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MEHAR'S METHOD FOR ANALYZING THE FUZZY RELIABILITY OF PISTON MANUFACTURING SYSTEM

METODA MEHAR DO ANALIZY ROZMYTEJ NIEZAWODNOŚCI SYSTEMU PRODUKCJI TŁOKÓW

To the best of our knowledge till now there are only two analytical methods for finding the exact solution of fuzzy differential equations. In this paper, the shortcoming of one of these existing methods is pointed out. To overcome the shortcoming of the existing method, a new method, named as Mehar's method, is proposed for solving fuzzy differential equations. To show the advantage of Mehar's method over existing method the fuzzy Kolmogorov's differential equations, developed by using fuzzy Markov model of piston manufacturing system, are solved by using the existing and Mehar's method and it is shown that the results, obtained by using the existing method, may or may not be fuzzy number while the results, obtained by using Mehar's method, are always fuzzy number.

Keywords: *Fuzzy differential equations, fuzzy reliability, trapezoidal fuzzy number.*

Wedle naszej najlepszej wiedzy, do tej pory stworzono jedynie dwie metody analityczne precyzyjnego rozwiązywania rozmytych równań różniczkowych. W artykule wskazano wady jednej z istniejących metod oraz zaproponowano nową metodę rozwiązywania równań różniczkowych, nazwaną metodą Mehar, w której wady te zostały wyeliminowane. Aby wykazać przewagę metody Mehar nad istniejącą metodą, rozwiązano za pomocą obu tych metod rozmyte równania różniczkowe Kołmogorowa wyprowadzone przy użyciu rozmytego markowowskiego modelu systemu produkcji tłoków. Wykazano, że wyniki otrzymane z wykorzystaniem istniejącej metody, mogą ale nie muszą być liczbami rozmytymi, natomiast wyniki otrzymane przy pomocy metody Mehar zawsze stanowią liczbę rozmytą.

Słowa kluczowe: *rozmyte równania różniczkowe, rozmyta niezawodność, trapezoidalna liczba rozmyta.*

1. Introduction

Fuzzy differential equations are utilized for the purpose of modelling problems in science and engineering. The concept of a fuzzy derivative was first introduced by Chang and Zadeh [18] it was followed by Dubois and Prade [21], who defined and used the extension principle. Buckley and Feuring [14] introduced two analytical methods for solving n^{th} order linear differential equations with fuzzy initial conditions. Their first method of solution was to fuzzify the crisp solution and then checked to see if it satisfies the differential equation with fuzzy initial conditions and the second method was the reverse of the first method, in that they first solved the fuzzy initial value problem and then checked to see if it defines a fuzzy function.

In the last few years, lot of work has been done by several authors in theoretical and applied fields of fuzzy differential equations [1-10, 12, 13, 15-17, 19, 20, 22-24, 30-37, 39, 40].

In this paper, the shortcoming of one of these existing methods is pointed out. To overcome the shortcoming of the existing method, a new method, named as Mehar's method, is proposed for solving fuzzy differential equations. To show the advantage of Mehar's method over existing method the fuzzy Kolmogorov's differential equations, developed by using fuzzy Markov model of piston manufacturing system, are solved by using the existing and Mehar's method and it is shown that the results, obtained by using the existing method, may or may not be fuzzy number while the results, obtained by using Mehar's method, are always fuzzy number.

This paper is organized as follows: In Section 2, some basic definitions, arithmetic operations between intervals, arithmetic operations between trapezoidal fuzzy numbers and arithmetic operations between *JMD* trapezoidal fuzzy numbers are presented. In Section 3, the existing method for solving fuzzy differential equations is presented. The shortcoming of the existing method is discussed in Section 4. In Section 5, a new method, named as Mehar's method, is proposed to find the exact solution of fuzzy differential equations with the help of *JMD* representation of trapezoidal fuzzy numbers. Advantages of the proposed method over the existing method is shown in Section 6. In Section 7, advantages of *JMD* representation of trapezoidal fuzzy numbers over existing representation of trapezoidal fuzzy numbers is presented. In Section 8, fuzzy reliability of piston manufacturing system is evaluated. The conclusion is discussed in Section 9.

2. Preliminaries

In this section, some basic definitions, arithmetic operations between intervals, arithmetic operations between trapezoidal fuzzy numbers and arithmetic operations between *JMD* trapezoidal fuzzy numbers are presented.

2.1. Basic definitions

In this section, some basic definitions are presented [25].

2.1.1. *α*-cut

In this section, α-cut of a fuzzy number, zero α-cut and equality of α-cut are presented.

Definition 2.1. An α-cut of a fuzzy number $\tilde{A}$ is defined as a crisp set $A_\alpha = \{x : \mu_{\tilde{A}}(x) \geq \alpha, x \in X\}$, where $\alpha \in [0,1]$.

Definition 2.2. An α-cut $A_\alpha = [a, b]$ is said to be zero α-cut iff $a = 0$ and $b = 0$.

Definition 2.3. Two α-cuts $A_\alpha = [a_1,b_1]$ and $B_\alpha = [a_2,b_2]$ are said to be equal i.e., $A_\alpha = B_\alpha$ iff $a_1 = a_2$ and $b_1 = b_2$.

2.1.2. Trapezoidal fuzzy number

In this section, definitions of trapezoidal fuzzy number, zero trapezoidal fuzzy number and equality of trapezoidal fuzzy numbers are presented [25].

Definition 2.4 A fuzzy number $\tilde{A} = (a,b,c,d)$ is said to be a trapezoidal fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & -\infty < x \leq a, \\ \frac{(x-a)}{(b-a)}, & a \leq x < b, \\ 1, & b \leq x \leq c, \\ \frac{(x-d)}{(c-d)}, & c < x \leq d, \\ 0, & d \leq x < \infty. \end{cases}$$

Definition 2.5 A trapezoidal fuzzy number $\tilde{A} = (a,b,c,d)$ is said to be zero trapezoidal fuzzy number iff $a=0$, $b=0$, $c=0$, $d=0$.

Definition 2.6 A trapezoidal fuzzy number $\tilde{A} = (a,b,c,d)$ is said to be non-negative trapezoidal fuzzy number iff $a \geq 0$.

Definition 2.7 Two trapezoidal fuzzy numbers $\tilde{A} = (a_1,b_1,c_1,d_1)$ and $\tilde{B} = (a_2,b_2,c_2,d_2)$ are said to be equal i.e., $\tilde{A} = \tilde{B}$ iff $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$.

2.1.3. *JMD* representation of trapezoidal fuzzy number

Kumar and Kaur [28] proposed *JMD* representation of trapezoidal fuzzy number and proved that it is better to use the proposed representation of trapezoidal fuzzy numbers instead of existing representation of trapezoidal fuzzy numbers for finding the fuzzy optimal solution of fuzzy transportation problems. In this section, definitions of *JMD* trapezoidal fuzzy number, zero *JMD* trapezoidal fuzzy number and equality of *JMD* trapezoidal fuzzy numbers are presented.

Definition 2.8. Let (a,b,c,d) be a trapezoidal fuzzy number then its *JMD* representation is $(x,\alpha,\beta,\gamma)_{JMD}$, where $x = a$, $\alpha = b - a \geq 0$, $\beta = c - b \geq 0$, $\gamma = d - c \geq 0$.

Definition 2.9. A trapezoidal fuzzy number $\tilde{A} = (x,\alpha,\beta,\gamma)_{JMD}$ is said to be zero trapezoidal fuzzy number if and only if $x=0$, $\alpha=0$, $\beta=0$, $\gamma=0$.

Definition 2.10. A trapezoidal fuzzy number $\tilde{A} = (x,\alpha,\beta,\gamma)_{JMD}$ is said to be non-negative trapezoidal fuzzy number if and only if $x \geq 0$.

Definition 2.11. Two trapezoidal fuzzy numbers $\tilde{A} = (x_1,\alpha_1,\beta_1,\gamma_1)_{JMD}$ and $\tilde{B} = (x_2,\alpha_2,\beta_2,\gamma_2)_{JMD}$ are said to be equal i.e., $\tilde{A} = \tilde{B}$ if and only if $x_1 = x_2$, $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$.

2.2. Arithmetic operations

In this section, arithmetic operations between intervals, trapezoidal fuzzy numbers and *JMD* trapezoidal fuzzy number are presented.

2.2.1. Arithmetic operations between intervals

In this section, some arithmetic operations between intervals are presented [25].

Let $A = [a_1, b_1]$, $B = [a_2, b_2]$ be two intervals then

(i) $A + B = [a_1 + a_2, b_1 + b_2]$

(ii) $A - B = [a_1 - b_2, b_1 - a_2]$

(iii) $\lambda A = \begin{cases} (\lambda x_1, \lambda y_1, \lambda z_1, \lambda w_1), & \lambda \geq 0 \\ (\lambda w_1, \lambda z_1, \lambda y_1, \lambda x_1), & \lambda \leq 0 \end{cases}$

(iv) $AB = [a,b]$, where, a = minimum $(a_1a_2, a_1b_2, a_2b_1, b_1b_2)$ and b = maximum $(a_1a_2, a_1b_2, a_2b_1, b_1b_2)$

2.2.2. Arithmetic operations between trapezoidal fuzzy numbers

In this section, arithmetic operations between trapezoidal fuzzy numbers are presented [25].

