_d(t)$ stresses, whereas its strength, $Y(t)$, is a single random process. Let us initially assume that the stresses are independent random processes having continuous cumulative distribution functions $F_i(x) = P(X_i(t) \leq x), i=1,2,\ldots,n$ and the strength has the marginal distribution function $G(x)=P(Y(t) \leq x)$.

In our method, we first form the KL divergence $D_{KL}(Y(t) || X_1(t), \ldots, X_d(t))=D_{ KL(\xi_1)}^{(t)}$ of $X(t)$ from $Y(t)$ by using (3) for $\xi=1,2,\ldots,n$. After this, we calculate the $D_{ KL(\xi_1)}^{(t)}$ for selected values of the parameters of marginal lifetime distributions of the stress and
strength random variables. Using these values the operation level of the component, depending on the number of stresses, can be defined as follows:

\[
\begin{aligned}
&n, \\&n-1, \\&n-2, \\&\vdots \\&2, \\&1, \\&0,
\end{aligned}
\begin{aligned}
t < t_0 \\t_n \leq t < t_{n-1} \\t_{n-1} \leq t < t_{n-2} \\\vdots \\t_3 \leq t < t_2 \\t_2 \leq t < t_1 \\t_1 \leq t
\end{aligned}
\]

where \( t_i \) denotes the time when \( D_i^{(t)} (\xi) \) is equal to zero, \( i=1,2,\ldots,n \).

Also using \( D_i^{(t)} (\xi) \) and \( t_i \) values we can define the following equations:

\[
\xi_{y}^{j} = \begin{cases} 
D_{i}^{(t)} (\xi) & \text{if } t_{j+1} < t \leq t_{j} \\
0 & \text{otherwise}
\end{cases}
\]

where \( j=1,2,\ldots,n-1 \) and:

\[
\xi_{t}^{j} = \begin{cases} 
D_{i}^{(t)} (\xi) & \text{if } t \leq t_{j} \\
0 & \text{otherwise}
\end{cases}
\]

Now with the help of the above equations, the new reliability score \( Y(t) \) for the component can be expressed as follows:

\[
Y(t) = \sum_{j=1}^{n} X_{j}^{(t)} \left( 1 + \frac{\xi_{j}^{t}}{u_{j}} \right)
\]

where:

\[
X_{j}^{(t)} = \begin{cases} 
1, & \text{if } \xi_{j}^{t} > 0 \\
0, & \text{otherwise}
\end{cases}
\]

and \( u_{j} = \sup_{t} D_{i}^{(t)} (\xi) \).

In probabilistic design it is common to use parametric statistical models to compute the reliability obtained from stress-strength interference theory. In the following section we apply our method to a Weibull distributional example.

### 4. A Weibull distributional example

In this section, we apply the proposed method to find the component operation performance. Assume that the component is subject to \( X_1(t), X_2(t) \) and \( X_3(t) \) stresses, which remain fixed over time, whereas its strength, \( Y(t) \), is a single random variable, which is stochastically decreasing in time.

A Weibull process is a useful model for events that are changing over time. Here, let \( G \) be a Weibull cumulative distribution function and its shape parameter \( \beta > 0 \) is constant with aging time, while its scale parameter \( \theta(t) \) decreases over time.

Then, its cumulative distribution function can be written as:

\[
G'(x) = 1 - \exp \left( -\left( \frac{x}{\alpha(t)} \right)^{\beta} \right), x > 0.
\]

Similarly, assume that \( X_{i}, X_{2} \) and \( X_{3} \) are Weibull random variables with cumulative distribution functions:

\[
F_{i}(x) = 1 - \exp \left( -\left( \frac{x}{\theta_{i}} \right)^{\beta} \right), x > 0,
\]

where \( \beta > 0 \) is the shape parameter, \( \theta_i > 0 \) is the scale parameter of the distributions and \( i=1,2,3 \). Also both \( \beta \) and \( \theta_i \) are constant with aging time.

