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OPTIMAL NUMBER OF MINIMAL REPAIRS UNDER A CUMULATIVE DAMAGE MODEL WITH CUMULATIVE REPAIR COST LIMIT

OPTYMALNA LICZBA NAPRAW MINIMALNYCH W ŚWIETLE MODELU SUMOWANIA USZKODZEŃ PRZY OGRANICZONYM ŁĄCZNYM KOSZCIE NAPRAW

In this paper, we consider a repair number counting replacement policy under a cumulative damage model, in which the policy includes the concept of a cumulative repair cost limit. The system experiences two kinds of shocks: a type I shock causes a random amount of damage to the system leading to a serious failure when the total damage exceeds a failure level; or a type II shock causes the system into minor failure which can be corrected by minimal repair. When a minor failure occurs, the repair cost will be evaluated and minimal repair is executed if the accumulated repair cost is less than a predetermined limit L. The system is replaced anticipatively at n-th minor failure, or at the j-th minor failure (j < n) at which the accumulated repair cost exceeds a predetermined limit L, or any serious failure. In order to assess the performance of the proposed maintenance policy and to minimize the long-term expected cost per unit time, a mathematical model for the maintained system cost is derived. By minimizing that cost, the optimal number n is also verified finite and unique under certain conditions. Analyses based on numerical results are conducted to highlight the properties of the proposed maintenance policy in respect to the different parameters.*

***Keywords**: Cumulative damage model, Cumulative repair cost limit, Replacement policy, Minimal repair.*

W przedstawionym artykule omawiamy politykę wymiany systemu opartą na modelu sumowania uszkodzeń polegającą na obliczaniu liczby napraw. Polityka ta obejmuje koncepcję limitu łącznego kosztu napraw. System może być narażony na działanie dwóch rodzajów szkodliwych czynników: czynniki I-ego typu powodują losowo określony zakres uszkodzeń systemu, prowadząc do poważnej awarii, gdy łącznie uszkodzenia przekraczają poziom awarii; lub czynniki typu II-ego powodujące drobne uszkodzenia, które można skorygować poprzez minimalną naprawę. Gdy dochodzi do niewielkiego uszkodzenia, wtedy szacuje się koszt naprawy i realizuje minimalną naprawę, jeśli łączny koszt naprawy jest niższy od uprzednio ustalonego limitu L. System zostaje prewencyjnie wymieniony albo przy n-tej drobnej awarii albo przy j-tej drobnej awarii (j <n), przy której łączny koszt naprawy przekracza uprzednio ustalony limit L lub też przy jakimkolwiek poważnym uszkodzeniu. W celu oceny skuteczności proponowanej polityki obsługiwania i zminimalizowania przewidywanego długoterminowego kosztu przypadającego na jednostkę czasu, wyprowadzono model matematyczny kosztów dla obsługiwanego systemu. Poprzez minimalizację tych kosztów, określono również optymalną liczbę napraw n, która w pewnych warunkach jest liczbą skończoną i niepowtarzalną. W oparciu o wyniki numeryczne, przeprowadzono analizy mające na celu naświetlenie właściwości proponowanej polityki obsługiwania w odniesieniu do różnych parametrów.*

***Słowa kluczowe**: Model sumowania uszkodzeń, limit łącznych kosztów napraw, polityka wymiany, naprawa minimalna.*

1. Introduction.

Most production systems suffer increasing wear with usage or age and are subject to random failures resulting from this deterioration (Wang (2002)) and most of them are maintained or repairable systems. Moreover, for some systems, such as aircrafts, submarines, military systems, and nuclear systems, it is very important to maintain a system to prevent failures because it can be dangerous or disastrous. The growing importance of maintenance has generated an increasing interest in the development and implementation of optimal maintenance strategies for improving system availability, preventing the occurrence of system failures, and reducing maintenance costs of deteriorating systems. In the deteriorating system, the level of deterioration is represented by a degradation process, such as corrosion, wear out, material fatigue, and fatigue-crack-growth in engineering applications or markers of health status and quality of life data in medical settings. Cumulative damage models are often used to describe these above situations.

