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# STRUCTURAL RELIABILITY ANALYSIS USING FUZZY SETS THEORY

# ANALIZA NIEZAWODNOŚCIOWA KONSTRUKCJI Z WYKORZYSTANIEM TEORII ZBIORÓW ROZMYTYCH

*Prediction of structural performance is a complex problem because of the existence of randomness and fuzziness in engineering practice. In this area, reliability analyses have been performed using probabilistic methods. This work investigates reliability analysis of structure involving fuzziness and randomness. In particular, the safety state of the structure is defined by a fuzzy state variable, fuzzy random allowable interval, or fuzzy random generalized strength. Because the membership function of the fuzzy safety state is the key to structural reliability analysis using the fuzzy sets theory, this work proposes useful methods to determine the membership functions and develops a structural reliability analysis method based on the fuzzy safety state. Several examples are provided to illustrate the proposed methods.*

***Keywords***: *reliability, structure, fuzzy safety state, membership function, fuzzy random generalized stress, fuzzy random generalized strength.*

*Przewidywanie zachowania konstrukcji stanowi złożone zagadnienie ze względu na istnienie w praktyce inżynierskiej losowości i rozmytości. Na tym obszarze, analizy niezawodnościowe prowadzono dotąd przy pomocy metod probabilistycznych. W niniejszej pracy przedstawiono metodę niezawodnościowej analizy konstrukcji uwzględniającą rozmytość i losowość. Dokładniej, stan bezpieczeństwa konstrukcji określano za pomocą rozmytej zmiennej stanu, rozmytego losowego przedziału dozwolonego lub rozmytej losowej uogólnionej wytrzymałości. Ponieważ funkcja przynależności rozmytego stanu bezpieczeństwa stanowi klucz do niezawodnościowej analizy konstrukcji wykorzystującej teorię zbiorów rozmytych, w niniejszej pracy zaproponowano przydatne metody wyznaczania funkcji przynależności oraz opracowano metodę niezawodnościowej analizy konstrukcji opartą na rozmytym stanie bezpieczeństwa. Zaproponowane metody zilustrowano kilkoma przykładami.*

***Słowa kluczowe***: *niezawodność, konstrukcja, rozmyty stan bezpieczeństwa, funkcja przynależności, rozmyte losowe uogólnione naprężenie, rozmyta losowa uogólniona wytrzymałość.*

## 1. Introduction

Stress Strength Interference (SSI) is a fundamental model for structural reliability-based design and has been widely used in engineering practice [1, 6, 24, 25]. In an SSI model, a limit state function must be determined. In most studies, the limit state function is usually assumed to be exact without considering fuzziness. This means that the corresponding theory used to determine the limit states is perfect, which may not be realistic in real-world applications. Indeed, unvoidable errors could be resulted if such limit state function is used for reliability analysis. As a result, it is necessary to develop a new reliability model that takes fuzziness into consideration [27, 28]. To this end, the following question must be answered: what fundamental issue needs to be addressed for this purpose? The answer is that more data (i.e., experimental results) should be collected. As a matter of fact, if the exact value of the actual strength of a structure cannot be determined, we need to rely on more data to give additional information necessary to correct the theoretical model used [18].

When it is expensive to obtain experimental data or there are a few but poorly documented instances of failure of the prototype system, it would be difficult to correct the theoretical model. There is another extreme case where there are no data at all for calculating the probability of failure at the early design stage. For these circumstances, using engineering judgment or experience for similar structures in SSI modeling becomes a very useful alternative.

Uncertainties and ambiguities in structural performance have been dealt with using probability theory. However, it is worth pointing out that some uncertainties, which are not random in nature, may play important roles in the safety assessment of engineering structures [17]. In other words, the probability-based reliability provides a solution different from the observed failure rate which is inferred from the statistics of structural accidents [4, 28]. A more fundamental argument against the conventional approach to parameterizing model uncertainties is provided by Blockley [2, 3].

Fuzziness could be produced due to some factors, such as omissions, human error, inadequate modeling, experience, and intuition of the engineers. Such uncertainties are called "subjective uncertainties", because they could be evaluated solely by an engineer's experience and judgment. Fuzzy sets theory, which was proposed by Zadeh in 1965, is available to deal with the subjective uncertainties in a quantitative way. Moreover, this theory makes it possible to define safety events in a more flexible form than the probabilistic approach.

The first known theoretical approach to using fuzzy logic for failure diagnosis belongs to Tsukamato and Tarano [23]. Brown [4] and Blockley [3] applied the fuzzy sets theory in an attempt to explain the difference between the calculated and observed failure probabilities. Savchuk [18] suggested some improvement of reliability estimation in the framework of the SSI model, which was essentially based on a limit-state model. So far, only initial attempts have been made [14, 15], but sophisticated formulations and algorithms for numerical treatments and applications to complex structural reliability analysis have not been reported.

The approach proposed in this work aims to overcome some of the problems of the conventional treatment mentioned above. Specifically, the main purpose is to perform reliability estimation based on a limit-state model considering both the error of the limit-state model and the fuzziness of data.

## 2. Fuzzy safety state of structure

In a traditional SSI model, the generalized strength of a structure $R$ and the generalized stress $S$ are both considered to be random variables. The safety margin, also called the state variable of the structure, is defined as:

$$Z \overset{\Delta}{=} R - S \cdot \tag{1}$$

It is obvious that the state variable is also a random variable. The random event that the structure works satisfactorily during its service life $T$, denoted by $A$, is defined as:

$$A \overset{\Delta}{=} \{S < R\} \cdot \tag{2}$$

This event is also called the state of safe operation, or simply the safety state, of the structure. Let $p_S(x)$ denote the probability density function (pdf) of the generalized stress $S \in (-\infty, \infty)$, $p_R(y)$ the pdf of the generalized strength $R \in [x, \infty)$, $p_{S,R}(x,y)$ the joint pdf of $S$ and $R$, and $p_Z(z)$ the pdf of the state variable Z. The reliability of the structure is simply the probability of the safety state of the structure and can be expressed by integrating the pdf of the random event $A$, with respect to the domains of $x$ and $y$:

$$P_r = P(A) = \int_0^{\infty} p_Z(z)\mathrm{d}z = \int_{-\infty}^{\infty}\int_{x}^{\infty} p_{S,R}(x,y)\mathrm{d}y\mathrm{d}x \,. \tag{3}$$

The probability of structural failure is then given by:

$$P_f = 1 - P(A) \cdot \tag{4}$$

When the random variables $S$ and $R$ are independent, their joint pdf can be simply expressed as the multiplication of their individual pdf:

$$p_{S,R}(x,y) = p_S(x)p_R(y) \,. \tag{5}$$

According to the SSI model, the structure is safe as long as the generalized stress $S$ is lower than the generalized strength $R$. However, the accuracy of this theory has been widely questioned [13, 14, 15, 19, 21]. First, the safety criterion should be fuzzy in practice [7, 8, 9, 10, 26]. Second, the generalized strength is usually not known precisely and thus is fuzzy [11, 19]. For example, cracks had been found in the tail rotor components of a CH-149 Cormorant helicopter [22], which had been created by the generalized strength during its operations. Third, the load is also fuzzy [14, 19, 11]. In the space shuttle Columbia, the debris hitting it has led to its demise during the re-entry [13]. Moreover, a turbine blade was fractured and traveled through subsequent sections of the turbine, while a shroud that dropped into the turbine air path caused excessive wear to several turbine blades at Langley AFB [16]. Such impacts generated fuzzy random loads causing structural failures. Of course, the behavior of the structure under study and the stress developed may be fuzzy too. Solid rocket seal leakage during the launch of space shuttle Challenger was undetected and precipitated its disintegration [13]. This event has been most probability originated from the fuzzy random generalized stresses in the inter-connections. These observations have led to the fuzzy versions of SSI models.

When failure modes, such as fatigue, abrasion, and erosion, are considered, the safety state of operation of the structure under consideration may exhibit both fuzziness and randomness [21]. Therefore, both fuzziness and randomness need to be considered in reliability analysis of the structure.

In consideration of the above observations, the safety state of the structure, i.e., the state of satisfactory operation, is often treated as a fuzzy set, denoted by $\tilde{A}$, which is a subset of the universe of discourse of the state of the structure. This means that the event of the safe operation is considered to be a fuzzy event and we should use the probability of this fuzzy event to measure the reliability of the structure.