Let $\tilde{A}_1 = (x_1,y_1,z_1,w_1)$ and $\tilde{A}_2 = (x_2,y_2,z_2,w_2)$ be two trapezoidal fuzzy numbers, then

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2, w_1 + w_2)$

(ii) $\tilde{A}_1 \ominus \tilde{A}_2 = (x_1 - w_2, y_1 - z_2, z_1 - y_2, w_1 - x_2)$

(iii) $\lambda \tilde{A}_1 = \begin{cases} (\lambda x_1, \lambda y_1, \lambda z_1, \lambda w_1), & \lambda \geq 0 \\ (\lambda w_1, \lambda z_1, \lambda y_1, \lambda x_1), & \lambda \leq 0 \end{cases}$

(iv) $\tilde{A}_1 \otimes \tilde{A}_2 \simeq$ (minimum (x), minimum (y), maximum (y), maximum (x)),
where $x = (x_1x_2, x_1w_2, w_1x_2, w_1w_2)$ and $y = (y_1y_2, y_1z_2, z_1y_2, z_1z_2)$.

2.2.3. Arithmetic operations between *JMD* trapezoidal fuzzy numbers

In this section, arithmetic operations between *JMD* trapezoidal fuzzy numbers are presented [28].

Let $\tilde{A}_1 = (x_1, \alpha_1, \beta_1, \gamma_1)_{JMD}$ and $\tilde{A}_2 = (x_2, \alpha_2, \beta_2, \gamma_2)_{JMD}$ be two *JMD* trapezoidal fuzzy numbers, then

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = (x_1 + x_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2)_{JMD}$

(ii) $\tilde{A}_1 \ominus \tilde{A}_2 = (x_1 - x_2 - \alpha_2 - \beta_2 - \gamma_2, \alpha_1 + \gamma_2, \beta_1 + \beta_2, \alpha_2 + \gamma_1)_{JMD}$

(iii) $\lambda A_1 = \begin{cases} (\lambda x_1, \lambda \alpha_1, \lambda \beta_1, \lambda \gamma_1)_{JMD}, & \lambda \geq 0 \\ (\lambda x_1 + \lambda \alpha_1 + \lambda \beta_1 + \lambda \gamma_1, -\lambda \gamma_1, -\lambda \beta_1, -\lambda \alpha_1)_{JMD}, & \lambda \leq 0. \end{cases}$

(iv) $\widetilde{A}_1 \otimes \widetilde{A}_2 \simeq$ (minimum (x), minimum (y) $-$ minimum (x), maximum (y) $-$ minimum (y), maximum (x) $-$ maximum (y)), where

$$x = (x_1x_2, x_1x_2 + x_1\alpha_2 + x_1\beta_2 + x_1\gamma_2, x_1x_2 + x_2\alpha_1 + x_2\beta_1 + x_2\gamma_1,$$
$$x_1x_2 + x_1\alpha_2 + x_1\beta_2 + x_1\gamma_2 + x_2\alpha_1 + \alpha_1\alpha_2 + \alpha_1\beta_2 + \alpha_1\gamma_2 +$$
$$+ x_2\beta_1 + \beta_1\alpha_2 + \beta_1\beta_2 + \beta_1\gamma_2 + x_2\gamma_1 + \gamma_1\alpha_2 + \gamma_1\beta_2 + \gamma_1\gamma_2)$$

and

$$y = (x_1x_2 + x_1\alpha_2 + x_2\alpha_1 + \alpha_1\alpha_2, x_1x_2 + x_1\alpha_2 + x_1\beta_2 + x_2\alpha_1 +$$
$$+ \alpha_1\alpha_2 + \alpha_1\beta_2, x_1x_2 + x_1\alpha_2 + x_2\alpha_1 + \alpha_1\alpha_2 +$$
$$x_2\beta_1 + \beta_1\alpha_2, x_1x_2 + x_1\alpha_2 + x_1\beta_2 + x_2\alpha_1 + \alpha_1\alpha_2 +$$
$$+ \alpha_1\beta_2 + x_2\beta_1 + \beta_1\alpha_2 + \beta_1\beta_2)$$

Remark 2.1. Let $\widetilde{A}_1 = (x_1, y_1, z_1, w_1)$ be a *JMD* trapezoidal fuzzy number and $\widetilde{A}_2 = (x_2, y_2, z_2, w_2)$ be a non-negative *JMD* trapezoidal fuzzy number, then

$$\widetilde{A}_1 \otimes \widetilde{A}_2 \cong \begin{cases} (x_1x_2, y_1y_2, z_1z_2, w_1w_2) & x_1 \geq 0 \\ (x_1w_2, y_1y_2, z_1z_2, w_1w_2) & x_1 < 0 \text{ and } y_1 \geq 0 \\ (x_1w_2, y_1z_2, z_1z_2, w_1w_2) & y_1 < 0 \text{ and } z_1 \geq 0 \\ (x_1w_2, y_1z_2, z_1y_2, w_1w_2) & z_1 < 0 \text{ and } w_1 \geq 0 \\ x_1w_2, y_1z_2, z_1y_2, w_1x_2) & \text{otherwise} \end{cases}$$

Remark 2.2. Let $\tilde{A}_1 = (x_1, \alpha_1, \beta_1, \gamma_1)_{JMD}$ be a *JMD* trapezoidal fuzzy number and $\tilde{A}_2 = (x_2, \alpha_2, \beta_2, \gamma_2)_{JMD}$ be a non-negative *JMD* trapezoidal fuzzy number, then

$$\widetilde{A}_1 \otimes \widetilde{A}_2 \cong$$

$$\begin{cases} (x_1x_2, x_1\alpha_2 + x_2\alpha_1 + \alpha_1\alpha_2, x_1\beta_2 + \alpha_1\beta_2 + x_2\beta_1 + \alpha_2\beta_1 + \\ \beta_1\beta_2, x_1\gamma_2 + \alpha_1\gamma_2 + \beta_1\gamma_2 + x_2\gamma_1 + \alpha_2\gamma_1 + \beta_2\gamma_1 + \gamma_1\gamma_2)_{JMD}, & x_1 \geq 0 \\ x_1x_2 + x_1\alpha_2 + x_1\beta_2 + x_1\gamma_2, x_2\alpha_1 + \alpha_1\alpha_2 - x_1\beta_2 - x_1\gamma_2, \\ x_1\beta_2 + \alpha_1\beta_2 + x_2\beta_1 + \beta_1\alpha_2 + \beta_1\beta_2, x_1\gamma_2 + \alpha_1\gamma_2 + \beta_1\gamma_2 + \\ x_2\gamma_1 + \gamma_1\alpha_2 + \gamma_1\beta_2 + \gamma_1\gamma_2)_{JMD}, & x_1 < 0 \text{ and } x_1 + \alpha_1 \geq 0 \\ x_1x_2 + x_1\alpha_2 + x_1\beta_2 + x_1\gamma_2, x_1\beta_2 + x_2\alpha_1 + \alpha_1\alpha_2 + \alpha_1\beta_2 - \\ x_1\beta_1 - x_1\gamma_2, x_2\beta_1 + \beta_1\alpha_2 + \beta_1\beta_2, x_1\gamma_2 + \alpha_1\gamma_2 + \beta_1\gamma_2 + \\ x_2\gamma_1 + \gamma_1\alpha_2 + \gamma_1\beta_2 + \gamma_1\gamma_2)_{JMD}, & x_1 + \alpha_1 < 0 \text{ and } x_1 + \alpha_1 + \beta_1 \geq 0 \\ x_1x_2 + x_1\alpha_2 + x_1\beta_2 + x_1\gamma_2, x_2\alpha_1 + \alpha_1\alpha_2 + \alpha_1\beta_2 - x_1\gamma_2, \\ x_2\beta_1 + \beta_1\alpha_2 - x_1\beta_2 - \alpha_1\beta_2, x_1\beta_2 + x_1\gamma_2 + \alpha_1\beta_2 + \alpha_1\gamma_2 + \\ \beta_1\beta_2 + \beta_1\gamma_2 + x_2\gamma_1 + \gamma_1\alpha_2 + \gamma_1\beta_2 + \gamma_1\gamma_2)_{JMD}, & x_1 + \alpha_1 + \beta_1 < 0 \text{ and } x_1 + \alpha_1 + \beta_1 + \gamma_1 \geq 0 \\ x_1x_2 + x_1\alpha_2 + x_1\beta_2 + x_1\gamma_2, x_2\alpha_1 + \alpha_1\alpha_2 + \alpha_1\beta_2 - x_1\gamma_2, \\ x_2\beta_1 + \beta_1\alpha_2 - x_1\beta_2 - \alpha_1\beta_2, x_2\gamma_1 - x_1\alpha_2 - \alpha_1\alpha_2 - \beta_1\alpha_2)_{JMD}, & \text{otherwise.} \end{cases}$$

3. Existing method

Buckley and Feuring [14] introduced two analytical methods for solving fuzzy initial value problem for n^{th} order linear differential equations. In this section, one of these existing methods for solving fuzzy differential equations is presented.