For computing the operation performance of a component at first we must form KL divergence \( D_{KL}(l) \) of \( X_{i} \) from \( Y(t) \) for \( l=1,2,3 \). The KL divergence (3) can also be written for \( X_{i} \) \((l=1,2,3)\) and \( Y(t) \) as:

\[
D_{KL}(l) = H(y,x_{l}) - H(y)
\]

where:

\[
H(y,x_{l}) = \int_{0}^{\infty} g_{l}(x) \log \frac{1}{f_{l}(x)} dx
\]

and:

\[
H(y) = \int_{0}^{\infty} g_{l}(x) \log \frac{1}{g_{l}(x)} dx
\]

Here, \( H(y) \) is the differential entropy of a continuous random variable \( Y(t) \) with density \( g_{l}(x) \). Let \( Y(t), X_{1}, X_{2}, X_{3} \) are independent. Now, using probability density functions of (5) and (6) in (8), we have:

\[
H(y,x_{l}) = \frac{\beta}{\theta_{l}} \int_{0}^{\infty} \frac{x^{\beta-1} e^{-\left( \frac{x}{\alpha(t)} \right)^{\beta}}}{\alpha(t)^{\beta-1} e^{-\left( \frac{x}{\alpha(t)} \right)^{\beta}}} \log \beta \left( \frac{\theta_{l}}{\theta} \right)^{\beta} \frac{x^{\beta-1} e^{-\left( \frac{x}{\alpha(t)} \right)^{\beta}}}{\alpha(t)^{\beta-1} e^{-\left( \frac{x}{\alpha(t)} \right)^{\beta}}} \frac{1}{\alpha(t)^{\beta-1} e^{-\left( \frac{x}{\alpha(t)} \right)^{\beta}}} dx
\]

\[
= - \log \frac{\beta}{\theta_{l}} - \frac{\beta (\beta-1)}{\theta_{l}} \int_{0}^{\infty} \frac{x^{\beta-1} e^{-\left( \frac{x}{\alpha(t)} \right)^{\beta}}}{\alpha(t)^{\beta-1} e^{-\left( \frac{x}{\alpha(t)} \right)^{\beta}}} \log dx + \frac{\beta}{\theta_{l}} \int_{0}^{\infty} \frac{x^{\beta-1} e^{-\left( \frac{x}{\alpha(t)} \right)^{\beta}}}{\alpha(t)^{\beta-1} e^{-\left( \frac{x}{\alpha(t)} \right)^{\beta}}} \frac{1}{\alpha(t)^{\beta-1} e^{-\left( \frac{x}{\alpha(t)} \right)^{\beta}}} dx
\]

By making the substitution \( u = \left( \frac{x}{\alpha(t)} \right)^{\beta} \) in (10) and then using following integral:

\[
\int_{0}^{\infty} e^{-t} \log t dt = -C,
\]

\[
C = \int_{0}^{\infty} e^{-t} \log t dt
\]
where $C=0.577215$ is the Euler’s constant (Eq. 8.367.4 in Gradshteyn and Ryzhik, [17]), $H(y; x_l)$ can be obtained as:

$$H(y, x_l) = -\log \frac{\beta}{\alpha(t)} + (\beta - 1) \left( \frac{C}{\beta} - \log \alpha(t) \right) + \left( \frac{\alpha(t)}{\theta_l} \right)^\beta,$$  

(12)

Similarly, using probability density function of (5) in (9), we have:

$$H(y) = -\log \frac{\beta}{\alpha(t)} + (\beta - 1) \left( \frac{C}{\beta} - \log \alpha(t) \right) + 1,$$  

(13)

where suitable transformations and simplifications have been applied and also (11) used.

Table 1. Numerical values obtained from Equation (15) for $\beta=0.9$, $\theta_1=0.01$, $\theta_2=0.02$ and $\theta_3=0.05$