Using the equation for the probability of a fuzzy event [7, 30], we can define the reliability of the structure by multiplying the membership function of the state variable $Z$, $\mu_{\tilde{A}}(z)$, belonging to the fuzzy safety state $\tilde{A}$ with the pdf of $Z$ as:

$$P_r = P(\tilde{A}) = \int_{-\infty}^{+\infty} \mu_{\tilde{A}}(z) p_Z(z)\mathrm{d}z \,, \tag{6}$$

Similarly, if we use the random variables $S$ and $R$ defined earlier, we can utilize the joint pdf of $S$ and $R$ to define the reliability of the structure as follows:

$$P_r = P(\tilde{A}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mu_{\tilde{A}}(x,y) p_{S,R}(x,y)\mathrm{d}x\mathrm{d}y \,, \tag{7}$$

where $\mu_{\tilde{A}}(x,y)$ is the membership function of the fuzzy safety state $\tilde{A}$ in terms of realizations $x$ and $y$ of $S$ and $R$, respectively. Note that we have the relationship between $\mu_{\tilde{A}}(z)$ and $\mu_{\tilde{A}}(x,y)$:

$$\mu_{\tilde{A}}(z) = \mu_{\tilde{A}}(x, z - x) \,. \tag{8}$$

Apparently, if $S$ and $R$ are independent, we can apply Eq. (5) to Eq. (7) and have

$$P_r = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mu_{\tilde{A}}(x,y) p_S(x) p_R(y)\mathrm{d}x\mathrm{d}y \,. \tag{9}$$

It is easy to show that the reliability of a structure reduces to its conventional reliability, if we use the following membership function for the safety state of the structure,

$$\mu_{\tilde{A}}(x,y) = \begin{cases} 1, & x < y \\ 0, & x \ge y \end{cases} \,. \tag{10}$$

As a result, Eq. (7) reduces to Eq. (3), and Eq. (9) can be written into

$$P_r = \int_{-\infty}^{+\infty} p_R(y)\left[\int_{-\infty}^{y} p_S(x)\mathrm{d}x\right]\mathrm{d}y \,. \tag{11}$$

Evidently, Eq. (11) is the same as the traditional formula for structural reliability evaluation when the stress and strength are independent. This means that the proposed model of structural reliability analysis is consistent with the model of conventional reliability analysis, and the latter is a special case of the former. The fuzzy safety state of a structure may be defined in one of three different forms.

The fuzzy safety state is defined by the state variable $Z$:

$$\tilde{A} \overset{\Delta}{=} \{Z \tilde{>} 0\}, \tag{12}$$

where $Z \tilde{>} 0$ indicates that the state variable $Z$ is larger than 0 in a fuzzy sense. Here the membership function of $\tilde{A}$ is shown in Fig. 1(a). The transition curve from 0 to 1 may take a proportional, parabola, or in other forms.

The fuzzy safety state is defined by the fuzzy random generalized strength $\tilde{R}$:

$$\tilde{A} \overset{\Delta}{=} \{\tilde{S} < \tilde{R}\}, \quad \tilde{A} \overset{\Delta}{=} \{S < \tilde{R}\} \,. \tag{13}$$

The fuzzy random generalized strength $\tilde{R}$ is an index representing the generalized strength of the material of structure when both fuzziness and randomness are considered, and its membership function generally follows the shape illustrated in Fig. 1 (b).

The fuzzy safety state is defined by a fuzzy random allowable interval of the stress, $[\tilde{S}]$ :

$$\tilde{A} \overset{\Delta}{=} \{\tilde{S} \subset [\tilde{S}]\}, \quad \tilde{A} \overset{\Delta}{=} \{S \subset [\tilde{S}]\}, \tag{14}$$

where $\tilde{S}$ is the fuzzy random generalized stress. On the other hand, the fuzzy random allowable interval of the generalized stress reflects the fuzziness of the safety criteria used. In defining the safety criteria, one needs to consider the fuzziness of the structural responses, including the stress, deflection, deformation, frequency, etc., and the fuzziness of the allowable interval of the structural response. In other words, there is no clear boundary between what is allowed and what is not allowed. Thus, the allowable interval of the generalized stress possesses fuzziness, and its membership function generally has the shape as illustrated in Fig. 1(c). In the following sections, we discuss how to evaluate the reliability of a structure when one of the three forms of the fuzzy safety state is used.

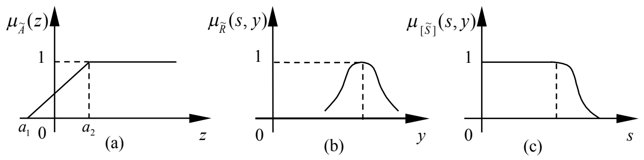


*Fig. 1. Membership function*

## 3. Form 1: The fuzzy safety state defined by the state variable

In this section, we provide a method to analyze the reliability of a structure when the fuzzy safety state is defined by the state variable. In this form, the reliability is computed simply by integrating the membership function of $\tilde{A}$ times the pdf of *Z*. Under this definition, we consider two shapes of the membership function of the fuzzy safety state, namely, the rising half-trapezoidal distribution and the rising half-ridge distribution. When the shape of the membership function is specified, we consider the commonly used pdf of exponential distribution, normal distribution, lognormal distribution, or Weibull distribution.

### 3.1. The membership function of the fuzzy safety state follows a rising half-trapezoidal distribution

In this case, the membership function of the fuzzy safety state $\tilde{A}$ is illustrated in Fig.1(a), and its mathematical form is given below, which represents a proportional type transition,

$$\mu_{\tilde{A}}(z) = \begin{cases} 0 & , \ z < a_1 \\ \dfrac{z-a_1}{a_2-a_1} & , \ a_1 \le z \le a_2 \\ 1 & , \ z > a_2 \end{cases} \tag{15}$$

#### 3.1.1. The state variable follows the exponential distribution

In this case, the pdf of *Z* is exponential function with failure rate $\lambda$

$$p_Z(z) = \lambda e^{-\lambda z}. \tag{16}$$

With Eq. (6), we have the following expression for the reliability of the structure:

$$\begin{aligned} P_r = P(\tilde{A}) &= \int_{-\infty}^{+\infty} \mu_{\tilde{A}}(z) p_Z(z) \mathrm{d}z \\ &= \int_{a_1}^{a_2} \frac{z-a_1}{a_2-a_1} \lambda e^{-\lambda z} \mathrm{d}z + \int_{a_2}^{+\infty} \lambda e^{-\lambda z} \mathrm{d}z \\ &= \frac{1}{\lambda(a_2-a_1)} \left( e^{-\lambda a_1} - e^{-\lambda a_2} \right) \end{aligned} \tag{17}$$

#### 3.1.2. The state variable follows a normal distribution

In this case, the pdf of *Z* is expressed as:

$$p_Z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}, \tag{18}$$

where $\mu$ and $\sigma^2$ are the mean and variance of $Z$ , respectively. With Eq. (6), the reliability of the structure can be expressed as:

$$\begin{aligned} P_r = P(\tilde{A}) &= \int_{-\infty}^{+\infty} \mu_{\tilde{A}}(z) p_Z(z) \mathrm{d}z \\ &= \int_{a_1}^{a_2} \frac{z-a_1}{a_2-a_1} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \mathrm{d}z + \int_{a_2}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \mathrm{d}z \\ &= 1 + \left( \frac{\mu-a_2}{a_2-a_1} \right) \Phi\left( \frac{a_2-\mu}{\sigma} \right) - \left( \frac{\mu-a_1}{a_2-a_1} \right) \Phi\left( \frac{a_1-\mu}{\sigma} \right) \\ &\quad + \frac{\sigma}{(a_2-a_1)\sqrt{2\pi}} \left( e^{-\frac{(a_1-\mu)^2}{2\sigma^2}} - e^{-\frac{(a_2-\mu)^2}{2\sigma^2}} \right) \end{aligned} \tag{19}$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal random variable.

#### 3.1.3. The state variable follows a lognormal distribution

Herein, the pdf of *Z* is written as the lognormal form

$$p_Z(z) = \frac{1}{\sqrt{2\pi} z \sigma_{\ln z}} e^{-\frac{(\ln z - \mu_{\ln z})^2}{2\sigma_{\ln z}^2}}, \tag{20}$$

where $\mu_{\ln z}$ and $\sigma_{\ln z}^2$ are the mean and variance of $\ln Z$ . Then the reliability can be obtained by

$$\begin{aligned} P_r = P(\tilde{A}) &= \int_{-\infty}^{+\infty} \mu_{\tilde{A}}(z) p_Z(z) \mathrm{d}z \\ &= \int_{a_1}^{a_2} \frac{z-a_1}{a_2-a_1} \frac{1}{\sqrt{2\pi} z \sigma_{\ln z}} e^{-\frac{(\ln z - \mu_{\ln z})^2}{2\sigma_{\ln z}^2}} \mathrm{d}z + \int_{a_2}^{+\infty} \frac{1}{\sqrt{2\pi} z \sigma_{\ln z}} e^{-\frac{(\ln z - \mu_{\ln z})^2}{2\sigma_{\ln z}^2}} \mathrm{d}z \\ &= 1 + e^{\mu_{\ln z} + \frac{1}{2}\sigma_{\ln z}^2} \left[ \Phi\left( \frac{\ln a_2 - \mu_{\ln z} - \sigma_{\ln z}^2}{\sigma_{\ln z}} \right)^2 - \Phi\left( \frac{\ln a_1 - \mu_{\ln z} - \sigma_{\ln z}^2}{\sigma_{\ln z}} \right) \right] \\ &\quad + \frac{a_1}{a_2-a_1} \Phi\left( \frac{\ln a_1 - \mu_{\ln z}}{\sigma_{\ln z}} \right) - \frac{a_2}{a_2-a_1} \Phi\left( \frac{\ln a_2 - \mu_{\ln z}}{\sigma_{\ln z}} \right) \end{aligned} \tag{21}$$

#### 3.1.4. The state variable follows the 3-parameter Weibull distribution

In this case, the pdf of *Z* is

$$p_Z(z) = \frac{\beta}{\eta} \left( \frac{z-\gamma}{\eta} \right)^{\beta-1} e^{-\left( \frac{z-\gamma}{\eta} \right)^{\beta}}. \tag{22}$$

Then the reliability of the structure can be expressed as:

$$P_r = P(\tilde{A}) = \int_{-\infty}^{+\infty} \mu_{\tilde{A}}(z) p_Z(z) \mathrm{d}z$$

$$= \int_{a_1}^{a_2} \frac{z-a_1}{a_2-a_1} \frac{\beta}{\eta}\left(\frac{z-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{z-\gamma}{\eta}\right)^{\beta}} \mathrm{d}z + \int_{a_2}^{+\infty} \frac{\beta}{\eta}\left(\frac{z-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{z-\gamma}{\eta}\right)^{\beta}} \mathrm{d}z$$

$$= \frac{1}{a_2-a_1} \int_{a_1}^{a_2} e^{-\left(\frac{z-\gamma}{\eta}\right)^{\beta}} \mathrm{d}z \,, \tag{23}$$

where the integral can be solved numerically.