Cumulative damage models are a special class of mathematical models within reliability theory that describe the probability of failure of a given system under the impact of a damaging environment. The system suffers damage due to shocks and fails when the total amount of damage exceeds a failure level K, and it generates cumulative damage process. Recently, Nakagawa (2007) summarized sufficiently PM policies and their optimization problems for cumulative damage models. The replacement models where a system is replaced when the total damage exceeds a threshold level k can refer Feldman (1976), Nagakawa (1976) and Satow et al. (2000). The replacement models where a unit is replaced at a planned time T were proposed in Taylor (1975), Mizuno (1986), Nakagawa (1980, 2007), Qian *et al.* (1999) and Perry (2000). Furthermore, the replacement models where a system is replaced at shock N were proposed in Nagakawa (1984).

Nakagawa and Kijima (1989) considered a standard cumulative damage model with minimal repair at failure to obtain the optimal values T^*, N^*, and k^*, individually. Kijima and Nakagawa (1991) considered a cumulative damage shock model with imperfect PM policy. Satow and Nakagawa (1997) considered a modified cumulative damage model that the damage can be produced by shocks or increased with time at constant rate a. The optimal values T^*, N^*, and k^* are obtained individually. Qian *et al.* (1999) presented an extended cumulative damage model with two kinds of shocks: failure shock and damage shock. This model is applied to the backup of files in a database system and the optimal replacement period is obtained. Satow *et al.* (2000) considered a cumulative damage model with two types of damages that are both from external shocks and deterioration with time. The optimal threshold k^* is obtained.

Qian *et al.* (2003) considers an extended cumulative damage model with maintenance at each shock when the total damage does not exceed a failure level *K*; with minimal repair at each shock when the total damage exceeds a failure level *K*, and with replacement at time T or at failure *N*. The optimal values T^* and N^* are obtained. Qian *et al.* (2005) applied cumulative damage model for a used system with initial damage level. A unique optimal T^* or managerial level k^* which minimizes the expected cost rate are obtained. Ito and Nakagawa (2011) compared the standard cumulative damage model with two other cumulative damage models: (1) the amount of damage due to shocks is measured only at periodic time; and (2) the amount of damage increases linearly with time. This cumulative damage model can be applied to the garbage collection policies for a database system in Satow et al.(1996) and applied to obtain the optimal full and cumulative backup policies successfully for a database system in Qian et al. (1999, 2005). And, it is also applied to describe the cumulative damage of a fibrous carbon composite in Padgett(1998).

Zhao *et al.* (2012) considered a periodical replacement model that the unit is replaced at a planned time or when the total damage exceeds a failure level, whichever occurs first, and undergoes minimal repair when independent damage occurs. Furthermore, they considered a modified model that the total damage is measured at periodic times and increases approximately with time linearly. Zhao and Nakagawa (2012) considered age and periodic replacement last models with working cycles and applied this type of replacement policy to a standard cumulative damage model. Zhao *et al.* (2013) applied the notion of maintenance last to a standard cumulative damage model, in which the unit undergoes preventive maintenances before failure at a planned time *T*, at a damage level *k*, or at a shock number *N*, whichever occurs last.

With regard to repair-cost-limit policies allowing minimal repairs, Lai (2007) applied the concept of cumulative repair cost limit into replacement model that included the information of all repair costs to decide whether the system should be repaired or replaced. Following the work of Lai (2007), Chien, *et al.* (2009) extended the work of Lai (2007) by introducing the random lead time for replacement delivery. Chien, *et al.* (2010) modified the work of Chien, et al. (2009) by adding an age-dependent type of failure. Chang, *et al.* (2010) presented a model for determining the optimal number of minimal repairs before replacement. Chang, *et al.* (2013) modified the work of Chang, *et al.* (2010) by allowing an age-dependent failure type. Sheu, *et al.* (2010) presented a generalized model for determining the optimal replacement policy based on multiple factors (or more information) such as the number of minimal repairs before replacement and the cumulative repair cost limit.