The solution of fuzzy initial value problem for n^{th} order fuzzy linear differential equation

$$\tilde{a}_n\tilde{y}^{(n)} \oplus \tilde{a}_{n-1}\tilde{y}^{(n-1)} \oplus ... \oplus \tilde{a}_1\tilde{y}^{(1)} \oplus \tilde{a}_0\tilde{y} =$$
$$= \tilde{g}(x), \tilde{y}(0) = \tilde{\gamma}_0, \tilde{y}^{(1)}(0) = \tilde{\gamma}_1, ..., \tilde{y}^{(n-1)}(0) = \tilde{\gamma}_{n-1} \qquad (1)$$

where, $\tilde{y}^{(i)} = \frac{d^i\tilde{y}}{dx^i}$ for $i = n, n\text{-}1, ..., 1$, $\widetilde{a}_n$ is a non zero trapezoidal fuzzy number and $\widetilde{a}_{n-1}, \widetilde{a}_{n-2}, ..., \widetilde{a}_1, \widetilde{a}_0$ are any type of trapezoidal fuzzy numbers, can be obtained by using the following steps of the existing method:

Step 1: Find the α-cut $[a_{n(1)}(x,\alpha), a_{n(2)}(x,\alpha)]$, $[a_{n-1(1)}(x,\alpha), a_{n-1(2)}(x,\alpha)], ...,$ $[a_{1(1)}(x,\alpha), a_{1(2)}(x,\alpha)], [a_{0(1)}(x,\alpha), a_{0(2)}(x,\alpha)]$, $[y_1^{(n)}(x,\alpha), y_2^{(n)}(x,\alpha)]$, $[y_1^{(n-1)}(x,\alpha), y_2^{(n-1)}(x,\alpha)]$, ..., $[y_1^{(1)}(x,\alpha), y_2^{(1)}(x,\alpha)]$, $[y_1(x,\alpha), y_2(x,\alpha)]$ and $[\gamma_{0(1)}(0,\alpha), \gamma_{0(2)}(0,\alpha)], [\gamma_{1(1)}(0,\alpha), \gamma_{1(2)}(0,\alpha)], ...,$ $[\gamma_{n-1(1)}(0,\alpha), \gamma_{n-1(2)}(0,\alpha)]$ corresponding to fuzzy parameters $\tilde{a}_n, \tilde{a}_{n-1}, \tilde{a}_{n-2}, ..., \tilde{a}_1, \tilde{a}_0, \tilde{y}^{(n)}, \tilde{y}^{(n-1)}, ..., \tilde{y}^{(1)}, \tilde{y}$ and $\tilde{\gamma}_0, \tilde{\gamma}_1, ..., \tilde{\gamma}_{n-1}$ respectively.

Step 2: Convert the fuzzy initial value problem for n^{th} order fuzzy linear differential equation (1), into the following n^{th} order differential equation:

$$[a_{n(1)}(x,\alpha), a_{n(2)}(x,\alpha)][y_1^{(n)}(x,\alpha), y_2^{(n)}(x,\alpha)] + [a_{n-1(1)}(x,\alpha), a_{n-1(2)}(x,\alpha)]$$
$$[y_1^{(n-1)}(x,\alpha), y_2^{(n-1)}(x,\alpha)] + ... +$$
$$+ [a_{1(1)}(x,\alpha), a_{1(2)}(x,\alpha)][y_1^{(1)}(x,\alpha), y_2^{(1)}(x,\alpha)] +$$
$$+ [a_{0(1)}(x,\alpha), a_{0(2)}(x,\alpha)]\,[y_1(x,\alpha), y_2(x,\alpha)] = [g(x), g(x)],$$
$$[y_1(0,\alpha), y_2(0,\alpha)] = [\gamma_{0(1)}(0,\alpha), \gamma_{0(2)}(0,\alpha)],$$
$$[y_1^{(1)}(0,\alpha), y_2^{(1)}(0,\alpha)] = [\gamma_{1(1)}(0,\alpha), \gamma_{1(2)}(0,\alpha)] ...,$$
$$[y_1^{(n-1)}(0,\alpha), y_2^{(n-1)}(0,\alpha)] = [\gamma_{n-1(1)}(0,\alpha), \gamma_{n-1(2)}(0,\alpha)].$$

Step 3: Convert the n^{th} order differential equation, obtained from Step 2, into the following ordinary differential equations

$$b_ny^{(n)} + b_{n-1}y^{(n-1)} + ... + b_1y^{(1)} + b_0y = g(x)$$
$$y_1(0,\alpha) = \gamma_{0(1)}(0,\alpha), y_1^{(1)}(0,\alpha) = \gamma_{1(1)}(0,\alpha), ..., y_1^{(n-1)}(0,\alpha) = \gamma_{n-1(1)}(0,\alpha)$$
$$c_ny^{(n)} + c_{n-1}y^{(n-1)} + ... + c_1y^{(1)} + c_0y = g(x)$$
$$y_2(0,\alpha) = \gamma_{0(2)}(0,\alpha), y_2^{(1)}(0,\alpha) = \gamma_{2(1)}(0,\alpha), ..., y_2^{(n-1)}(0,\alpha) = \gamma_{n-1(2)}(0,\alpha)$$

where,

$$b_iy^{(i)} = \text{minimum } (a_{i(1)}(x,\alpha)y_1^{(i)}(x,\alpha), a_{i(1)}(x,\alpha)y_2^{(i)}(x,\alpha),$$
$$a_{i(2)}(x,\alpha)y_1^{(i)}(x,\alpha), a_{i(2)}(x,\alpha)y_2^{(i)}(x,\alpha))$$

and

$$c_iy^{(i)} = \text{maximum } (a_{i(1)}(x,\alpha)y_1^{(i)}(x,\alpha), a_{i(1)}(x,\alpha)y_2^{(i)}(x,\alpha),$$
$$a_{i(2)}(x,\alpha)y_1^{(i)}(x,\alpha), a_{i(2)}(x,\alpha)y_2^{(i)}(x,\alpha))$$

for $i = n, n\text{-}1, ..., 1, 0$.

Step 4: Solve the ordinary differential equations, obtained from Step 3, to find the values of $y_1(x_0, \alpha)$ and $y_2(x_0, \alpha)$ corresponding to $x=x_0$, where x_0 is any real number.

Step 5: Check that $[y_1(x_0,\alpha), y_2(x_0,\alpha)]$ defines the α-cut of a fuzzy number or not i.e., for the values of $y_1(x_0,\alpha)$ and $y_2(x_0,\alpha)$, the following conditions are satisfied or not.
(i) $y_1(x_0,\alpha)$ a monotonically increasing function for $\alpha \in [0,1]$
(ii) $y_2(x_0,\alpha)$ a monotonically decreasing function for $\alpha \in [0,1]$
(iii) $y_1(x_0,1) = y_2(x_0,1)$

Case 1: If $[y_1(x_0,\alpha)\ y_2(x_0,\alpha)]$ defines the α-cut of a fuzzy number then the fuzzy solution $\tilde{y}(x_0)$ of fuzzy differential equation (1) exist and $[y_1(x_0,\alpha), y_2(x_0,\alpha)]$ represents the α-cut corresponding to fuzzy solution $\tilde{y}(x_0)$.

Case 2: If $[y_1(x_0,\alpha)\ y_2(x_0,\alpha)]$ does not define the α-cut of a fuzzy number then the fuzzy solution $\tilde{y}(x_0)$ of fuzzy differential equation (1) does not exist.

4. Shortcoming of existing method in real life problems

Several authors have proposed different methods for analyzing the fuzzy reliability of industrial systems. One of the existing method for analyzing the fuzzy reliability is by using the fuzzy Markov model [11, 26, 27, 29, 38], in which fuzzy Kolomogorov's differential equations are developed with the help of fuzzy Markov model and the fuzzy reliability is evaluated by solving the obtained fuzzy Kolomogorov's differential equations.

In this section, the set of fuzzy Kolomogorov's differential equations, obtained by using fuzzy Markov model of a piston manufacturing system, is solved by using one of the analytical methods [14] and it is shown that the obtained solution may or may not be a fuzzy number. Due to which the solution of fuzzy differential equations, obtained by using the existing method, can not be used to analyze the fuzzy reliability of piston manufacturing system.

4.1. Fuzzy Markov modeling of piston manufacturing system

Piston manufacturing system consists of two sub-systems namely R_1 and R_2 , which are connected in series. Further the sub-system R_1 consists of six sub-systems namely *A,B,C,D,E* and *F* and similarly, six sub-systems namely *G,H,I,J,K* and *L* constitute the sub-system R_2. Markov models for the sub-systems R_1 and R_2 are shown in Figure 1 and Figure 2 respectively.

The operations that are performed on these machines or sub-systems are as follows:

1. **Sub-system** ***A*** **(Fixture Seat Machine):** This machine is used to clamp the piston.
2. **Sub-system** ***B*** **(Rough Grooving and Turning Machine):** On this machine, rough grooves are made on piston. Turning operation is performed on this machine i.e., to bring the dia of piston to proper size.
3. **Sub-system** ***C*** **(Rough Pin Hole Boring Machine):** Pin hole boring operation is performed using this machine i.e., proper size is given to holes.
4. **Sub-system** ***D*** **(Oil Hole Drilling Machine):** On this machine, one hole is made on the piston to supply the oil. The oil is used to move piston in cylinder smoothly.
5. **Sub-system** ***E*** **(Finishing Grooving Machine):** On this machine, the finishing is given to rough grooves which are prepared using sub-system *B*.
6. **Sub-system** ***F*** **(Finish Profile Turning Machine):** Oval shape is given to piston using this machine.
7. **Sub-system** ***G*** **(Finish Pin Hole Boring Machine):** On this machine, finishing is given to the pin hole portion which is prepared using sub-system *C*.