### 3.2. The membership function of the fuzzy safety state follows a rising half-ridge distribution

In this case, the membership function of the fuzzy safety state $\tilde{A}$ is illustrated in Fig. 2, where the incremental process is in a sinusoidal form given by:

$$\mu_{\tilde{A}}(z) = \begin{cases} 0 & , z < a_1 \\ \frac{1}{2} + \frac{1}{2}\sin\frac{\pi}{a_2-a_1}\left(z - \frac{a_1+a_2}{2}\right) & , a_1 \le z \le a_2 \\ 1 & , z > a_2 \end{cases} \tag{24}$$

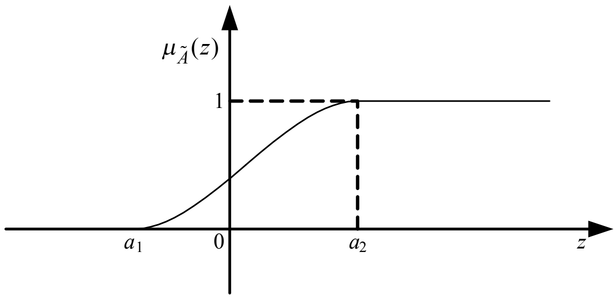


*Fig. 2. Rising half-ridge distribution*

#### 3.2.1. The state variable follows an exponential distribution

The pdf of $Z$ is

$$p_Z(z) = \lambda e^{-\lambda z}$$

Then the reliability of the structure can be expressed as:

$$P_r = P(\tilde{A}) = \int_{-\infty}^{+\infty} \mu_{\tilde{A}}(z) p_Z(z) \mathrm{d}z$$

$$= \int_{a_1}^{a_2} \left[\frac{1}{2} + \frac{1}{2}\sin\frac{\pi}{a_2-a_1}\left(z - \frac{a_1+a_2}{2}\right)\right] \lambda e^{-\lambda z} \mathrm{d}z + \int_{a_2}^{+\infty} \lambda e^{-\lambda z} \mathrm{d}z$$

$$= \frac{\pi^2}{2(\lambda^2 (a_2-a_1)^2 + \pi^2)} \left(e^{-\lambda a_1} + e^{-\lambda a_2}\right) \tag{25}$$

#### 3.2.2. The state variable follows a normal distribution

In this case, the pdf of $Z$ is given by

$$p_Z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$$

Then the reliability of the structure is

$$P_r = P(\tilde{A}) = \int_{-\infty}^{+\infty} \mu_{\tilde{A}}(z) p_Z(z) \mathrm{d}z$$

$$= \int_{a_1}^{a_2} \left[\frac{1}{2} + \frac{1}{2}\sin\frac{\pi}{a_2-a_1}\left(z - \frac{a_1+a_2}{2}\right)\right] \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \mathrm{d}z + \int_{a_2}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \mathrm{d}z$$

$$= 1 - \frac{1}{2}\Phi\left(\frac{a_1-\mu}{\sigma}\right) - \frac{1}{2}\Phi\left(\frac{a_2-\mu}{\sigma}\right)$$

$$+ \frac{1}{2\sqrt{2\pi}\sigma} \int_{a_1}^{a_2} \sin\frac{\pi}{a_2-a_1}\left(z - \frac{a_1+a_2}{2}\right) e^{-\frac{(z-\mu)^2}{2\sigma^2}} \mathrm{d}z \,, \tag{26}$$

where the integral can be solved by a numerical method.

#### 3.2.3. The state variable follows a lognormal distribution

In this case, the pdf of $Z$ is expressed as:

$$p_Z(z) = \frac{1}{\sqrt{2\pi} z \sigma_{\ln z}} e^{-\frac{(\ln z - \mu_{\ln z})^2}{2\sigma_{\ln z}^2}}$$

Then the reliability of the structure is written as:

$$P_r = P(\tilde{A}) = \int_{-\infty}^{+\infty} \mu_{\tilde{A}}(z) p_Z(z) \mathrm{d}z$$

$$= \int_{a_1}^{a_2} \left[\frac{1}{2} + \frac{1}{2}\sin\frac{\pi}{a_2-a_1}\left(z - \frac{a_1+a_2}{2}\right)\right] \frac{1}{\sqrt{2\pi} z \sigma_{\ln z}} e^{-\frac{(\ln z - \mu_{\ln z})^2}{2\sigma_{\ln z}^2}} \mathrm{d}z$$

$$+ \int_{a_2}^{+\infty} \frac{1}{\sqrt{2\pi} z \sigma_{\ln z}} e^{-\frac{(\ln z - \mu_{\ln z})^2}{2\sigma_{\ln z}^2}} \mathrm{d}z$$

$$= 1 - \frac{1}{2}\Phi\left(\frac{\ln a_1 - \mu_{\ln z}}{\sigma_{\ln z}}\right) - \frac{1}{2}\Phi\left(\frac{\ln a_2 - \mu_{\ln z}}{\sigma_{\ln z}}\right)$$

$$+ \frac{1}{2\sqrt{2\pi}\sigma_{\ln z}} \int_{a_1}^{a_2} \frac{1}{z} e^{-\frac{(\ln z - \mu_{\ln z})^2}{2\sigma_{\ln z}^2}} \sin\frac{\pi}{a_2-a_1}\left(z - \frac{a_1+a_2}{2}\right) \mathrm{d}z \,, \tag{27}$$

where the integral can be solved by a numerical method.

#### 3.2.4. The state variable follows a Weibull distribution

The pdf of $Z$ is written as

$$p_Z(z) = \frac{\beta}{\eta}\left(\frac{z-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{z-\gamma}{\eta}\right)^{\beta}} .$$

Then the reliability of the structure is given by

$$P_r = P(\tilde{A}) = \int_{-\infty}^{+\infty} \mu_{\tilde{A}}(z) p_Z(z) \mathrm{d}z \tag{28}$$

$$= \int_{a_1}^{a_2} \left[\frac{1}{2} + \frac{1}{2}\sin\frac{\pi}{a_2-a_1}\left(z - \frac{a_1+a_2}{2}\right)\right] \frac{\beta}{\eta}\left(\frac{z-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{z-\gamma}{\eta}\right)^{\beta}} \mathrm{d}z + \int_{a_2}^{+\infty} \frac{\beta}{\eta}\left(\frac{z-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{z-\gamma}{\eta}\right)^{\beta}} \mathrm{d}z$$

$$= \frac{1}{2}\left(e^{-\left(\frac{a_1-\gamma}{\eta}\right)^{\beta}} + e^{-\left(\frac{a_2-\gamma}{\eta}\right)^{\beta}}\right) + \frac{\beta}{2\eta} \int_{a_1}^{a_2} \sin\left[\frac{\pi}{a_2-a_1}\left(z - \frac{a_1+a_2}{2}\right)\right]\left(\frac{z-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{z-\gamma}{\eta}\right)^{\beta}} \mathrm{d}z \,,$$

where the integral can be solved via a numerical method.

### 3.3. Simply Supported Beam under Stress

Consider a beam which is supported at both ends carrying uniformly distributed load as shown in Fig. 3. All concerned variables, including the dimensions of the beam, the load distribution, and the strength of the material, are assumed to follow the normal distributions,

$$q \sim \mathrm{N}(210, 7^2) \text{ N/mm},$$

$$b \sim \mathrm{N}(120, 10^2) \text{ mm},$$

$$l \sim N(4000, 150^2) \text{ mm},$$

$$h \sim N(240, 10^2) \text{ mm},$$

$$R \sim N(623, 23^2) \text{ MPa}.$$

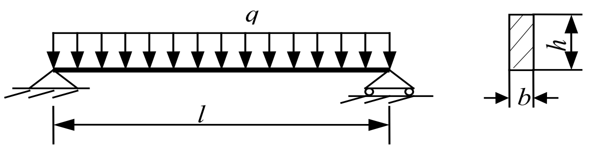


*Fig. 3. The simply supported beam*

We first apply the form 1 model where the fuzzy safety state is defined by the state variable $Z$. The membership function of the fuzzy safety state $\tilde{A}$ is given by the following half-trapezoidal distribution:

$$\mu_{\tilde{A}}(z) = \begin{cases} 0 & , z < -40 \\ \dfrac{z+40}{80} & , -40 \le z \le 40 \\ 1 & , z > 40 \end{cases}$$

Then the reliability of this beam can be calculated as follows.