In this study, we present a repair number counting replacement policy with cumulative repair cost limit where the system is subject to a cumulative damage model. The concept of cumulative repair cost limit adopts the entire repair cost history to make decision for repairing or replacing the system. The remainder of the paper is organized as follows: Section 2 presents the model formulation and optimization. In Section 3, the long-term expected cost per unit time $\overline{C}(n,L)$ is derived and the conditions characterize the optimal n^* is developed. A computational example is provided to demonstrate the above results in Section 4. Section 5 provides conclusions

2. Problem formulation

Assume that the system is subject to shocks which randomly occur according to a non-homogeneous Poisson process $\{N(t)\}_{t\geq 0}$ with intensity rate $\lambda(t)$. Whenever a shock occurs, it will be type-I shock with probability p $(0<p\leq 1)$ and type-II shock with probability q $(p+q=1)$. By using the decomposition theorem of Poisson process, it is noted that type-I and type-II shocks occur according to two non-homogeneous Poisson processes $\{N_1(t)\}_{t\geq 0}$ and $\{N_2(t)\}_{t\geq 0}$ with intensity rates $p\lambda(t)$ and $q\lambda(t)$, respectively. And, $N_1(t)$ and $N_2(t)$ denote the numbers of type-I and type-II shocks occurred during [0, *t*], respectively.

The type-I shocks whenever occur cause some damage to the system and these damages are additive. When a type-I shock occurs, a random amount D_i of damage from *i-th* type-I shock has a probability distribution $H(d)=P(D_i\leq d)$ and a finite mean μ_d, *i*=1,2,3,…. Then the accumulated damage to the the *j-th* type I shock after the installation $W_j=\sum_{i=1}^{j}D_i$ has a distribution function:

$$P(W_j\leq w)=H^{(j)}(w)=\begin{cases}1 & j=0\\ H_1*H_2*\cdots*H_j(w), & j=1,2,3,\cdots\end{cases}, \quad (1)$$

where the "*" mark is denoted the *Stieltjes* convolution of the distribution *H*(*d*) with itself. The probability of *j* type-I shocks in [0, *t*] is given by:

$$P(N_1(t)=j)=\frac{\left(m_1(t)\right)^j\exp\left(-m_1(t)\right)}{j!}=P_{1,j}(t), \quad (2)$$

where $m_1(t)=\int_0^t p\lambda(x)dx$ denote the mean number of type-I shock in [0,*t*].

If the total damage exceeds a failure level *K,* a serious failure occurs. The probability that a serious failure occurs at the *j-th* type-I shock is $H^{(j-1)}(K)-H^{(j)}(K)$. Let a random variable *Z* denote the occurrence time of the first serious failure, so the survival function of *Z* is given by:

$$\overline{F_z}(t)=P(Z>t)=P(Y_{N_1(t)}<K)=\sum_{j=0}^{\infty}P(N_1(t)=j,Y_j<K)=\sum_{j=0}^{\infty}P_{1,j}(t)H^{(j)}(K) \quad (3)$$

and the density function of Z is $f_z(t)=p\lambda(t)\sum_{j=0}^{\infty}\left(H^{(j)}(K)-H^{(j+1)}(K)\right)P_{1,j}(t)$.

Each type-II shock makes the system into minor failure. Hence, the probability of *j* minor failures in [0, *t*] is given by:

$$P(N_2(t)=j)=\frac{\left(m_2(t)\right)^j\exp(-m_2(t))}{j!}=P_{2,j}(t) \quad (4)$$

where $m_2(t)=\int_0^t q\lambda(x)dx$ denote the mean number of minor failures in [0,*t*].

Moreover, let S_{2j} (*j*=1, 2, 3, …) denote the occurrence time of the *j-th* minor failure, where $S_{20}=0$, then the distribution function of a random variable S_{2j} is given by:

$$P(S_{2j} \le t) = P(N_2(t) \ge j) = \sum_{i=j}^{\infty} P_{2,i}(t), \quad j = 1, 2, 3, ...,$$

and

$$f_{s_{2j}}(t) = \frac{d}{dt} P(S_{2j} \le t) = \frac{d}{dt} \sum_{i=j}^{\infty} \frac{(m_2(t))^i \exp(-m_2(t))}{i!} = q\lambda(t) P_{2,j-1}(t).$$