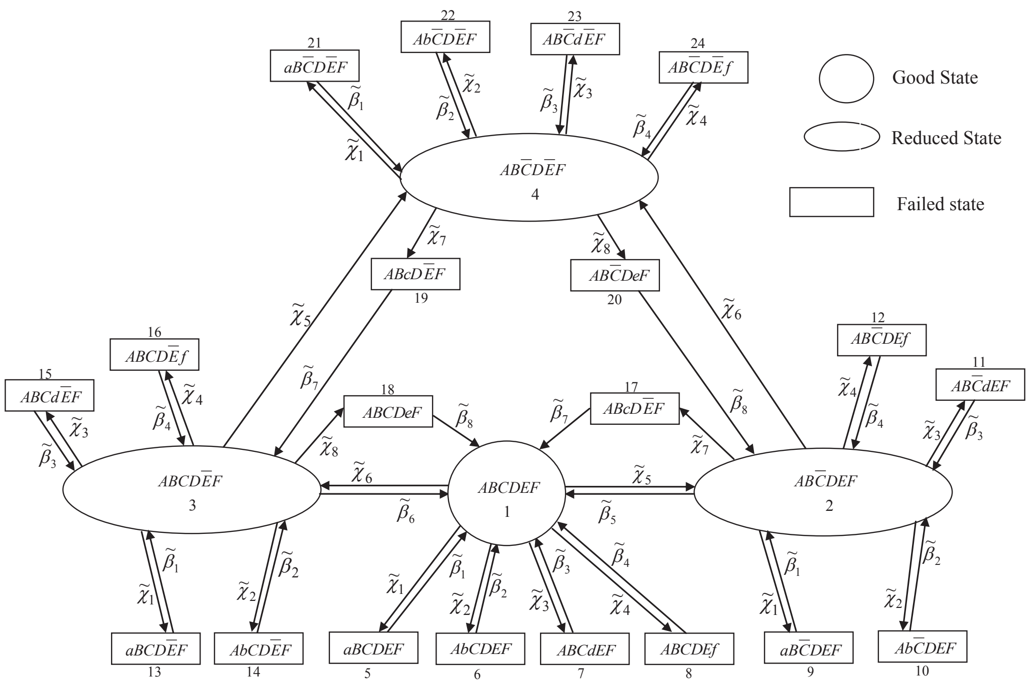

Fig. 1. Fuzzy Markov model of sub-system R_1

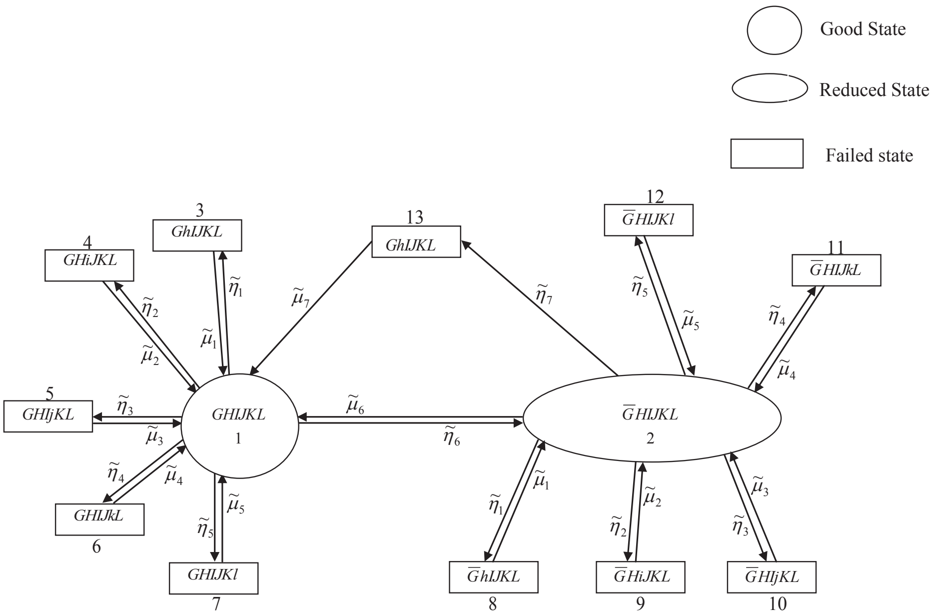

Fig. 2. Fuzzy Markov model of sub-system R_2

8. **Sub-system** *H* **(Finish Crown and Cavity Machine):** On this machine, finishing operation is performed on the crown of piston.
9. **Sub-system** *I* **(Valve Milling Machine):** On this machine, valve recess is made on the piston.
10. **Sub-system** *J* **(Chamfering or Radius Machine):** This machine rounds off the corners of the piston, so that it can run smoothly in the cylinder.
11. **Sub-system** *K* **(Circlip Grooving Machine):** On this machine, circlip grooves are made on the piston.
12. **Sub-system** *L* **(Piston Cleaning Machine):** This machine is used to clean the inside and outside portion of the piston.

4.2. Notation

In this section, notation that is used to analyze the fuzzy reliability of piston manufacturing system are presented:

1. *A,B,C,D,E,F* and *G,H,I,J,K,L* denote good conditions of sub-systems of R_1 and R_2 respectively.
2. The symbols *a,b,c,d,e,f,g,h,i,j,k* and *l* represent the failed state of the sub-systems *A,B,C,D,E,F,G,H,I,J,K* and *L* respectively.
3. $\bar{C}, \bar{E}$ and $\bar{G}$ indicate that the sub-systems *C,E* and *G* are working in reduced state.
4. $\tilde{\chi}_i$ (i=1 to 8) represents the fuzzy failure rates of the relevant sub-systems, when the transition is from *A* to *a*, *B* to *b*, *D* to *d*, *F* to *f*, *C* to $\bar{C}$, *E* to $\bar{E}$, $\bar{C}$ to *c* and $\bar{E}$ to *e* respectively.
5. $\tilde{\beta}_i$ (i=1 to 8) represents the fuzzy repair rates of the relevant sub-systems, when the transition is from *a* to *A*, *b* to *B*, *d* to *D*, *f* to *F*, $\bar{C}$ to *C*, $\bar{E}$ to *E*, *c* to *C* and *e* to *E* respectively.
6. $\tilde{\eta}_i$ (i=1 to 7) represents the fuzzy failure rates of the relevant sub-systems, when the transition is from *H* to *h*, *I* to *i*, *J* to *j*, *K* to *k*, *L* to *l*, *G* to $\bar{G}$ and $\bar{G}$ to *g* respectively.
7. $\tilde{\mu}_i$ (i=1 to 7) represents the fuzzy repair rates of the relevant sub-systems, when the transition is from *h* to *H*, *i* to *I*, *j* to *J*, *k* to *K*, *l* to *L*, $\bar{G}$ to *G* and *g* to *G* respectively.
8. $\tilde{P}_j(t), j = 1,2,...,n$ represents the fuzzy probability that the system is in state S_j at time *t*, where *n* is number of states. $\tilde{P}_j'(t), j = 1,2,...,n$ represents derivative of $\tilde{P}_j(t)$ with respect to *t*.
9. $\tilde{R}_1(t)$ and $\tilde{R}_2(t)$ denote the fuzzy reliability of the sub-systems R_1 and R_2 respectively.
10. $\tilde{R}(t)$ represents the fuzzy reliability of the whole system.

4.3. Assumptions

In this section, the assumptions that are used for analyzing the fuzzy reliability of piston manufacturing system are presented:

(i) Fuzzy failure rates and fuzzy repair rates are independent with each other and their unit is per hour.
(ii) There are no simultaneous failures among the sub-systems.
(iii) Sub-systems *C,E* and *G* fails through reduced states only.

4.4. Data

On the basis of the perception of the experts, the appropriate failure rates and repair rates for the different sub-systems of R_1 and R_2, represented by trapezoidal fuzzy numbers, are shown in table 1 and table 2 respectively.

4.5. Fuzzy Kolmogorov's differential equations for the sub-systems R_1 and R_2

In this section, fuzzy Kolmogorov's differential equations are developed by using the Markov model for the sub-systems R_1 and R_2.

Fuzzy Kolmogorov's differential equations for the sub-system R_1 associated with the Markov model (Figure 1) are:

Tab. 1. Fuzzy failure rates and fuzzy repair rates for the different sub-systems of R_1

Fuzzy failure rate	Fuzzy repair rate
$\tilde{\chi}_1$=(0.00105,0.00126,0.00154,0.00175)	$\tilde{\beta}_1$=(1.026,1.0584,1.1016,1.134)
$\tilde{\chi}_2$=(0.00045,0.00054,0.00066,0.00075)	$\tilde{\beta}_2$=(0.04085,0.04214,0.04386,0.04515)
$\tilde{\chi}_3$=(0.000675,0.00081,0.00099,0.001125)	$\tilde{\beta}_3$=(0.475,0.49,0.51,0.525)
$\tilde{\chi}_4$=(0.000675,0.00081,0.00099,0.001125)	$\tilde{\beta}_4$=(0.2717,0.28028,0.29172,0.3003)
$\tilde{\chi}_5$=(0.0156,0.01872,0.02288,0.026)	$\tilde{\beta}_5$=(0.1463,0.15092,0.15702,0.1617)
$\tilde{\chi}_6$=(0.0156,0.01872,0.02288,0.026)	$\tilde{\beta}_6$=(0.2375,0.245,0.255,0.2625)
$\tilde{\chi}_7$=(0.000675,0.00081,0.00099,0.001125)	$\tilde{\beta}_7$=(0.05605,0.05782,0.06018,0.06195)
$\tilde{\chi}_8$=(0.002925,0.00351,0.00429,0.004875)	$\tilde{\beta}_8$=(0.08265,0.08526,0.08874,0.09135)

Tab. 2. Fuzzy failure rates and fuzzy repair rates for the different sub-systems of R_2