The maximum stress in the simple supported beam is given by

$$S = \frac{0.75 q l^2}{b h^2}.$$

Since $q, l, b, h$ are all normal random variables, we can obtain the mean and the approximate standard deviation using the Taylor expansion of the maximum stress $S$ with the following equations:

$$\mu_S = \frac{0.75 \mu_q \mu_l^2}{\mu_b \mu_h^2} = \frac{0.75 \times 210 \times 4000^2}{120 \times 240^2} = 364.6\text{MPa}$$

$$\sigma_S \overset{\Delta}{=} \left\{ \left[ \frac{\partial S}{\partial q}\Big|_{X=\mu_X} \right]^2 \sigma_q^2 + \left[ \frac{\partial S}{\partial l}\Big|_{X=\mu_X} \right]^2 \sigma_l^2 + \left[ \frac{\partial S}{\partial b}\Big|_{X=\mu_X} \right]^2 \sigma_b^2 + \left[ \frac{\partial S}{\partial h}\Big|_{X=\mu_X} \right]^2 \sigma_h^2 \right\}^{\frac{1}{2}}$$

$= 52.4$ MPa,

where $X = (X_1, X_2, X_3, X_4) = (q, l, b, h)$ and $\mu_X = (\mu_q, \mu_l, \mu_b, \mu_h)$.

The mean and the standard deviation of the state variable are

$$\mu_Z = \mu_R - \mu_S = 623 - 364.6 = 258.4\text{MPa}$$

$$\sigma_Z = \left(\sigma_R^2 + \sigma_S^2\right)^{\frac{1}{2}} = (23^2 + 52.4^2)^{\frac{1}{2}} = 57.2\text{MPa}$$

The pdf of $Z$ can be considered to follow the normal distribution

$$p_Z(z) = \frac{1}{\sqrt{2\pi}\sigma_Z} e^{-\frac{(z-\mu_Z)^2}{2\sigma_Z^2}} = \frac{1}{\sqrt{2\pi} \times 57.2} e^{-\frac{(z-258.4)^2}{2 \times 57.2^2}}$$

Using the obtained $\mu_{\tilde{A}}(z)$, $p_Z(z)$, and Eq. (6), we have $P_r \approx 0.99928$. If we use the conventional reliability analysis method, the reliability of the beam is

$$P_r = \Phi(\frac{\mu_Z}{\sigma_Z}) = \Phi(4.517) = 0.99999.$$

From this example, we see that the reliability of the beam obtained with the conventional reliability analysis method is higher than that with the proposed method. The conventional reliability method over-estimates the reliability of the beam, and thus the obtained design is risker than that obtained through the proposed method. This is because the conventional reliability method does not consider the fuzziness of the safety criterion.

## 4. Form 2: The fuzzy safety state defined by the fuzzy random generalized strength

In this form, the reliability is computed by integrating the membership functions of $\tilde{A}$ which is the weighted sum of $\mu_{\tilde{A}_1}(x,y)$ and $\mu_{\tilde{A}_2}(x,y)$. $\mu_{\tilde{A}_1}(x,y)$ is the area ratio of $\mu_{\tilde{R}}(s,y)$, while $\mu_{\tilde{A}_2}(x,y)$ is the area ratio of $\mu_{\tilde{R}_{\text{peak}}}(s,y)$ in the case of random generalized stress and fuzzy random generalized strength. The reliability is computed by integrating the membership functions of $\tilde{A}$, which equals the weighted sum of $\mu_{\tilde{A}_1'}(x,y)$ and $\mu_{\tilde{A}_2'}(x,y)$. $\mu_{\tilde{A}_1'}(x,y)$ is the area ratio between the difference of $\mu_{\tilde{S}}(s,x)$, $\mu_{\tilde{R}}(s,y)$ and $\mu_{\tilde{S}}(s,x)$, while $\mu_{\tilde{A}_2'}(x,y)$ is the membership function of $\mu_{\tilde{R}}(S_{\text{peak}}(x),y)$ in the case of fuzzy random generalized stress and fuzzy random generalized strength.

### 4.1. Two characteristic values of a fuzzy number

A real fuzzy number $\tilde{N}$ is defined as a fuzzy set in the domain of real numbers $\mathbb{R}$, and its membership function (shown in Fig. 4) has the following characteristics:

(1) It is a continuous mapping from $\mathbb{R}$ to the closed interval $[0, w]$, $0 < w \le 1$;

(2) It is equal to 0 in $(-\infty, c]$, i.e., $\mu_{\tilde{N}}(s) = 0$, when $-\infty < s \le c$;

(3) It is strictly increasing in $[c, a]$;

(4) It is equal to 1 in $[a, b]$, i.e., $\mu_{\tilde{N}}(s) = 1$, when $a \le s \le b$;

(5) It is strictly decreasing in $[b, d]$;

(6) It is equal to 0 in $[d, \infty)$, i.e. $\mu_{\tilde{N}}(s) = 0$, when $d \le s < \infty$;

where $a$, $b$, $c$ and $d$ are real numbers and $c \le a \le b \le d$. Among other choices, we may set $c = -\infty$, $a = b$, $c = a$, $b = d$, and $d = +\infty$ individually or in various combinations. When the generalized strength and the generalized stress in structural safety analysis are treated as fuzzy variables, their membership functions often exhibit the shape depicted in Fig. 5. From detailed analysis of the membership function of a fuzzy variable, one can see that it can be well represented by two characteristic values, one is the area distribution of the membership function and the other is the position of the peak value. The area distribution of a fuzzy variable, i.e., to a certain point $s$, the area on its left and right sides under the membership function curve, is analogous to the pdf of a random variable. The peak value of a fuzzy number shown in Fig. 4 is

$$s_{\text{peak}} = (a+b)/2 \tag{29}$$

### 4.2. Determination of the membership function of fuzzy safety state in the case of random generalized stress and fuzzy random generalized strength

The fuzzy safety event in the case of random generalized stress and fuzzy random generalized strength is defined in Eq. (13), with fuzzy safety state in this case being denoted by $\tilde{A}$. According to the area distribution of the membership function $\mu_{\tilde{R}}(s,y)$ of fuzzy random generalized strength $\tilde{R}$ and the position of the peak value $R_{\text{peak}}(y)$ of $\mu_{\tilde{R}}(s,y)$, two partial expressions of $\mu_{\tilde{A}}(x,y)$, $\mu_{\tilde{A}_1}(x,y)$ and $\mu_{\tilde{A}_2}(x,y)$, can be obtained. Multiplying these two factors by weights $w_1$ and $w_2$ respectively ($w_1 + w_2 = 1$), $\mu_{\tilde{A}}(x,y)$ becomes the weighted sum of these two partial expressions:

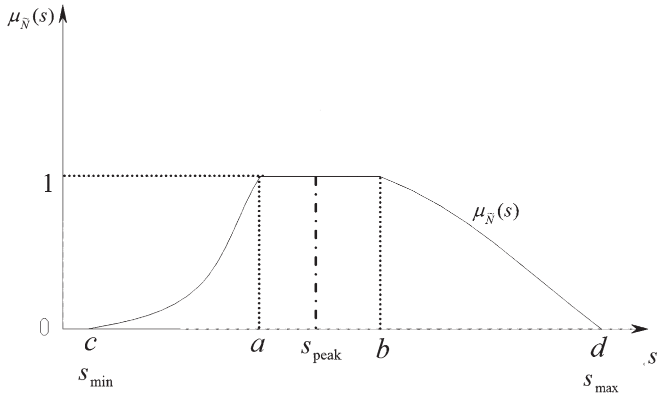


*Fig. 4. The membership function of a fuzzy number*

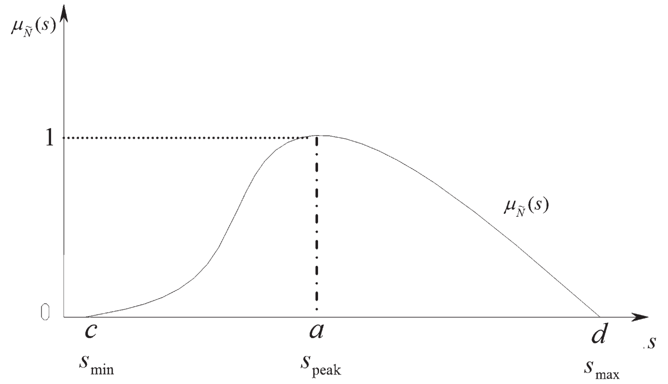


*Fig. 5. The membership function of generalized strength and generalized stress*

$$\mu_{\tilde{A}}(x,y) = w_1\mu_{\tilde{A}_1}(x,y) + w_2\mu_{\tilde{A}_2}(x,y) \tag{30}$$