When a minor failure occurs, the repair cost due to this minor failure is evaluated. Suppose that a minimal repair cost X_i due to the *i-th* minor failure has a nonnegative independent and identical distribution function $G(x) = P(X_i \le x)$ and a finite mean μ_x, *i*=1,2,3,…. Then, the accumulated repair cost till to *j-th* minor failure $Y_j = \sum_{i=1}^{j} X_i$ has a distribution function:

$$P(Y_j \le y) = G^{(j)}(y) = \begin{cases} 1 & j = 0 \\ G_1 * G_2 * \cdots * G_j(y), & j = 1,2,3,\cdots \end{cases} \quad (5)$$

If the accumulated repair cost exceeds a predetermined limit *L*, then the system must be replaced at this minor failure. Let a random variable *U* denote the occurrence time when the accumulated repair cost exceeds a predetermined limit *L*, so the survival function of *U* is given by:

$$\overline{F_u}(t) = P(U > t) = P(Y_{N_2(t)} < L) = \sum_{j=0}^{\infty} P(N_2(t) = j, Y_j < L) = \sum_{j=0}^{\infty} P_{2,j}(t) G^{(j)}(L) \quad (6)$$

and the density function of *U* is $f_u(t) = q\lambda(t) \sum_{j=0}^{\infty} \left(G^{(j)}(L) - G^{(j+1)}(L) \right) P_{2,j}(t)$.

In this model, preventive maintenance policy is executed according to the following scheme. Preventive replacement is carried out at the *n-th* minor failure or at the occurrence time of one minor failure, in which the accumulated repair cost at this moment exceeds a predetermined limit *L*, and failure replacement is executed at the occurrence time of a serious failure. According to the above scheme, the replacement of the system can occur at three different situations and the probabilities of three situations will be introduced as follows.

First, if the accumulated repair cost till to (*n-1*)*-th* minor failure is less than *L* and the *n-th* minor failure precedes a serious failure, then preventive replacement is executed at the *n-th* minor failure. Therefore, the probability of situation 1 is given by:

$$\int_0^{\infty} P(Z > s_{2n}, \ Y_{n-1} < L) dF(s_{2n}) = G^{(n-1)}(L) \int_0^{\infty} \overline{F_z}(t) P_{2,n-1}(t) q\lambda(t) dt. \quad (7)$$

Because the occurrences of minor and serious failures are mutually independent.

Second, if the *j-th* (*j*<*n*) minor failure occurs and the accumulated repair cost till to this failure exceeds *L*, and no serious failure has occurred, then the system will be replaced by a new one at time S_{2j}, *j*=1,2,3,…, n-1. Therefore, the probability of situation 2 is given by:

$$\begin{aligned} \sum_{j=1}^{n-1} P(Y_{j-1} < L < Y_j, W_1 > S_{2j}) &= \sum_{j=1}^{n-1} \left(G^{(j-1)}(L) - G^{(j)}(L) \right) \int_0^{\infty} P(W_1 > s_{2j}) dF(s_{2j}) \\ &= \sum_{j=1}^{n-1} \left(G^{(j-1)}(L) - G^{(j)}(L) \right) \int_0^{\infty} \overline{F_z}(t) P_{2,j-1}(t) q\lambda(t) dt \end{aligned} \quad (8)$$

Finally, if a serious failure occurs before time S_{2j}, *j*=1,2,…, *n*-1 and the accumulated repair cost till to this serious failure is less than *L*, then the system will be replaced at serious failure. Therefore, the probability of situation 3 is given by:

$$\sum_{j=0}^{n-1} P(N_2(z) = j, Y_j < L) = \sum_{j=0}^{n-1} G^{(j)}(L) \int_0^{\infty} P_{2,j}(t) dF_z(t) \quad (9)$$

More specifically, we also require the following assumptions:

(a1) The system is monitored continuously so that minor or serious failures can be detected instantaneously.

(a2) The times taken for minimal repair or replacement are very smaller than the mean time between failures. As a consequence, we can ignore those and treat those as being zero.

(a3) The steady state case is considered.

Finally, Replacement at *n-th* minor failure or at which the accumulated repair cost exceeds limit *L* costs C_0 and is called preventive replacement, while replacement at serious failure costs C_1 and is called as failure replacement in which $C_1 > C_0$. This problem is just to find an optimal n^* to minimize the long-term expected cost per unit time $\overline{C}(n, L)$ in the steady state case.