Fuzzy failure rate	Fuzzy repair rate
$\tilde{\eta}_1$=(0.00105,0.00126,0.00154,0.00175)	$\tilde{\mu}_1$=(0.3135,0.3234,0.3366,0.3465)
$\tilde{\eta}_2$=(0.00023,0.00027,0.00033,0.00038)	$\tilde{\mu}_2$=(0.475,0.49,0.51,0.525)
$\tilde{\eta}_3$=(0.00008,0.00009,0.00011,0.00013)	$\tilde{\mu}_3$=(0.6365,0.6566,0.6834,0.7035)
$\tilde{\eta}_4$=(0.00023,0.00027,0.00033,0.00038)	$\tilde{\mu}_4$=(0.03325,0.0343,0.0357,0.03675)
$\tilde{\eta}_5$=(0.00008,0.00009,0.00011,0.00013)	$\tilde{\mu}_5$=(2.8785,2.9694,3.0906,3.1815)
$\tilde{\eta}_6$=(0.0156,0.01872,0.02288,0.026)	$\tilde{\mu}_6$=(0.2109,0.21756,0.22644,0.2331)
$\tilde{\eta}_7$=(0.003,0.0036,0.0044,0.005)	$\tilde{\mu}_7$=(0.11875,0.1225,0.1275,0.13125)

$$\begin{aligned}
&\tilde{P}_1^{(1)}(t)\oplus\tilde{\lambda}_1\tilde{P}_1(t)=\tilde{\beta}_1\tilde{P}_5(t)\oplus\tilde{\beta}_2\tilde{P}_6(t)\oplus\tilde{\beta}_3\tilde{P}_7(t)\oplus\tilde{\beta}_4\tilde{P}_8(t)\\
&\qquad\oplus\tilde{\beta}_5\tilde{P}_2(t)\oplus\tilde{\beta}_6\tilde{P}_3(t)\oplus\tilde{\beta}_7\tilde{P}_{17}(t)\oplus\tilde{\beta}_8\tilde{P}_{18}(t)\\
&\tilde{P}_2^{(1)}(t)\oplus\tilde{\lambda}_2\tilde{P}_2(t)=\tilde{\beta}_1\tilde{P}_9(t)\oplus\tilde{\beta}_2\tilde{P}_{10}(t)\oplus\tilde{\beta}_3\tilde{P}_{11}(t)\oplus\tilde{\beta}_4\tilde{P}_{12}(t)\\
&\qquad\oplus\tilde{\beta}_8\tilde{P}_{20}(t)\oplus\tilde{\chi}_5\tilde{P}_1(t)\\
&\tilde{P}_3^{(1)}(t)\oplus\tilde{\lambda}_3\tilde{P}_3(t)=\tilde{\beta}_1\tilde{P}_{13}(t)\oplus\tilde{\beta}_2\tilde{P}_{14}(t)\oplus\tilde{\beta}_3\tilde{P}_{15}(t)\oplus\tilde{\beta}_4\tilde{P}_{16}(t)\\
&\qquad\oplus\tilde{\beta}_7\tilde{P}_{19}(t)\oplus\tilde{\chi}_6\tilde{P}_1(t)\\
&\tilde{P}_4^{(1)}(t)\oplus\tilde{\lambda}_4\tilde{P}_4(t)=\tilde{\beta}_1\tilde{P}_{21}(t)\oplus\tilde{\beta}_2\tilde{P}_{22}(t)\oplus\tilde{\beta}_3\tilde{P}_{23}(t)\oplus\tilde{\beta}_4\tilde{P}_{24}(t)\\
&\qquad\oplus\tilde{\chi}_5\tilde{P}_3(t)\oplus\tilde{\chi}_6\tilde{P}_2(t)\\
&\tilde{P}_{4+i}^{(1)}(t)\oplus\tilde{\beta}_i\tilde{P}_{4+i}(t)=\tilde{\chi}_i\tilde{P}_1(t), i=1,2,3,4\\
&\tilde{P}_{8+i}^{(1)}(t)\oplus\tilde{\beta}_i\tilde{P}_{8+i}(t)=\tilde{\chi}_i\tilde{P}_2(t), i=1,2,3,4 \qquad \text{(S1)}\\
&\tilde{P}_{12+i}^{(1)}(t)\oplus\tilde{\beta}_i\tilde{P}_{12+i}(t)=\tilde{\chi}_i\tilde{P}_3(t), i=1,2,3,4\\
&\tilde{P}_{17}^{(1)}(t)\oplus\tilde{\beta}_7\tilde{P}_{17}(t)=\tilde{\chi}_7\tilde{P}_2(t)\\
&\tilde{P}_{18}^{(1)}(t)\oplus\tilde{\beta}_8\tilde{P}_{18}(t)=\tilde{\chi}_8\tilde{P}_3(t)\\
&\tilde{P}_{19}^{(1)}(t)\oplus\tilde{\beta}_7\tilde{P}_{19}(t)=\tilde{\chi}_7\tilde{P}_4(t)\\
&\tilde{P}_{20}^{(1)}(t)\oplus\tilde{\beta}_8\tilde{P}_{20}(t)=\tilde{\chi}_8\tilde{P}_4(t)\\
&\tilde{P}_{20+i}^{(1)}(t)\oplus\tilde{\beta}_i\tilde{P}_{20+i}(t)=\tilde{\chi}_i\tilde{P}_4(t), i=1,2,3,4
\end{aligned}$$

where,

$$\tilde{P}_i^{(1)}(t)=\frac{d\tilde{P}_i}{dt} \text{ for } i=1 \text{ to } 24$$

$$\begin{aligned}
&\tilde{\lambda}_1=\tilde{\chi}_1\oplus\tilde{\chi}_2\oplus\tilde{\chi}_3\oplus\tilde{\chi}_4\oplus\tilde{\chi}_5\oplus\tilde{\chi}_6\\
&\tilde{\lambda}_2=\tilde{\chi}_1\oplus\tilde{\chi}_2\oplus\tilde{\chi}_3\oplus\tilde{\chi}_4\oplus\tilde{\chi}_6\oplus\tilde{\chi}_7\oplus\tilde{\beta}_5\\
&\tilde{\lambda}_3=\tilde{\chi}_1\oplus\tilde{\chi}_2\oplus\tilde{\chi}_3\oplus\tilde{\chi}_4\oplus\tilde{\chi}_5\oplus\tilde{\chi}_8\oplus\tilde{\beta}_6\\
&\tilde{\lambda}_4=\tilde{\chi}_1\oplus\tilde{\chi}_2\oplus\tilde{\chi}_3\oplus\tilde{\chi}_4\oplus\tilde{\chi}_7\oplus\tilde{\chi}_8
\end{aligned}$$

with fuzzy initial conditions $\tilde{P}_1(0)$=(0.94,0.945,0.955,0.96), $\tilde{P}_2(0)$=(0.006,0.0065,0.0075,0.008), $\tilde{P}_3(0)$=(0.004,0.0045,0.0055,0.006), $\tilde{P}_4(0)$=(0.002,0.0025,0.0035,0.004) and $\tilde{P}_j(0)$=(0,0,0,0), *j*=4 to 24. (C1)

Fuzzy Kolmogorov's differential equations for the sub-system R_2 associated with the Markov model (Figure 2) are:

$$\begin{aligned}
&\tilde{P}_1^{(1)}(t)\oplus\tilde{\delta}_1\tilde{P}_1(t)=\tilde{\mu}_1\tilde{P}_3(t)\oplus\tilde{\mu}_2\tilde{P}_4(t)\oplus\tilde{\mu}_3\tilde{P}_5(t)\oplus\tilde{\mu}_4\tilde{P}_6(t)\\
&\qquad\oplus\tilde{\mu}_5\tilde{P}_7(t)\oplus\tilde{\mu}_6\tilde{P}_2(t)\oplus\tilde{\mu}_7\tilde{P}_{13}(t)\\
&\tilde{P}_2^{(1)}(t)\oplus\tilde{\delta}_2\tilde{P}_2(t)=\tilde{\mu}_1\tilde{P}_8(t)\oplus\tilde{\mu}_2\tilde{P}_9(t)\oplus\tilde{\mu}_3\tilde{P}_{10}(t)\oplus\tilde{\mu}_4\tilde{P}_{11}(t)\\
&\qquad\oplus\tilde{\mu}_5\tilde{P}_{12}(t)\oplus\tilde{\eta}_6\tilde{P}_1(t)\\
&\tilde{P}_{2+i}^{(1)}(t)\oplus\tilde{\mu}_i\tilde{P}_{2+i}(t)=\tilde{\eta}_i\tilde{P}_1(t), i=1,2,3,4,5 \qquad \text{(S2)}\\
&\tilde{P}_{7+i}^{(1)}(t)\oplus\tilde{\mu}_i\tilde{P}_{7+i}(t)=\tilde{\eta}_i\tilde{P}_2(t), i=1,2,3,4,5\\
&\tilde{P}_{13}^{(1)}(t)\oplus\tilde{\mu}_7\tilde{P}_{13}(t)=\tilde{\eta}_7\tilde{P}_2(t)
\end{aligned}$$

where,

$$\tilde{P}_i^{(1)}(t)=\frac{d\tilde{P}_i}{dt} \text{ for } i=1 \text{ to } 13$$

$$\begin{aligned}
&\tilde{\delta}_1=\tilde{\eta}_1\oplus\tilde{\eta}_2\oplus\tilde{\eta}_3\oplus\tilde{\eta}_4\oplus\tilde{\eta}_5\oplus\tilde{\eta}_6\\
&\tilde{\delta}_2=\tilde{\eta}_1\oplus\tilde{\eta}_2\oplus\tilde{\eta}_3\oplus\tilde{\eta}_4\oplus\tilde{\eta}_5\oplus\tilde{\eta}_7\oplus\tilde{\mu}_6
\end{aligned}$$

with fuzzy initial conditions $\tilde{P}_1(0)$=(0.95,0.955,0.965,0.97), $\tilde{P}_2(0)$=(0.004,0.0045,0.0055,0.006) and $\tilde{P}_j(0)$=(0,0,0,0), *j*=3 to 13. (C2)

4.6. Solution of fuzzy Kolmogorov's differential equations of sub-system R_1 and R_2

The solution of fuzzy Kolmogorov's differential equations of sub-system R_1 and R_2, developed in Section 4.5, are obtained by using the existing method [14], discussed in Section 3, for

α = 0, 0.2, 0.4, 0.6, 0.8, 1 at t = 360 hours and the solution is shown in table 3 and table 4 respectively.