#### 4.2.1. Determination of $\mu_{\tilde{A}_1}(x,y)$

As shown in Fig. 6, according to the area distribution of the membership function $\mu_{\tilde{R}}(s,y)$ of fuzzy random generalized strength $\tilde{R}$, the partial expression $\mu_{\tilde{A}_1}(x,y)$ of $\mu_{\tilde{A}}(x,y)$ has the expression of integral value of $\mu_{\tilde{R}}(s,y)$ in the domain $\left[s(x), R_{\max}(y)\right]$ divided by the total integral value of $\mu_{\tilde{R}}(s,y)$ in the full domain $\left[R_{\min}(y), R_{\max}(y)\right]$

$$\mu_{\tilde{A}_1}(x,y) = \frac{\int_{s(x)}^{R_{\max}(y)} \mu_{\tilde{R}}(s,y)\mathrm{d}s}{\int_{R_{\min}(y)}^{R_{\max}(y)} \mu_{\tilde{R}}(s,y)\mathrm{d}s} \tag{31}$$

#### 4.2.2. Determination of $\mu_{\tilde{A}_2}(x,y)$

According to the relative position between generalized stress $s(x)$ and $R_{\text{peak}}(y)$, the partial expression $\mu_{\tilde{A}_2}(x,y)$ of $\mu_{\tilde{A}}(x,y)$ can be determined. If $R_{\text{peak}}(y)$ is considered to be a deterministic value, $\mu_{\tilde{A}_2}(x,y)$ varies from 1 to 0 at $R_{\text{peak}}(y)$ when $s(x)$ passes through $R_{\text{peak}}(y)$ from left to right, which leads to the discontinuity of the membership function $\mu_{\tilde{A}}(x,y)$ at $R_{\text{peak}}(y)$. To overcome this problem, $R_{\text{peak}}(y)$ is converted to a fuzzy set $\tilde{R}_{\text{peak}}$, whose membership function $\mu_{\tilde{R}_{\text{peak}}}(s,y)$ is a normal or symmetric triangular membership function. The range $\delta_2$ of $\tilde{R}_{\text{peak}}$ is determined according to how steep the change of $\mu_{\tilde{A}_2}(x,y)$ should be near $R_{\text{peak}}(y)$: the steeper the change of $\mu_{\tilde{A}_2}(x,y)$ is, the smaller $\delta_2$ is. As shown in Fig. 7, $\mu_{\tilde{A}_2}(x,y)$ can be determined as the ratio between the integral of $\mu_{\tilde{R}_{\text{peak}}}(s,y)$ in the domain $\left[s(x), R_{\text{peak}}(y)+\delta_2\right]$ and the integral of $\mu_{\tilde{R}_{\text{peak}}}(s,y)$ in the domain $\left[R_{\text{peak}}(y)-\delta_2, R_{\text{peak}}(y)+\delta_2\right]$ as

$$\mu_{\tilde{A}_2}(x,y) = \frac{\int_{s(x)}^{R_{\min}(y)+\delta_1+\delta_2} \mu_{\tilde{R}_{\text{peak}}}(s,y)\mathrm{d}s}{\int_{R_{\min}(y)+\delta_1-\delta_2}^{R_{\min}(y)+\delta_1+\delta_2} \mu_{\tilde{R}_{\text{peak}}}(s,y)\mathrm{d}s}, \tag{32}$$

where $R_{\text{peak}}(y) = R_{\min}(y) + \delta_1$.

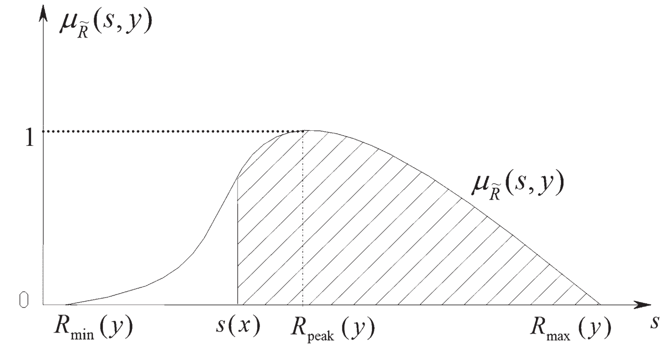


*Fig. 6. Determine $\mu_{\tilde{A}_1}(x,y)$ according to the area distribution of $\mu_{\tilde{R}}(s,y)$*

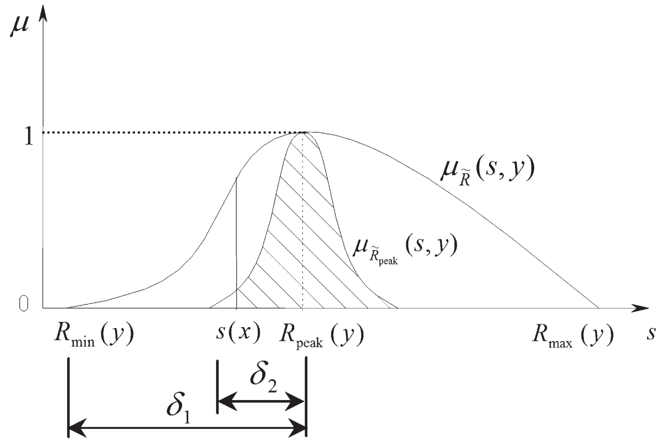


*Fig. 7. Determine $\mu_{\tilde{A}_2}(x,y)$ according to the position of $R_{\text{peak}}(y)$*

#### 4.2.3. Determination of weights $w_1$ and $w_2$

The weights $w_1$ and $w_2$ denote the relative influence of the corresponding characteristic factors on $\mu_{\tilde{A}}(x,y)$. $w_1$ and $w_2$ can be determined based on experience and through other methods that may be problem-specific. Generally, the larger is the dissymmetry of the membership function $\mu_{\tilde{R}}(s,y)$, the larger is $w_2$. Based on Eqs. (30-32), the membership function of fuzzy safety state in the case of random generalized stress and fuzzy random generalized strength, $\mu_{\tilde{A}}(x,y)$, takes the form

$$\mu_{\tilde{A}}(x,y) = w_1 \frac{\int_{s(x)}^{R_{\max}(y)} \mu_{\tilde{R}}(s,y)\mathrm{d}s}{\int_{R_{\min}(y)}^{R_{\max}(y)} \mu_{\tilde{R}}(s,y)\mathrm{d}s} + w_2 \frac{\int_{s(x)}^{R_{\min}(y)+\delta_1+\delta_2} \mu_{\tilde{R}_{\mathrm{peak}}}(s,y)\mathrm{d}s}{\int_{R_{\min}(y)+\delta_1-\delta_2}^{R_{\min}(y)+\delta_1+\delta_2} \mu_{\tilde{R}_{\mathrm{peak}}}(s,y)\mathrm{d}s}. \quad (33)$$

Therefore the reliability can be computed using this membership function as

$$P_r = P(\tilde{A}) = \int_{R_{\min}}^{R_{\max}} \int_{S_{\min}}^{S_{\max}} \mu_{\tilde{A}}(x,y) p_S(x) p_R(y) \mathrm{d}x\mathrm{d}y, \quad (34)$$

where $p_S(x)$ is the pdf of the random generalized stress and $p_R(y)$ is the pdf of the fuzzy random generalized strength.

### 4.3. Determination of the membership function of fuzzy safety state in the case of fuzzy random generalized stress and fuzzy random generalized strength

Fuzzy safety state in the case of fuzzy random generalized stress and fuzzy random generalized strength is denoted by $\tilde{A}'$, and the corresponding membership function is $\mu_{\tilde{A}'}(x,y)$. Based on the membership function of fuzzy safety state $\tilde{A}'$, $\mu_{\tilde{A}}(x,y)$, the area distribution of the membership function of fuzzy random generalized stress $\tilde{S}$, $\mu_{\tilde{S}}(s,x)$, and position of the peak value of $\mu_{\tilde{S}}(s,x)$, $S_{\mathrm{peak}}(x)$, two partial expressions of $\mu_{\tilde{A}'}(x,y)$, $\mu_{\tilde{A}_1'}(x,y)$ and $\mu_{\tilde{A}_2'}(x,y)$ can be obtained. Multiplying these two factors by weights $w_1'$ and $w_2'$ respectively ($w_1' + w_2'$=1), and $\mu_{\tilde{A}'}(x,y)$ equals to the weighted sum of the two partial expressions:

$$\mu_{\tilde{A}'}(x,y) = w_1' \mu_{\tilde{A}_1'}(x,y) + w_2' \mu_{\tilde{A}_2'}(x,y) \quad (35)$$

Agin, the weights $w_1'$ and $w_2'$ denote the relative influence of the corresponding characteristic factors on $\mu_{\tilde{A}'}(x,y)$. $w_1'$ and $w_2'$ can be determined based on experience and through other problem-specific methods. Generally, the larger is the asymmetry of the membership function $\mu_{\tilde{S}}(s,x)$, the larger is $w_2'$.