3. Long-term expected cost per unit time

It is well known that if a replacement is performed, a new replacement cycle will restart. Therefore, the continuous replacement cycles will constitute a renewal process. Let $E(V_1)$ and $E(R_1)$ denote the mean length of a replacement cycle and the expected total cost incurred during a replacement cycle, respectively. Using the renewal-reward theorem, we can observe that the long-term expected cost per unit time in the steady-state case is given by (Ross (1983)):

$$\overline{C}(n, L) = E(R_1) / E(V_1).$$

Under our defined preventive maintenance policy, the expected length of a replacement cycle $E(V_1)$ is given by:

$$\begin{aligned} E(V_1) &= G^{(n-1)}(L) \int_0^{\infty} t \times \overline{F_z}(t) P_{2,n-1}(t) q\lambda(t) dt + \sum_{j=1}^{n-1} \left(G^{(j-1)}(L) - G^{(j)}(L) \right) \int_0^{\infty} t \times \overline{F_z}(t) P_{2,j-1}(t) q\lambda(t) dt \\ &\quad + \sum_{j=0}^{n-1} G^{(j)}(L) \int_0^{\infty} t \times P_{2,j}(t) dF_z(t) \\ &= \sum_{j=0}^{n-1} G^{(j)}(L) \int_0^{\infty} \overline{F_z}(t) P_{2,j}(t) dt \end{aligned} \quad (10)$$

If the system is replaced preventively following the *n-th* minor failure, the total cost is $C_0 + \sum_{i=1}^{n-1} X_i$. When the system is replaced preventively following the *j-th* (*j*<*n*) minor failure because the accumulated repair cost exceeds *L*, the total cost will be $C_0 + \sum_{i=1}^{j-1} X_i, \ j = 1, 2, ..., n-1$. However, if the replacement is executed at serious failure, then the total cost is $C_1 + \sum_{i=1}^{N_2(Z)} X_i$. Therefore, the expected total cost $E(R_1)$ can be derived as follows:

$$\begin{aligned} E(R_1) &= G^{(n-1)}(L)\int_0^\infty E\left[C_0+\sum_{i=1}^{n-1}X_i\right]\times\overline{F_z}(t)P_{2,n-1}(t)q\lambda(t)dt \\ &+\sum_{j=1}^{n-1}\left(G^{(j-1)}(L)-G^{(j)}(L)\right)\int_0^\infty E\left[C_0+\sum_{i=1}^{j-1}X_i\right]\times\overline{F_z}(t)P_{2,j}(t)q\lambda(t)dt \\ &+\sum_{j=0}^{n-1}G^{(j)}(L)\int_0^\infty E\left[C_1+\sum_{i=1}^{N_2(t)}X_i\right]\times P_{2,j}(t)dF_z(t) \\ &= C_0+(C_1-C_0)\sum_{j=0}^{n-1}G^{(j)}(L)\int_0^\infty P_{2,j}(t)dF_z(t)+\mu_x\sum_{j=1}^{n-1}G^{(j)}(L)\int_0^\infty\overline{F_z}(t)P_{2,j-1}(t)q\lambda(t)dt \end{aligned}, \quad (11)$$

where $\mu_x = E(X_i)$.

Combining (10) and (11), the long-term expected cost per unit time $\overline{C}(n,L)$ can be obtained as follows:

$$\overline{C}(n,L)=\frac{C_0+(C_1-C_0)\sum_{j=0}^{n-1}G^{(j)}(L)\int_0^\infty P_{2,j}(t)dF_z(t)+\mu_x\sum_{j=1}^{n-1}G^{(j)}(L)\int_0^\infty\overline{F_z}(t)P_{2,j-1}(t)q\lambda(t)dt}{\sum_{j=0}^{n-1}G^{(j)}(L)\int_0^\infty\overline{F_z}(t)P_{2,j}(t)dt} \quad (12)$$

In the steady-state case, we want to find an optimal number n^* that minimises $\overline{C}(n,L)$ under the following assumptions:

(a1) $\lambda(t)$ is a continuous and increasing function of t with $\lambda(\infty)=\lim_{t\to\infty}\lambda(t)$, which may be infinite.

(a2) $G^{(n)}(y)$ is PF2 (a Polya frequency function of order 2).