It is obvious from the results, shown in Table 3 and Table 4, that $[p_{j1}(t,\alpha), p_{j2}(t,\alpha)]$ does not define the α - cut of a fuzzy number $\tilde{p}_j(t)$ for j = 1. So, the solution obtained by using the existing method [14] cannot be used to analyze the fuzzy reliability of piston manufacturing system.

Remark 4.1. In Section 4.6, it is shown that the solution obtained by using the existing method [14] can not be used to analyze the fuzzy reliability of piston manufacturing system. Similarly, several real life problems may be found for which the results of the existing method may or may not be valid.

5. Mehar's method with *JMD* representation of trapezoidal fuzzy numbers

To overcome the shortcoming of the existing method, discussed in Section 4, a new method, named as Mehar's method, is proposed to find the exact solution of fuzzy differential equations with the help of *JMD* representation of trapezoidal fuzzy numbers.

The solution of fuzzy initial value problem for n^{th} order fuzzy linear differential equation (1), where $\tilde{a}_n$ is a non zero *JMD* trapezoidal fuzzy number and $\tilde{a}_{n-1}, \tilde{a}_{n-2}, ..., \tilde{a}_1, \tilde{a}_0$ are *JMD* trapezoidal fuzzy numbers, can be obtained by using the following steps of Mehar's method:

Step 1: Convert all the parameters of fuzzy differential equations, represented by trapezoidal fuzzy number (a,b,c,d), into *JMD* trapezoidal fuzzy number $(x,\alpha,\beta,\gamma)_{JMD}$, where $\alpha = b - a \geq 0$, $\beta = c - b \geq 0$, $\gamma = d - c \geq 0$. Assuming $\tilde{a}_n = (a_{n(1)}, \beta_{n(1)}, \beta_{n(2)}, \beta_{n(3)})_{JMD}$,

$$\tilde{a}_{n-1} = (a_{n-1(1)}, \beta_{n-1(1)}, \beta_{n-1(2)}, \beta_{n-1(3)})_{JMD}, \tilde{a}_{n-2} = = (a_{n-2(1)}, \beta_{n-2(1)}, \beta_{n-2(2)}, \beta_{n-2(3)})_{JMD}, ...,$$

$$\tilde{a}_1 = (a_{1(1)}, \beta_{1(1)}, \beta_{1(2)}, \beta_{1(3)})_{JMD}, \tilde{a}_0 = = (a_{0(1)}, \beta_{0(1)}, \beta_{0(2)}, \beta_{0(3)})_{JMD}, \tilde{y}^{(n)} = (y_1^{(n)}, \alpha_1^{(n)}, \alpha_2^{(n)}, \alpha_3^{(n)})_{JMD},$$

$$\tilde{y}^{(n-1)} = (y_1^{(n-1)}, \alpha_1^{(n-1)}, \alpha_2^{(n-1)}, \alpha_3^{(n-1)})_{JMD}, ..., \tilde{y}^{(1)} = = (y_1^{(1)}, \alpha_1^{(1)}, \alpha_2^{(1)}, \alpha_3^{(1)})_{JMD}, \tilde{y} = (y_1, \alpha_1, \alpha_2, \alpha_3)_{JMD}$$

and

$$\tilde{\gamma}_0 = (\gamma_{0(1)}, \zeta_{0(1)}, \zeta_{0(2)}, \zeta_{0(3)})_{JMD}, \tilde{\gamma}_1 = (\gamma_{1(1)}, \zeta_{1(1)}, \zeta_{1(2)}, \zeta_{1(3)})_{JMD}, ...,$$

$$\tilde{\gamma}_{n-1} = (\gamma_{n-1(1)}, \zeta_{n-1(1)}, \zeta_{n-1(2)}, \zeta_{n-1(3)})_{JMD}$$

Tab. 3. Solution of fuzzy Kolmogorov's differential equations for sub-system R_1 obtained by using the existing method [14]

	$\tilde{p}_j(t)$ for $\alpha = 0$		$\tilde{p}_j(t)$ for $\alpha = 0.2$		$\tilde{p}_j(t)$ for $\alpha = 0.4$		$\tilde{p}_j(t)$ for $\alpha = 0.6$		$\tilde{p}_j(t)$ for $\alpha = 0.8$		$\tilde{p}_j(t)$ for $\alpha = 1$	
j	$\tilde{p}_{j1}(t,\alpha)$	$\tilde{p}_{j2}(t,\alpha)$	$\tilde{p}_{j1}(t,\alpha)$	$\tilde{p}_{j2}(t,\alpha)$	$\tilde{p}_{j1}(t,\alpha)$	$\tilde{p}_{j2}(t,\alpha)$	$\tilde{p}_{j1}(t,\alpha)$	$\tilde{p}_{j2}(t,\alpha)$	$\tilde{p}_{j1}(t,\alpha)$	$\tilde{p}_{j2}(t,\alpha)$	$\tilde{p}_{j1}(t,\alpha)$	$\tilde{p}_{j2}(t,\alpha)$
1	0.525421	0.396102	0.515146	0.401844	0.504872	0.407586	0.494597	0.413328	0.484323	0.419071	0.474049	0.424813
2	0.055996	0.065514	0.05667	0.065057	0.057344	0.064601	0.058018	0.064144	0.058692	0.063687	0.059366	0.063231
3	0.032869	0.036752	0.033158	0.036576	0.033447	0.036401	0.033737	0.036226	0.034026	0.036051	0.034316	0.035876
4	0.306812	0.424067	0.315888	0.419049	0.324964	0.414032	0.33404	0.409015	0.343116	0.403998	0.352192	0.398981
5	0.000537	0.000611	0.000542	0.000607	0.000547	0.000604	0.000553	0.000601	0.000558	0.000597	0.000564	0.000594
6	0.005867	0.006627	0.005923	0.006592	0.00598	0.006557	0.006036	0.006522	0.006093	0.006487	0.00615	0.006453
7	0.000747	0.000849	0.000754	0.000844	0.000761	0.000839	0.000769	0.000834	0.000776	0.000829	0.000784	0.000825
8	0.001307	0.001485	0.00132	0.001476	0.001333	0.001468	0.001346	0.001459	0.001359	0.001451	0.001372	0.001443
9	0.000057	0.000101	0.000059	0.000098	0.000062	0.000095	0.000065	0.000093	0.000068	0.000091	0.000071	0.000088
10	0.000623	0.001093	0.000651	0.001065	0.000681	0.001038	0.000709	0.001011	0.000738	0.000984	0.000767	0.000957
11	0.000079	0.00014	0.000082	0.000136	0.000086	0.000133	0.00009	0.000129	0.000094	0.000126	0.000098	0.000122
12	0.000139	0.000245	0.000145	0.000238	0.000151	0.000232	0.000158	0.000226	0.000164	0.00022	0.000171	0.000214
13	0.000033	0.000056	0.000034	0.000054	0.000036	0.000053	0.000037	0.000052	0.000039	0.000051	0.000041	0.00005
14	0.000366	0.000614	0.000381	0.0006	0.000397	0.000586	0.000412	0.000572	0.000428	0.000558	0.000444	0.000544
15	0.000046	0.000078	0.000048	0.000076	0.00005	0.000074	0.000052	0.000072	0.000054	0.000071	0.000056	0.000069
16	0.000081	0.000137	0.000084	0.000134	0.000088	0.000131	0.000091	0.000127	0.000095	0.000124	0.000099	0.000121
17	0.000679	0.001193	0.00071	0.001163	0.000741	0.001133	0.000773	0.001103	0.000804	0.001073	0.000836	0.001044
18	0.00117	0.001967	0.00122	0.001921	0.00127	0.001876	0.00132	0.001831	0.00137	0.001786	0.00142	0.001741
19	0.003628	0.007661	0.003876	0.007431	0.004125	0.007201	0.004373	0.006972	0.004622	0.006742	0.004871	0.006513
20	0.010731	0.022557	0.01146	0.021884	0.01219	0.021211	0.012919	0.020538	0.013649	0.019865	0.014379	0.019193
21	0.000313	0.000654	0.000334	0.000634	0.000355	0.000615	0.000376	0.000595	0.000397	0.000576	0.000419	0.000557
22	0.003293	0.006991	0.00352	0.00678	0.003748	0.006569	0.003975	0.006358	0.004203	0.006147	0.004431	0.005936
23	0.000435	0.000908	0.000464	0.000881	0.000493	0.000853	0.000522	0.000826	0.000551	0.000799	0.000581	0.000772
24	0.000759	0.001587	0.00081	0.00154	0.000861	0.001493	0.000912	0.001446	0.000963	0.001399	0.001015	0.001352

Tab. 4. Solution of fuzzy Kolmogorov's differential equations for sub-system R_2 obtained by using the existing method [14]