#### 4.3.1. Determination of $\mu_{\tilde{A}_1'}(x,y)$

As shown in Fig. 8, according to the area distribution of the membership function of fuzzy random generalized stress $\tilde{S}$, $\mu_{\tilde{S}}(s,x)$, the partial expression of $\mu_{\tilde{A}'}(x,y)$, $\mu_{\tilde{A}_1'}(x,y)$ is the ratio of between the integral of $\mu_{\tilde{S}}(s,x)\mu_{\tilde{R}}(s,y)$ and the integral of $\mu_{\tilde{S}}(s,x)$ in the domain of $\left[s, S_{\max}(y)\right]$

$$\mu_{\tilde{A}_1'}(x,y) = \frac{\int_{s}^{S_{\max}} \mu_{\tilde{S}}(s,x)\mu_{\tilde{R}}(s,y)\mathrm{d}s}{\int_{S_{\min}}^{S_{\max}} \mu_{\tilde{S}}(s,x)\mathrm{d}s}. \quad (36)$$

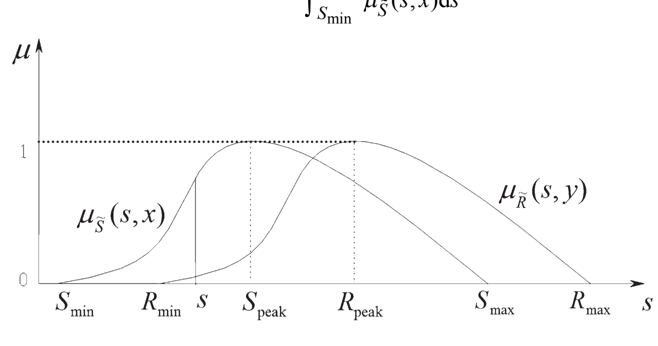


*Fig. 8. Membership function of fuzzy random generalized stress and fuzzy random generalized strength*

For a certain element $s$ of fuzzy set $\tilde{S}$, the membership function value of $s$ in fuzzy safety state $\tilde{A}'$ is $\mu_{\tilde{A}'}(s,y)$, and the membership function value of $s$ in $\tilde{S}$ is $\mu_{\tilde{S}}(s,x)$, which can be considered to be the weight. Thus, the method to determine the expression $\mu_{\tilde{A}_1'}(x,y)$ is essentially a weighted-average method, as shown in Eq. (36).

#### 4.3.2. Determination of $\mu_{\tilde{A}_2'}(x,y)$

According to the relative position between $S_{\mathrm{peak}}(x)$ and $\tilde{R}$, $\mu_{\tilde{A}_2'}(x,y)$ takes the expression

$$\mu_{\tilde{A}_2'}(x,y) = \mu_{\tilde{R}}(S_{\mathrm{peak}}(x), y) \quad (37)$$

On the basis of Eqs. (35–37), the membership function of fuzzy safety state in the case of fuzzy random generalized stress and fuzzy random generalized strength, $\mu_{\tilde{A}'}(x,y)$, takes the form

$$\mu_{\tilde{A}'}(x,y) = w_1' \frac{\int_{S_{\min}}^{S_{\max}} \mu_{\tilde{S}}(s,x)\mu_{\tilde{R}}(s,y)\mathrm{d}s}{\int_{S_{\min}}^{S_{\max}} \mu_{\tilde{S}}(s,x)\mathrm{d}s} + w_2' \mu_{\tilde{R}}(S_{\mathrm{peak}}(x), y) \quad (38)$$

### 4.4. Reliability of output axis of gearbox

In engineering, the overall structure of a gearbox is complex, which makes it more difficult to analyze the stress-strength relationship. For a gearbox, Form 1 is too simple and does not comply with this structure. In many applications, gearboxes are usually damaged by catastrophic loads such as impactions which are directly related to the strength. Therefore, damage may not be related to stress only, and form 3 may not be applicable in this situation. For simplicity, we consider a fuzzy reliability computation problem involving fuzzy random stress and fuzzy strength using the form 2 where the fuzzy safety state is defined by the fuzzy random generalized strength. It is known that the strength $\tilde{R}$ of the output axis of some gearbox is near 240MPa. As shown in Fig. 9, the membership function of $\tilde{R}$ is

$$\mu_{\tilde{R}}(s) = \begin{cases} 0, & s \le 220 \\ (s-220)/20, & 220 < s \le 240 \\ (280-s)/40, & 240 < s \le 280 \\ 0, & s > 280 \end{cases} \quad (39)$$

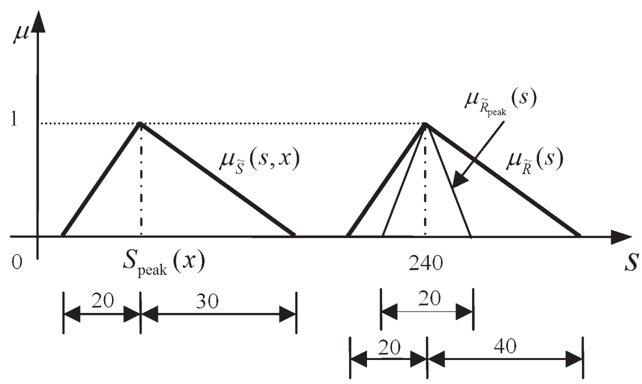


*Fig. 9. The membership function of fuzzy random stress and fuzzy strength*

The peak value of fuzzy random stress $\tilde{S}$, $S_{\mathrm{peak}}(x)$, follows the normal distribution $\sigma \sim \mathrm{N}(140, 20^2)$ MPa. The membership function of $\tilde{S}$ is

$$\mu_{\tilde{S}}(s,S_{\text{peak}})=\begin{cases}0, & s\le(S_{\text{peak}}-20)\\ (s-S_{\text{peak}}+20)/20, & (S_{\text{peak}}-20)<s\le S_{\text{peak}}\\ (S_{\text{peak}}+30-s)/30, & S_{\text{peak}}<s\le(S_{\text{peak}}+30)\\ 0, & s>(S_{\text{peak}}+30)\end{cases}\quad(40)$$

In the domain $[S_{\min}, S_{\max}]$, where $S_{\min}=S_{\text{peak}}(x)-20$ and $S_{\max}=S_{\text{peak}}(x)+30$, we now calculate the fuzzy reliability of the output axis. According to the asymmetry conditions of the membership functions of the fuzzy random stress and the fuzzy strength, the weight values are shown in Table 1. The membership functions of fuzzy random stress and fuzzy strength are both sectional functions, and it is difficult to obtain the analytical expressions for the membership function of fuzzy safety state and the fuzzy reliability. To overcome this technical difficulty, numerical methods are adopted, and MATLAB is used in implementing the computation of the membership function of fuzzy safety state and the fuzzy reliability.

*Table 1. The weight values of partial expressions`*

| $w_1$ | $w_2$ | $w_1'$ | $w_2'$ |
|---|---|---|---|
| 0.7 | 0.3 | 0.8 | 0.2 |

With different peak values of fuzzy random stress $\tilde{S}$, $S_{\text{peak}}$, the corresponding membership function values of fuzzy safety state are shown in Table 2. When $\tilde{S}$ is completely on the left-hand side of $\tilde{R}$, $\mu_{\tilde{A}'}(S_{\text{peak}})$ equals 1. When $\tilde{S}$ is completely on the right-hand side of $\tilde{R}$, $\mu_{\tilde{A}'}(S_{\text{peak}})$ equals 0. When $\tilde{S}$ moves through $\tilde{R}$ from the left to the right, $\mu_{\tilde{A}'}(S_{\text{peak}})$ decreases from 1 to 0 continuously and monotonically. Such a computational result is reasonable. Based on the known conditions, the pdf of the peak value $S_{\text{peak}}$ of the fuzzy random stress is

$$p(S_{\text{peak}})=\frac{1}{\sqrt{2\pi}\times 20}\exp\left(-\frac{(S_{\text{peak}}-140)^2}{2\times 20^2}\right)\quad(41)$$

From Eq. (6), the fuzzy reliability of the output axis of the reducing gearbox is

$$P_r=P\left(\tilde{A}'\right)=\int_{S_{\text{peak}}(x)-20}^{S_{\text{peak}}(x)+30}\mu_{\tilde{A}'}(S_{\text{peak}})\,p(S_{\text{peak}})\,\mathrm{d}S_{\text{peak}}=0.99998869\quad(42)$$

## 5. Form 3: The fuzzy safety state defined by fuzzy random allowable interval

When the generalized stress is a random variable and the allowable interval of generalized stress is a fuzzy random variable, fuzzy event $\tilde{A}$ is a special fuzzy event. When both the generalized stress and its allowable interval are fuzzy random variables, $\tilde{A}$ is a general fuzzy event.