From Lemma 3.7 in Barlow and Proschan (1975), it is known that $G^{(n)}(L)$ is decreasing in n for all $L > 0$. In addition, we can observe that $G^{(n)}(L)$ is PF2 if and only if $G^{(n)}(L)/G^{(n-1)}(L)$ is decreasing in n for all $L > 0$ (Gottlieb 1980, p. 749).

If an optimal n^* exists, then the inequalities $\overline{C}(n+1,L)\ge\overline{C}(n,L)$ and $\overline{C}(n,L)<\overline{C}(n-1,L)$ are both satisfied for some finite n. In the derivation of these inequalities, we can see that the inequalities $\overline{C}(n+1,L)\ge\overline{C}(n,L)$ and $\overline{C}(n,L)<\overline{C}(n-1,L)$ are equivalent to the inequalities $K(n)\ge C_0$ and $K(n-1)<C_0$, where:

$$K(n)=\begin{cases}\dfrac{(C_1-C_0)\left(G^{(n)}(L)\int_0^\infty P_{2,n}(t)dF_z(t)\right)+\mu_x\left(G^{(n)}(L)\int_0^\infty\overline{F_z}(t)P_{2,n-1}(t)q\lambda(t)dt\right)}{G^{(n)}(L)\int_0^\infty\overline{F_z}(t)P_{2,n}(t)dt}\times\left(\sum_{j=0}^{n-1}G^{(j)}(L)\int_0^\infty\overline{F_z}(t)P_{2,j}(t)dt\right) \\ \quad -(C_1-C_0)\left(\sum_{j=0}^{n-1}G^{(j)}(L)\int_0^\infty P_{2,j}(t)dF_z(t)\right)-\mu_x\left(\sum_{j=1}^{n-1}G^{(j)}(L)\int_0^\infty\overline{F_z}(t)P_{2,j-1}(t)q\lambda(t)dt\right) & n=1,2,3,\cdots \\ 0 & n=0\end{cases} \quad (13)$$

Therefore, if we can show that $K(n)$ is an increasing function of n and $\lim_{n\to\infty}K(n)>C_0$, then n^* *is* finite and unique. To show that $K(n)$ is an increasing function of n, the following lemma is required.

Lemma 1. Under assumptions (a1) and (a2), the following results are true:

(1) $A_n=\int_0^\infty P_{2,n}(t)dF_z(t)\Big/\int_0^\infty\overline{F_z}(t)P_{2,n}(t)dt$ is increasing in n, and $\lim_{n\to\infty}A_n=p\lambda(\infty)$.

(2) $B_n=\int_0^\infty\overline{F_z}(t)P_{2,n}(t)q\lambda(t)dt\Big/\int_0^\infty\overline{F_z}(t)P_{2,n+1}(t)dt$ is increasing in n, and $\lim_{n\to\infty}B_n=q\lambda(\infty)$.

A proof of Lemma 1 can be found in Chang *et al*. (2010). Moreover, $K(n)$ is an increasing function of n is equivalent to the condition $K(n+1)-K(n)>0$ for all n. Consequently,

$$K(n+1)-K(n)$$

$$=\left(\sum_{j=0}^{n}G^{(j)}(L)\int_0^{\infty}\overline{F_z}(t)P_{2,j}(t)dt\right)\times\left(\begin{array}{l}\dfrac{(C_1-C_0)\left(G^{(n+1)}(L)\int_0^{\infty}P_{2,n+1}(t)dF_z(t)\right)+\mu_x\left(G^{(n+1)}(L)\int_0^{\infty}\overline{F_z}(t)P_{2,n}(t)q\lambda(t)dt\right)}{G^{(n+1)}(L)\int_0^{\infty}\overline{F_z}(t)P_{2,n+1}(t)dt}\\ -\dfrac{(C_1-C_0)\left(G^{(n)}(L)\int_0^{\infty}P_{2,n}(t)dF_z(t)\right)+\mu_x\left(G^{(n)}(L)\int_0^{\infty}\overline{F_z}(t)P_{2,n-1}(t)q\lambda(t)dt\right)}{G^{(n)}(L)\int_0^{\infty}\overline{F_z}(t)P_{2,n}(t)dt}\end{array}\right)$$