<table>
<thead>
<tr>
<th>$t$</th>
<th>$D_{KL}^{(1)}$</th>
<th>$D_{KL}^{(2)}$</th>
<th>$D_{KL}^{(3)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>11.126</td>
<td>4.870</td>
<td>1.234</td>
</tr>
<tr>
<td>10</td>
<td>4.870</td>
<td>1.808</td>
<td>0.242</td>
</tr>
<tr>
<td>15</td>
<td>2.807</td>
<td>0.871</td>
<td>0.036</td>
</tr>
<tr>
<td>20</td>
<td>1.808</td>
<td>0.456</td>
<td>0.018</td>
</tr>
<tr>
<td>25</td>
<td>1.234</td>
<td>0.242</td>
<td>0.059</td>
</tr>
<tr>
<td>30</td>
<td>0.871</td>
<td>0.123</td>
<td>0.057</td>
</tr>
<tr>
<td>35</td>
<td>0.627</td>
<td>0.057</td>
<td>0.107</td>
</tr>
<tr>
<td>40</td>
<td>0.456</td>
<td>0.021</td>
<td>0.159</td>
</tr>
<tr>
<td>45</td>
<td>0.333</td>
<td>0.004</td>
<td>0.211</td>
</tr>
<tr>
<td>50</td>
<td>0.242</td>
<td>0.</td>
<td>0.263</td>
</tr>
<tr>
<td>55</td>
<td>0.174</td>
<td>0.003</td>
<td>0.312</td>
</tr>
<tr>
<td>60</td>
<td>0.123</td>
<td>0.012</td>
<td>0.360</td>
</tr>
<tr>
<td>65</td>
<td>0.085</td>
<td>0.025</td>
<td>0.406</td>
</tr>
<tr>
<td>70</td>
<td>0.057</td>
<td>0.041</td>
<td>0.451</td>
</tr>
<tr>
<td>75</td>
<td>0.036</td>
<td>0.059</td>
<td>0.493</td>
</tr>
<tr>
<td>80</td>
<td>0.021</td>
<td>0.078</td>
<td>0.534</td>
</tr>
<tr>
<td>85</td>
<td>0.011</td>
<td>0.097</td>
<td>0.574</td>
</tr>
<tr>
<td>90</td>
<td>0.004</td>
<td>0.118</td>
<td>0.611</td>
</tr>
<tr>
<td>95</td>
<td>0.001</td>
<td>0.138</td>
<td>0.648</td>
</tr>
<tr>
<td>100</td>
<td>0.</td>
<td>0.159</td>
<td>0.683</td>
</tr>
</tbody>
</table>

Now using (12) and (13) in (7) we have:

$$D_{KL}^{(t)} = \log \left( \frac{\theta_l}{\alpha(t)} \right)^\beta + \frac{\alpha(t)}{\theta_l} - 1,$$  

(14)

Because of $\alpha(t)$ decreases over time, in (14), let $\alpha(t)=1/t$ then we have:

$$D_{KL}^{(t)} = \log (\theta_l)^\beta + \frac{1}{\theta_l} - 1,$$  

(15)

where $l=1,2,3$.

Clearly, when values in Table 1 used, we have:
5. Conclusion

In the study, it is theoretically assumed that a component operates under \( n \) different stresses and when the component’s strength remains weak in all stresses the component is fail. Here, for reliability evaluation we provide a new approach for obtaining the component operation performance. The proposed method described here is a simple and can clearly show the chance of component operation performance depending on time while under all stresses. The evaluation of the component operation performance naturally depends on the probability distributions of stresses and selection of probability distribution of component strength. The method used in the study does not originally depend on probability distribution. Reliability function is a parametric method, but the reliability score proposed from this aspect is nonparametric method for the component. When different effect functions are used instead of probability functions of stress and strength, the recommended method can be easily used.

### References


### Table 2. New reliability score for the component when \( \beta=0.9, \theta_1=0.01, \theta_2=0.02 \) and \( \theta_3=0.05 \)

<table>
<thead>
<tr>
<th>( t )</th>
<th>( Y(t) )</th>
<th>( t )</th>
<th>( Y(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>3.</td>
<td>10</td>
<td>2.1961</td>
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<tr>
<td>15</td>
<td>2.0291</td>
<td>20</td>
<td>2.</td>
</tr>
<tr>
<td>25</td>
<td>1.0496</td>
<td>30</td>
<td>1.0252</td>
</tr>
<tr>
<td>35</td>
<td>1.0117</td>
<td>40</td>
<td>1.0043</td>
</tr>
<tr>
<td>45</td>
<td>1.0008</td>
<td>50</td>
<td>1.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( t )</th>
<th>( Y(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.</td>
</tr>
</tbody>
</table>


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