	$\tilde{p}_j(t)$ for $\alpha = 0$		$\tilde{p}_j(t)$ for $\alpha = 0.2$		$\tilde{p}_j(t)$ for $\alpha = 0.4$		$\tilde{p}_j(t)$ for $\alpha = 0.6$		$\tilde{p}_j(t)$ for $\alpha = 0.8$		$\tilde{p}_j(t)$ for $\alpha = 1$	
j	$\tilde{p}_{j1}(t,\alpha)$	$\tilde{p}_{j2}(t,\alpha)$	$\tilde{p}_{j1}(t,\alpha)$	$\tilde{p}_{j2}(t,\alpha)$	$\tilde{p}_{j1}(t,\alpha)$	$\tilde{p}_{j2}(t,\alpha)$	$\tilde{p}_{j1}(t,\alpha)$	$\tilde{p}_{j2}(t,\alpha)$	$\tilde{p}_{j1}(t,\alpha)$	$\tilde{p}_{j2}(t,\alpha)$	$\tilde{p}_{j1}(t,\alpha)$	$\tilde{p}_{j2}(t,\alpha)$
1	0.87807	0.862585	0.876803	0.86353	0.875536	0.864525	0.87427	0.86552	0.873003	0.866515	0.871737	0.867511
2	0.064038	0.094192	0.065987	0.092551	0.067937	0.09091	0.069887	0.08927	0.071837	0.087629	0.073787	0.085989
3	0.002941	0.004356	0.003032	0.004278	0.003123	0.004201	0.003214	0.004123	0.003305	0.004046	0.003396	0.003969
4	0.000425	0.000624	0.000436	0.000611	0.000447	0.000598	0.000458	0.000586	0.000469	0.000573	0.00048	0.000561
5	0.00011	0.000159	0.000111	0.000155	0.000113	0.000151	0.000115	0.000147	0.000117	0.000143	0.000119	0.000139
6	0.006073	0.008919	0.00623	0.008739	0.006388	0.008559	0.006546	0.008379	0.006704	0.008199	0.006862	0.008019
7	0.000024	0.000035	0.000024	0.000034	0.000024	0.000033	0.000025	0.000032	0.000025	0.000031	0.000026	0.000031
8	0.000228	0.000475	0.000229	0.000459	0.000243	0.000442	0.000257	0.000425	0.000272	0.000409	0.000287	0.000393
9	0.000031	0.000068	0.000033	0.000065	0.000035	0.000062	0.000037	0.00006	0.000039	0.000057	0.000041	0.000055
0	0.000008	0.000017	0.000008	0.000016	0.000009	0.000015	0.000009	0.000014	0.00001	0.000013	0.00001	0.000013
11	0.000442	0.000973	0.000469	0.000937	0.000497	0.000901	0.000525	0.000865	0.000553	0.000829	0.000581	0.000794
12	0.000001	0.000004	0.000001	0.000003	0.000001	0.000003	0.000002	0.000003	0.000002	0.000003	0.000002	0.000003
13	0.001617	0.003588	0.001727	0.003463	0.001837	0.003339	0.001947	0.003215	0.002057	0.003091	0.002168	0.002967

the fuzzy linear differential equation (1), can be written as

$$(a_{n(1)},\beta_{n(1)},\beta_{n(2)},\beta_{n(3)})_{JMD}\otimes(y_1^{(n)},\alpha_1^{(n)},\alpha_2^{(n)},\alpha_3^{(n)})_{JMD}\oplus(a_{n-1(1)},\beta_{n-1(1)},\beta_{n-1(2)},\beta_{n-1(3)})_{JMD}\otimes$$
$$(y_1^{(n-1)},\alpha_1^{(n-1)},\alpha_2^{(n-1)},\alpha_3^{(n-1)})_{JMD}\oplus\ldots\oplus(a_{1(1)},\beta_{1(1)},\beta_{1(2)},\beta_{1(3)})_{JMD}\otimes(y_1^{(1)},\alpha_1^{(1)},\alpha_2^{(1)},\alpha_3^{(1)})_{JMD}\oplus$$
$$(a_{0(1)},\beta_{0(1)},\beta_{0(2)},\beta_{0(3)})_{JMD}\otimes(y_1,\alpha_1,\alpha_2,\alpha_3)_{JMD}=(g,0,0,0)_{JMD}$$

$$(y_1,\alpha_1,\alpha_2,\alpha_3)_{JMD}=(\gamma_{0(1)},\zeta_{0(1)},\zeta_{0(2)},\zeta_{0(3)})_{JMD},(y_1^{(1)},\alpha_1^{(1)},\alpha_2^{(1)},\alpha_3^{(1)})_{JMD}=(\gamma_{1(1)},\zeta_{1(1)},\zeta_{1(2)},\zeta_{1(3)})_{JMD},\ldots,$$
$$(y_1^{(n-1)},\alpha_1^{(n-1)},\alpha_2^{(n-1)},\alpha_3^{(n-1)})_{JMD}=(\gamma_{n-1(1)},\zeta_{n-1(1)},\zeta_{n-1(2)},\zeta_{n-1(3)})_{JMD}$$

Step 2: Using Definition 2.11 and Section 2.2.3, the fuzzy differential equation, obtained from Step 1, is converted into the following four crisp linear differential equations

$$b_nz^{(n)}+b_{n-1}z^{(n-1)}+\ldots+b_1z^{(1)}+b_0z=g \qquad (2)$$
$$y_1=\gamma_{0(1)},y_1^{(1)}=\gamma_{1(1)},\ldots,y_1^{(n-1)}=\gamma_{n-1(1)}$$
$$c_nz^{(n)}+c_{n-1}z^{(n-1)}+\ldots+c_1z^{(1)}+c_0z=0 \qquad (3)$$
$$\alpha_1=\zeta_{0(1)},\alpha_1^{(1)}=\zeta_{1(1)},\ldots,\alpha_1^{(n-1)}=\zeta_{n-1(1)}$$
$$d_nz^{(n)}+d_{n-1}z^{(n-1)}+\ldots+d_1z^{(1)}+d_0z=0 \qquad (4)$$
$$\alpha_2=\zeta_{0(2)},\alpha_2^{(1)}=\zeta_{1(2)},\ldots,\alpha_2^{(n-1)}=\zeta_{n-1(2)}$$
$$e_nz^{(n)}+e_{n-1}z^{(n-1)}+\ldots+e_1z^{(1)}+e_0z=0 \qquad (5)$$
$$\alpha_3=\zeta_{0(3)},\alpha_3^{(1)}=\zeta_{1(3)},\ldots,\alpha_3^{(n-1)}=\zeta_{n-1(3)}$$

where,

$$b_iz^{(i)}=\text{minimum }(a_{i(1)}y_1^{(i)},a_{i(1)}y_1^{(i)}+a_{i(1)}\alpha_1^{(i)}+a_{i(1)}\alpha_2^{(i)}+$$
$$+a_{i(1)}\alpha_3^{(i)},a_{i(1)}y_1^{(i)}+\beta_{i(1)}y_1^{(i)}+\beta_{i(2)}y_1^{(i)}+\beta_{i(3)}y_1^{(i)},$$
$$a_{i(1)}y_1^{(i)}+a_{i(1)}\alpha_1^{(i)}+a_{i(1)}\alpha_2^{(i)}+a_{i(1)}\alpha_3^{(i)}+\beta_{i(1)}y_1^{(i)}+\beta_{i(1)}\alpha_1^{(i)}+$$
$$+\beta_{i(1)}\alpha_2^{(i)}+\beta_{i(1)}\alpha_3^{(i)}+\beta_{i(2)}y_1^{(i)}+\beta_{i(2)}\alpha_1^{(i)}+$$
$$\beta_{i(2)}\alpha_2^{(i)}+\beta_{i(2)}\alpha_3^{(i)}+\beta_{i(3)}y_1^{(i)}+\beta_{i(3)}\alpha_1^{(i)}+\beta_{i(3)}\alpha_2^{(i)}+\beta_{i(3)}\alpha_3^{(i)}),$$
$$c_iz^{(i)}=\text{minimum }(a_{i(1)}y_1^{(i)}+a_{i(1)}\alpha_1^{(i)}+\beta_{i(1)}y_1^{(i)}+\beta_{i(1)}\alpha_1^{(i)},a_{i(1)}y_1^{(i)}$$
$$+a_{i(1)}\alpha_1^{(i)}+a_{i(1)}\alpha_2^{(i)}+\beta_{i(1)}y_1^{(i)}+\beta_{i(1)}\alpha_1^{(i)}+\beta_{i(1)}\alpha_2^{(i)},a_{i(1)}y_1^{(i)}+a_{i(1)}\alpha_1^{(i)}+$$
$$\beta_{i(1)}y_1^{(i)}+\beta_{i(1)}\alpha_1^{(i)}+\beta_{i(2)}y_1^{(i)}+\beta_{i(2)}\alpha_1^{(i)},a_{i(1)}y_1^{(i)}+a_{i(1)}\alpha_1^{(i)}+$$
$$a_{i(1)}\alpha_2^{(i)}+\beta_{i(1)}y_1^{(i)}+\beta_{i(1)}\alpha_1^{(i)}+\beta_{i(1)}\alpha_2^{(i)}+$$
$$\beta_{i(2)}y_1^{(i)}+\beta_{i(2)}\alpha_1^{(i)}+\beta_{i(2)}\alpha_2^{(i)})-b_iz^{(i)},\ d_iz^{(i)}=\text{maximum}$$
$$(a_{i(1)}y_1^{(i)}+a_{i(1)}\alpha_1^{(i)}+\beta_{i(1)}y_1^{(i)}+\beta_{i(1)}\alpha_1^{(i)},$$
$$a_{i(1)}y_1^{(i)}+a_{i(1)}\alpha_1^{(i)}+a_{i(1)}\alpha_2^{(i)}+\beta_{i(1)}y_1^{(i)}+\beta_{i(1)}\alpha_1^{(i)}+$$
$$\beta_{i(1)}\alpha_2^{(i)},a_{i(1)}y_1^{(i)}+a_{i(1)}\alpha_1^{(i)}+\beta_{i(1)}y_1^{(i)}+\beta_{i(1)}\alpha_1^{(i)}+$$
$$\beta_{i(2)}y_1^{(i)}+\beta_{i(2)}\alpha_1^{(i)},a_{i(1)}y_1^{(i)}+a_{i(1)}\alpha_1^{(i)}+a_{i(1)}\alpha_2^{(i)}+\beta_{i(1)}y_1^{(i)}+$$
$$\beta_{i(1)}\alpha_1^{(i)}+\beta_{i(1)}\alpha_2^{(i)}+\beta_{i(2)}y_1^{(i)}+\beta_{i(2)}\alpha_1^{(i)}+\beta_{i(2)}\alpha_2^{(i)}-b_iz^{(i)}-c_iz^{(i)},$$
$$e_iz^{(i)}=\text{maximum}$$
$$(a_{i(1)}y_1^{(i)},a_{i(1)}y_1^{(i)}+a_{i(1)}\alpha_1^{(i)}+a_{i(1)}\alpha_2^{(i)}+a_{i(1)}\alpha_3^{(i)},a_{i(1)}y_1^{(i)}+$$
$$\beta_{i(1)}y_1^{(i)}+\beta_{i(2)}y_1^{(i)}+\beta_{i(3)}y_1^{(i)},a_{i(1)}y_1^{(i)}+a_{i(1)}\alpha_1^{(i)}+a_{i(1)}\alpha_2^{(i)}+$$
$$a_{i(1)}\alpha_3^{(i)}+\beta_{i(1)}y_1^{(i)}+\beta_{i(1)}\alpha_1^{(i)}+\beta_{i(1)}\alpha_2^{(i)}+$$
$$\beta_{i(1)}\alpha_3^{(i)}+\beta_{i(2)}y_1^{(i)}+\beta_{i(2)}\alpha_1^{(i)}+\beta_{i(2)}\alpha_2^{(i)}+\beta_{i(2)}\alpha_3^{(i)}+\beta_{i(3)}y_1^{(i)}+$$
$$\beta_{i(3)}\alpha_1^{(i)}+\beta_{i(3)}\alpha_2^{(i)}+\beta_{i(3)}\alpha_3^{(i)})-b_iz^{(i)}$$
$$-c_iz^{(i)}-d_iz^{(i)}\quad\text{for } i=n,n\text{-}1,\ldots,1,0$$