Using *Lemma* 1, $K(n+1)-K(n)$ can be obtained as follows:

$$K(n+1)-K(n)=\sum_{j=0}^{n}G^{(j)}(L)\int_0^{\infty}\overline{F_z}(t)P_{2,j}(t)dt\times\left[(C_1-C_0)\times\left(A_{n+1}-A_n\right)+\mu_x\times\left(B_{n+1}-B_n\right)\right]. \quad (14)$$

Because

$\sum_{j=0}^{n}G^{(j)}(L)\int_0^{\infty}\overline{F_z}(t)P_{2,j}(t)dt>0$, $C_1>C_0$, $A_{n+1}>A_n$ and $B_{n+1}>B_n$, then $K(n+1)-K(n)>0$ for all *n*. Thus, we can see that $K(n)$ is increasing in *n*. In summary, the conditions for the existence and the uniqueness of an optimal value n^* are expressed in the following theorem:

Theorem 1. Under assumptions (a1) and (a2), if $\lim_{n\to\infty}K(n)>C_0$, then there exists a finite and unique n^* that minimises $\overline{C}(n,L)$ and satisfies:

$$K(n^*)\geq C_0 \text{ and } K(n^*-1)<C_0,\ n^*=1,2,3,\cdots. \quad (15)$$

Proof. The inequalities $\overline{C}(n+1,L)\geq\overline{C}(n,L)$ and $\overline{C}(n,L)<\overline{C}(n-1,L)$ imply (15). Under assumptions (a1) and (a2), we can observe that $K(n)$ is increasing in *n* from ***Lemma*** 1. Furthermore,

$$\lim_{n\to\infty}K(n)=(C_1-C_0)\left[\sum_{j=0}^{\infty}G^{(j)}(L)\int_0^{\infty}\overline{F_z}(t)P_{2,j}(t)p\lambda(\infty)dt-\sum_{j=0}^{\infty}G^{(j)}(L)\int_0^{\infty}P_{2,j}(t)dF_z(t)\right]$$
$$+\mu_x\left(\sum_{j=0}^{\infty}G^{(j)}(L)\int_0^{\infty}\overline{F_z}(t)P_{2,j}(t)q\lambda(\infty)dt-\sum_{j=1}^{\infty}G^{(j)}(L)\int_0^{\infty}\overline{F_z}(t)P_{2,j-1}(t)q\lambda(t)dt\right)$$

In equation (13), we know that $K(0)=0<C_0$. If $\lim_{n\to\infty}K(n)>C_0$, we can observe that there is a finite *n* such that equation (15) is satisfied. In addition, the optimal value n^* is unique according to the fact that $K(n)$ is increasing in *n*.

In our model, if $K=0$, then $\overline{F_z}(t)=\exp\left(-\int_0^t p\lambda(x)dx\right)$ and $\overline{C}(n,L)$ is the same as $C(n)$ in Chien, *et al.* (2010).

4. Numerical example

We consider that the intensity rate $\lambda(t)$ of arrival shocks is taking as

$$\lambda(t)=\lambda t^{\beta-1},\quad \lambda>0,\ \beta>1. \quad (16)$$

We assume that the shape parameter is set at β=2, and that $\lambda(t)=\lambda t$ is an increasing function of *t*. Let two replacement costs C_0 and C_1 be 1000 and 1500, respectively. The amount of damage from consecutive type-I shocks are independently and identically exponential random variables with finite mean μ_d =100. And, the failure level of the system is set at *K*=800. The costs for consecutive minimal repair are also independently and identically exponential random variables with finite mean μ_x =50. In addition, the cumulative repair cost limit *L* is fixed to be 500.

Because $\lambda(t)$ is strictly increasing to ∞ as $t\to\infty$, $K(n)$ is increasing in *n*. Thus, n^* is finite and unique. Using the software MAPLE, n^* and the minimum long-term expected cost per unit time $\overline{C}(n^*,L)$ are computed for various values of the parameters λ and *p* are listed in Tables 1 and 2.