### 5.1. Membership function of the special fuzzy event

If any realization $x$ of the random generalized stress is within the interval of $\mu_{[\tilde{S}]}(s,y)=1$ where the domain of $[\tilde{S}]$ is $\left[[\tilde{S}]_{\min},[\tilde{S}]_{\max}\right]$ as shown in Fig.1 (c), the structure is absolutely safe, thus $\mu_{\tilde{A}}(x,y)=1$. When $x$ is within the interval of transition, that is, $0<\mu_{[\tilde{S}]}(s,y)<1$, the structure is safe to some extent (depending on the value of $\mu_{[\tilde{S}]}(s,y)$). When $x$ is completely out of $\mu_{[\tilde{S}]}(s,y)$, that is, $\mu_{[\tilde{S}]}(s,y)=0$, the structure will absolutely fail. Therefore the membership function of the special fuzzy event $\tilde{A}$ can be defined as

$$\mu_{\tilde{A}}(x,y)\overset{\Delta}{=}\mu_{[\tilde{S}]}(s,y)\big|_{s=x}=\mu_{[\tilde{S}]}(x,y)\quad(43)$$

When only the randomness of the generalized stress and fuzziness of the allowable interval are taken into account, $\mu_{[\tilde{S}]}(s,y)$ degenerates to be $\mu_{[\tilde{S}]}(s)$ and

$$\mu_{\tilde{A}}(x)\overset{\Delta}{=}\mu_{[\tilde{S}]}(s)\big|_{s=x}=\mu_{[\tilde{S}]}(x)$$

### 5.2. Membership function of the general fuzzy event

As for the general fuzzy event, the degree of structural safety also depends on the relative position of membership functions $\mu_{\tilde{S}}(s,x)$ and $\mu_{[\tilde{S}]}(s,y)$. By imitating Eq. (43), the membership function of the general fuzzy event is defined as

$$\mu_{\tilde{A}}(x,y)=\mu_{[\tilde{S}]}(s,y)\big|_{s=\tilde{S}}=\mu_{[\tilde{S}]}(\tilde{S},y)\,.\quad(44)$$

It is known that if $\mu_{\tilde{S}}(s,x)$ is completely covered by $\mu_{[\tilde{S}]}(s,y)$ the structure is thought to be absolutely safe, i.e., $\mu_{\tilde{A}}(x,y)=1$. In other words, if any generalized stress $s\in\tilde{S}$ satisfies

$$\mu_{[\tilde{S}]}(s,y)\ge\mu_{\tilde{S}}(s,x)$$

then $\mu_{\tilde{A}}(x,y)=1$. As a result, $\mu_{\tilde{A}}(x,y)$ can be defined as

$$\mu_{\tilde{A}}(x,y)=\frac{\int_{s}^{S_{\max}}\left[\mu_{[\tilde{S}]}(s,y)\wedge\mu_{\tilde{S}}(s,x)\right]\mathrm{d}s}{\int_{S_{\min}}^{S_{\max}}\mu_{\tilde{S}}(s,x)\mathrm{d}s}\quad(45)$$

where "∧" means "minimal". Thus, the determination of the membership function of the fuzzy event $\tilde{A}$ becomes the calculation of the membership functions of the fuzzy random generalized stress $\tilde{S}$ and fuzzy random generalized strength $\tilde{R}$ as in the form 2 model. Generally speaking, by using a system analysis method and applying the Extension Principle [30], the membership function $\mu_{\tilde{S}}(s,x)$ of $\tilde{S}$ can be calculated from the membership functions of the fuzzy load and the fuzzy geometric size of the structure. The membership function

*Table 2. The membership function values of fuzzy safety state*

| $S_{\text{peak}}$ | 140 | 160 | 180 | 200 | 210 | 220 | 230 | 240 |
|---|---|---|---|---|---|---|---|---|
| $\mu_{\tilde{A}'}(S_{\text{peak}})$ | 1 | 1 | 1 | 0.9995 | 0.9904 | 0.9401 | 0.8027 | 0.5397 |
| $S_{\text{peak}}$ | 250 | 260 | 270 | 280 | 290 | 300 | 310 | 320 |
| $\mu_{\tilde{A}'}(S_{\text{peak}})$ | 0.2699 | 0.1145 | 0.0367 | 0.0062 | 0.0004 | 0 | 0 | 0 |

$\mu_{\tilde{R}}(s,y)$ of $\tilde{R}$ could be in the form of semi-trapezoid distribution, or semi-normal distribution. By using the probability formula of a fuzzy event, the general expression for the reliability of a structure is

$$P_r = P(\tilde{A}) = \int_{R_{\min}}^{R_{\max}} \int_{S_{\min}}^{S_{\max}} \mu_{\tilde{A}}(x,y)p(x,y)\mathrm{d}x\mathrm{d}y\,, \quad (46)$$

where $p(x,y)$ is the joint pdf. When only the randomness of generalized stress and fuzziness of allowable interval is taken into account, then

$$P_r = P(\tilde{A}) = \int_{S_{\min}}^{S_{\max}} \mu_{\tilde{A}}(x)p(x)\mathrm{d}x \quad (47)$$

### 5.3. Simply supported beam under stress

Consider again the simply supported beam shown in Fig.3. Assume all random variables are normally distributed, that is

$q$~N(110,$7^2$) N/mm,
$l$~N(3600,$150^2$) mm,
$b$~N(120,$10^2$) mm,
$h$~N(240,$10^2$) mm.

Suppose the beam is made of #45 steel (a steel in China) and the membership function of its allowable bending stress is

$$\mu_{[\tilde{S}]}(s) = \begin{cases} 1 & , s \le 160 \\ (200-s)/(200-160), & 160 \le s \le 200 \\ 0 & , s > 200 \end{cases}$$

Now we determine the reliability of this simply supported beam. It is known from the strength of materials that the maximum stress of this simply supported beam is $S = 0.75ql^2/bh^2$. Because $q$, $l$, $b$ and $h$ are all normal random variables, the mean and standard deviation of the maximum stress are $\mu_S = 155\text{MPa}$, $\sigma_S = 24\text{MPa}$, respectively. It is known that the stress is a random variable and follows the normal distribution $S$~N(155,$24^2$) MPa. For the strength, only its fuzziness is taken into account. Therefore,

$$\mu_{\tilde{A}}(x) = \mu_{[\tilde{S}]}(s)|_{s=x} = \begin{cases} 1 & , x < 160 \\ \dfrac{200-x}{200-160} & , 60 \le x \le 200 \\ 0 & , x > 200 \end{cases}$$

where its domain is $\left[\left[\tilde{S}\right]_{\min}, \left[\tilde{S}\right]_{\max}\right] = [160, 200]$. The corresponding reliability is

$$P_r = P(\tilde{A}) = \int_{-\infty}^{+\infty} \mu_{\tilde{A}}(x)p(x)\mathrm{d}x$$

$$= \int_{S_{\min}}^{160} \frac{1}{\sqrt{2\pi}\times 24} e^{-\frac{(x-155)^2}{2\times 24^2}} \mathrm{d}x + \int_{160}^{200} \frac{200-x}{40} \frac{1}{\sqrt{2\pi}\times 24} e^{-\frac{(x-155)^2}{2\times 24^2}} \mathrm{d}x$$

$$= 0.825$$

## 6. Quantitative analysis of the influence of fuzzy factor on reliability

From the above mentioned analysis, we know that the reliability of the structure is conducted on the fuzzy safety state (fuzzy safety criteria). Moreover, the reason for introduction of the fuzzy safety state is that the structure state (functioning and failure) is ambiguous when the structure is on the limit state boundary ($z = 0$).

When the membership function of the fuzzy safety state follows the rising half-trapezoidal distribution, according to Eq. (15) with $z = 0$, we have

$$\mu_{\tilde{A}}(z)|_{z=0} = -\frac{a_1}{a_2 - a_1}\,. \quad (48)$$

When the membership function of fuzzy safety state follows the rising half-ridge distribution, according to Eq. (24) with $Z = 0$, we have

$$\mu_{\tilde{A}}(z)|_{z=0} = \frac{1}{2} + \frac{1}{2}\sin\frac{\pi}{a_2 - a_1}\left(-\frac{a_1 + a_2}{2}\right). \quad (48)$$

Let $\mu_{\tilde{A}}(z) = \alpha$ and $\alpha = [0,1]$, and then $\alpha$ is called the degree of confidence or confidence level of the structure safety when $z = 0$. The larger $\alpha$ becomes, the higher confidence level of structure safety at $z = 0$ will be. It is significant in practical applications to view this parameter as the criterion of confirming the fuzzy region, since the safety criterion is generally constructed on experiments or statistics. If statistics is quite comprehensive and experiment is highly reliable, then the safety criterion constructed on them has a higher confidence level. This shows that the fuzzy safety criterion can mostly consider uncertainty factors of human cognition. However, the conventional safety criterion is incapable of possessing this strong point. For a half-trapezoidal distribution, let $-\frac{a_1}{a_2 - a_1} = \alpha$ then

$$a_2 = \left(1 - \frac{1}{\alpha}\right)a_1 \quad (49)$$

For a half-ridge distribution, let $\frac{1}{2} + \frac{1}{2}\sin\frac{\pi}{a_2 - a_1}\left(-\frac{a_1 + a_2}{2}\right) = \alpha$

then

$$a_2 = \frac{2\sin^{-1}(2\alpha - 1) - \pi}{2\sin^{-1}(2\alpha - 1) + \pi} a_1 \quad (50)$$