From Tables 1 and 2, we have the following conclusions:

(1) As λ increases, the optimal n^* is unchanged, but the minimum $\overline{C}(n^*,L)$ increases. This situation is due to the denominator of the equation (12), i.e., the expected length of a replacement cycle decreases. A greater value of λ implies that the arriving shocks occur more frequently, so the replacement period must be shorter to prevent the occurrence of random failures.

(2) As *p* increases (i.e., (1-*p*) decreases), the optimal n^* decreases, but the minimum $\overline{C}(n^*,L)$ increases. A greater value of *p* implies that serious failures occur more easily, so the replacement period must also be shorter, i.e., the optimal n^* must be smaller, to prevent the occurrence of serious failures.

Table 1. Optimal n^* and $\overline{C}(n^*,L)$ at different λ and p, when L/μ_x =10 and K/μ_d =8

	p=0.3		p=0.4		p=0.5		p=0.6		p=0.7	
	n^*	$\overline{C}(n^*,L)$	n^*	$\overline{C}(n^*,L)$	n^*	$\overline{C}(n^*,L)$	n^*	$\overline{C}(n^*,L)$	n*	$\overline{C}(n^*,L)$
λ=1.0	12	292.9909184	9	294.9119827	7	301.3978537	5	310.5921124	4	321.3561456
λ=1.5	12	358.8391246	9	361.1919384	7	369.1354753	5	380.3960969	4	393.5792913
λ=2.0	12	414.3517304	9	417.0685255	7	426.2409323	5	439.2435780	4	454.4662197
λ=2.5	12	463.2593182	9	466.2967872	7	476.5518497	5	491.0892494	4	508.1086802

Table 2. Optimal n* and $\overline{C}(n^*,L)$ at different values of L/μ_x and K/μ_d when p=0.5, λ=2

	L/μ_x =6		L/μ_x =8		L/μ_x =10		L/μ_x =12	
	n^*	$\overline{C}(n^*,L)$	n^*	$\overline{C}(n^*,L)$	n^*	$\overline{C}(n^*,L)$	n^*	$\overline{C}(n^*,L)$
K/μ_d =6	6	476.5594134	6	473.0556584	6	472.1746264	6	471.9659801
K/μ_d =8	7	432.9678584	7	427.7229495	7	426.2409323	7	425.8473161
K/μ_d =10	8	406.0023083	8	398.6716537	8	396.3677583	8	395.6877385
K/μ_d =12	9	388.8043367	9	379.1945801	9	375.8645467	9	374.7818536

(3) When the ratio L/μ_x is larger, i.e. this model allows more minor failures before replacement, we can see that the optimal n^* is unchanged and the minimum $\overline{C}(n^*,L)$ decreases. For a fixed ratio K/μ_d, the decreasing magnitudes of the optimal $\overline{C}(n^*,L)$ are significantly smaller than the increment of L/μ_x.

(4) When the ratio K/μ_d is larger, i.e. this model allows more type I shocks to occur before serious failure, we can see that the optimal n^* increases but the minimum $\overline{C}(n^*,L)$ decreases. The variations in the optimal n^* and $\overline{C}(n^*,L)$ with regard to K/μ_d are significantly larger than that of L/μ_x. Therefore, the ratio K/μ_d is more important than the ratio L/μ_x when determining the optimal replacement period.

5. Conclusions

In this article, a repair number counting replacement policy based on a cumulative repair-cost limit under a standard cumulative damage model is introduced. The long-term expected cost per unit time $\overline{C}(n,L)$ in operating the system was developed which incorporating costs due to holding minimal repair and different forms of replacement state. The optimal number n^* of minimal repair, which minimizes the cost rate function under a fixed cumulative repair-cost limit L, was shown. The existence, uniqueness and structural properties were also proposed. This research verifies that under some specific conditions, the optimal number n^* of minimal repair is finite and unique under fixed L and K. This model provided a general framework for analyzing the maintenance policies for a system subject to cumulative damage models, so two previous models in the literature were the special cases of our model. We also demonstrated some numerical examples.

However, some assumptions are possible limitations in this research, such as the repair and replacement times were negligible, and the repairs are minimal and the repaired system is as bad as old. In some practical situations, it would seem to be more practical to consider the concept of imperfect repairs or multi-unit systems. Taking these realistic factors into consideration in the proposed policy is one direction for future research.

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