That is to say, given a confidence level $\alpha$, there exists the quantitative relation to a certain degree between the upper tolerance and lower tolerance, e.g., Eq. (49) and Eq. (50). Therefore, in case $a_1$ is given, $a_2$ can be confirmed accordingly. After parameter $\alpha$ is introduced, the variational relation of the fuzzy reliability of structure with the tolerance is as follows

$$P_r = f(\alpha, a_1) \quad (51)$$

### 6.1. Simply supported beam under uniformly distributed load

A simply supported beam under uniformly distributed load is shown in Fig. 3. All basic random variables are assumed to follow normal distributions, i.e. $q \sim \text{N}(210, 7^2)\text{N/mm}$, $l \sim \text{N}(4000, 150^2)\text{mm}$, $b \sim \text{N}(120, 10^2)\text{mm}$, $h \sim \text{N}(240, 10^2)\text{mm}$. The beam is #45 steel and its strength is assumed to follow a normal distribution, i.e. $R \sim \text{N}(623, 23^2)\text{MPa}$. The membership function of the fuzzy safety state $\tilde{A}$ with proportional transition is

$$\mu_{\tilde{A}}(z) = \begin{cases} 0 & , z < a_1 \\ \dfrac{z - a_1}{a_2 - a_1}, & a_1 \le z \le a_2 \\ 1 & , z > a_2 \end{cases} \quad (52)$$

with $a_2 = \left(1 - \frac{1}{\alpha}\right)a_1$. The relationship between the reliability and the tolerance is shown in Table 3, and the reliability curves are illustrated in Fig. 10. When the membership function of the fuzzy safety state $\tilde{A}$ is assumed to follow a half-ridge distribution, i.e.

$$\mu_{\tilde{A}}(z) = \begin{cases} 0 & , z < a_1 \\ \dfrac{1}{2} + \dfrac{1}{2}\sin\dfrac{\pi}{a_2 - a_1}\left(z - \dfrac{a_1 + a_2}{2}\right), & a_1 \le z \le a_2 \\ 1 & , z > a_2 \end{cases} \quad (53)$$

where $a_2 = \dfrac{2\sin^{-1}(2\alpha-1)-\pi}{2\sin^{-1}(2\alpha-1)+\pi} a_1$. The relationship between the reliability and the tolerance is shown in Table 4, and the reliability curves are illustrated in Fig. 11.

We can arrive at conclusions by analyzing Table 3, Table 4, and Figs. 10-11,

1) The reliability of structures is associated with the confidence level $\alpha$ and tolerance $a_1$ besides basic random variables of structures. With the increasing of confidence level, the reliability of structures increases continuously. However, when confidence level is close to 1, the rate of increasing becomes low. This is consistent with the qualitative analysis.
2) When the confidence level $\alpha$ is lower, the increasing rate of reliability reduces along with the increasing tolerance. Moreover, when confidence level $\alpha$ is higher, the decreasing rate of reliability reduces along with the increasing of tolerance.
3) When the confidence level $\alpha$ is lower, the value of tolerance $a_1$ should not be too large, and when confidence level $\alpha$ is higher, the greater value of tolerance $a_1$ can be selected. This is also consistent with the qualitative analysis.

## 7. Conclusion

Fuzziness always exists in actual structure analysis. Since it is impossible to analyze the influence of fuzziness on structural reliability using the conventional reliability method, the evaluation of the structural reliability using the conventional reliability method cannot completely describe the reality. This work presents an investigation on reliability method of structures involved with fuzziness. The fuzzy safety state of a structure is defined by the state variable, fuzzy random

*Table 3. The relation of reliability and tolerances*

| $a_1$ / $P_r$ / $\alpha$ | -0.05×623 | -0.1×623 | -0.15×623 | -0.2×623 | -0.25×623 | -0.3×623 | -0.35×623 | -0.4×623 | -0.45×623 | -0.5×623 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 0.999990 | 0.999971 | 0.999820 | 0.999271 | 0.997346 | 0.992250 | 0.981452 | 0.962851 | 0.936768 | 0.906099 |
| 0.6 | 0.999997 | 0.999991 | 0.999972 | 0.999919 | 0.999766 | 0.999309 | 0.998553 | 0.996859 | 0.993800 | 0.988653 |
| 0.7 | 0.999998 | 0.999996 | 0.999992 | 0.999986 | 0.999973 | 0.999956 | 0.999892 | 0.999791 | 0.999622 | 0.999315 |
| 0.8 | 0.999998 | 0.999998 | 0.999998 | 0.999997 | 0.999996 | 0.999993 | 0.999991 | 0.999987 | 0.99998 | 0.999973 |
| 0.9 | 0.999999 | 0.999999 | 0.999999 | 0.999999 | 0.999999 | 0.999999 | 0.999999 | 0.999998 | 0.999998 | 0.999998 |

*Table 4. The relation of reliability and tolerances*

| $a_1$ / $P_r$ / $\alpha$ | -0.05×623 | -0.1×623 | -0.15×623 | -0.2×623 | -0.25×623 | -0.3×623 | -0.35×623 | -0.4×623 | -0.45×623 | -0.5×623 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 0.999997 | 0.999983 | 0.999920 | 0.999677 | 0.999094 | 0.997647 | 0.994349 | 0.987739 | 0.978378 | 0.964035 |
| 0.6 | 0.999997 | 0.999991 | 0.999975 | 0.999929 | 0.999803 | 0.999506 | 0.998900 | 0.997781 | 0.995721 | 0.992396 |
| 0.7 | 0.999997 | 0.999996 | 0.999992 | 0.999981 | 0.999966 | 0.999911 | 0.999824 | 0.999674 | 0.999332 | 0.998745 |
| 0.8 | 0.999998 | 0.999998 | 0.999997 | 0.999996 | 0.999993 | 0.999988 | 0.999980 | 0.999961 | 0.999958 | 0.999909 |
| 0.9 | 0.999999 | 0.999999 | 0.999999 | 0.999999 | 0.999999 | 0.999998 | 0.999997 | 0.999996 | 0.999995 | 0.999994 |

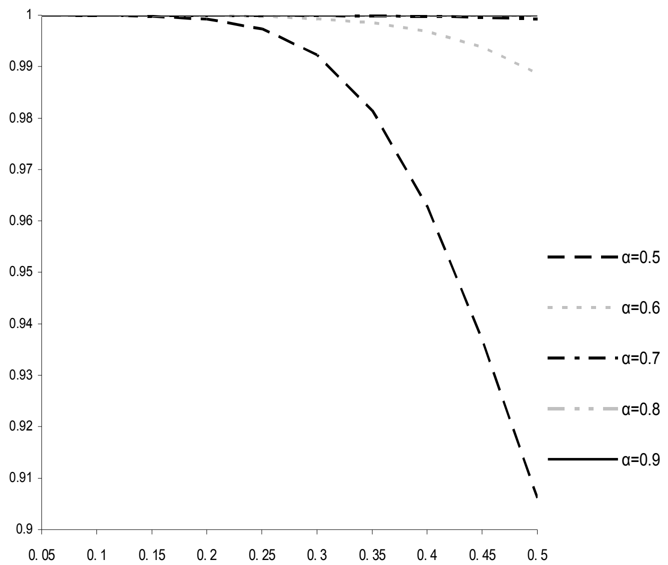


*Fig. 10. Relationship of fuzzy reliability and tolerances*

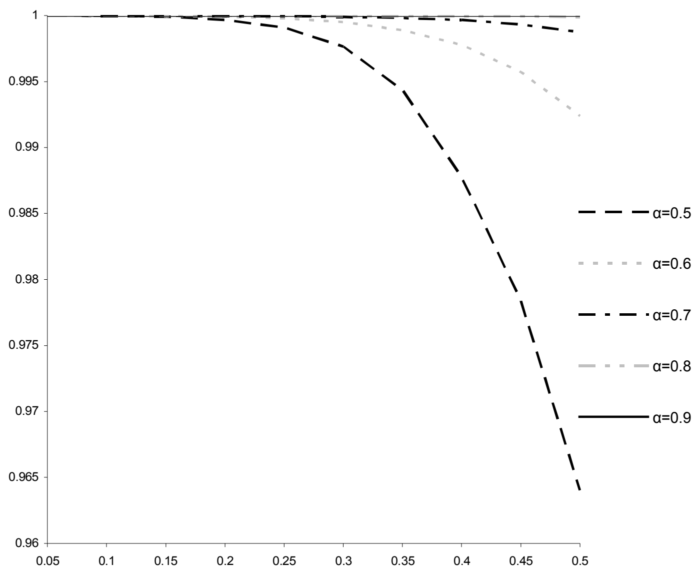


*Fig. 11. Relationship of reliability and tolerances*

allowable interval or fuzzy random generalized strength. Determination of the membership function of the fuzzy safety state is the key to the proposed method. This work concerns the fuzziness of the safety criterion, the fuzziness and randomness of generalized stresses and generalized strengths. The membership function of the fuzzy safety state is defined, and the structural reliability analysis method using fuzzy sets theory is proposed. Several examples are used to illustrate the proposed method.

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