Step 3. Solve the ordinary differential equations (2) to (5), obtained from Step 2, to find the values of y_1, α_1, α_2, and α_3.
Step 4. Put the values of $y_1,\alpha_1,\alpha_2,$ and α_3 in $\tilde{y}=(y_1,\alpha_1,\alpha_2,\alpha_3)_{JMD}$ to find the solution of fuzzy differential equation (1).
Step 5. Convert $\tilde{y}=(y_1,\alpha_1,\alpha_2,\alpha_3)_{JMD}$ into $\tilde{y}=(a,b,c,d)$, where, $a=y_1$, $b=y_1+\alpha_1$, $c=y_1+\alpha_1+\alpha_2$ and $d=y_1+\alpha_1+\alpha_2+\alpha_3$.

Remark 5.1. If $(y_1^{(i)},\alpha_1^{(i)},\alpha_2^{(i)},\alpha_3^{(i)})_{\text{JMD}}$ is non negative *JMD* trapezoidal fuzzy number, then the values of $b_iz^{(i)}$, $c_iz^{(i)}$, $d_iz^{(i)}$ and $e_iz^{(i)}$, defined in Step 2, can be written as

$$b_iz^{(i)}=\begin{cases}a_{i(1)}y_1^{(i)}, & a_{i(1)}\geq 0\\ a_{i(1)}y_1^{(i)}+a_{i(1)}\alpha_1^{(i)}+a_{i(1)}\alpha_2^{(i)}+a_{i(1)}\alpha_3^{(i)}, & \text{otherwise}\end{cases}$$

$$c_iz^{(i)}=\begin{cases}a_{i(1)}\alpha_1^{(i)}+\beta_{i(1)}y_1^{(i)}+\beta_{i(1)}\alpha_1^{(i)}, & a_{i(1)}\geq 0\\ \beta_{i(1)}y_1^{(i)}+\beta_{i(1)}\alpha_1^{(i)}-a_{i(1)}\alpha_2^{(i)}-a_{i(1)}\alpha_3^{(i)}, & a_{i(1)}\leq 0 \text{ and } a_{i(1)}+\beta_{i(1)}\geq 0\\ \beta_{i(1)}y_1^{(i)}+\beta_{i(1)}\alpha_1^{(i)}+\beta_{i(1)}\alpha_2^{(i)}-a_{i(1)}\alpha_3^{(i)}, & \text{otherwise.}\end{cases}$$

$$d_iz^{(i)}=\begin{cases}a_{i(1)}\alpha_2^{(i)}+\beta_{i(1)}\alpha_2^{(i)}+\beta_{i(2)}y_1^{(i)}+\beta_{i(2)}\alpha_1^{(i)}+\beta_{i(2)}\alpha_2^{(i)}, & a_{i(1)}+\beta_{i(1)}\geq 0\\ \beta_{i(2)}y_1^{(i)}+\beta_{i(2)}\alpha_1^{(i)}+\beta_{i(2)}\alpha_2^{(i)}, & a_{i(1)}+\beta_{i(1)}\leq 0 \text{ and } a_{i(1)}+\beta_{i(1)}+\beta_{i(2)}\geq 0\\ \beta_{i(2)}y_1^{(i)}+\beta_{i(2)}\alpha_1^{(i)}-a_{i(1)}\alpha_2^{(i)}-\beta_{i(1)}\alpha_2^{(i)}, & \text{otherwise.}\end{cases}$$

and

$$e_iz^{(i)}=\begin{cases}a_{i(1)}\alpha_3^{(i)}+\beta_{i(1)}\alpha_3^{(i)}+\beta_{i(2)}\alpha_3^{(i)}+\beta_{i(3)}y_1^{(i)}+\beta_{i(3)}\alpha_1^{(i)}+\\ \beta_{i(3)}\alpha_2^{(i)}+\beta_{i(3)}\alpha_3^{(i)}, & a_{i(1)}+\beta_{i(1)}+\beta_{i(2)}\geq 0\\ a_{i(1)}\alpha_2^{(i)}+a_{i(1)}\alpha_3^{(i)}+\beta_{i(1)}\alpha_2^{(i)}+\beta_{i(1)}\alpha_3^{(i)}+\\ \beta_{i(2)}\alpha_2^{(i)}+\beta_{i(2)}\alpha_3^{(i)}+\beta_{i(3)}y_1^{(i)}+\beta_{i(3)}\alpha_1^{(i)}+\\ \beta_{i(3)}\alpha_2^{(i)}+\beta_{i(3)}\alpha_3^{(i)}, & a_{i(1)}+\beta_{i(1)}+\beta_{i(2)}\leq 0 \text{ and } a_{i(1)}+\beta_{i(1)}+\beta_{i(2)}+\beta_{i(3)}\geq 0\\ a_{i(1)}\alpha_2^{(i)}+a_{i(1)}\alpha_3^{(i)}-\beta_{i(1)}y_1^{(i)}-\beta_{i(1)}\alpha_1^{(i)}-\\ \beta_{i(2)}y_1^{(i)}-\beta_{i(2)}\alpha_1^{(i)}, & \text{otherwise.}\end{cases}$$

for $i=n,n$-1,...,1,0

6. Advantages of Mehar's method

The main advantages of Mehar's method over existing method [14] is that on solving the fuzzy differential equations by using the existing method, the obtained results may or may not define the α-cut of a fuzzy number while on solving the fuzzy differential equations by using Mehar's method, the obtained solution are always fuzzy number.

In this section, to show the advantages of Mehar's method over existing method, the fuzzy Kolmogorov's differential equations for the sub-systems R_1 and R_2, developed in Section 4.5, for which by using the existing method the obtained results are not appropriate, are solved by using Mehar's method, proposed in Section 5 and the obtained results, after converting the obtained *JMD* trapezoidal fuzzy number $(x,\alpha,\beta,\gamma)_{JMD}$ into existing representation of trapezoidal fuzzy number (a,b,c,d), are shown in table 5 and table 6 respectively.

It is obvious from table 5 and table 6, that $(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$ defines a trapezoidal fuzzy number $\tilde{p}_j(t)$ for j = 1 to 24 and j = 1 to 13 respectively.

7. Advantages of *JMD* representation of trapezoidal fuzzy numbers over existing representation of trapezoidal fuzzy numbers

Kumar and Kaur [28] shown that it is better to use the proposed representation of trapezoidal fuzzy numbers instead of existing representation of trapezoidal fuzzy numbers for finding the fuzzy optimal solution of fuzzy transportation problems. In this section, it is shown that it is also better to use *JMD* representation of trapezoidal fuzzy numbers for solving fuzzy differential equations as compared to existing representation of trapezoidal fuzzy numbers.

To show the advantage of *JMD* representation of trapezoidal fuzzy numbers over existing representation of trapezoidal fuzzy numbers, the fuzzy Kolomorgov's differential equations, for the sub-systems R_1 and R_2 developed in Section 4.5, are solved by using Mehar's method with existing representation of trapezoidal fuzzy numbers and the obtained results are shown in table 7 and table 8 respectively.

It is obvious from table 7 and table 8, that $(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$ does not define a trapezoidal fuzzy number for j = 1.

Tab. 5. Solution of fuzzy Kolmogorov's differential equations for sub-system R_1 obtained by using Mehar's method

	$\tilde{p}_j(t)$ for t=60	$\tilde{p}_j(t)$ for t=120	$\tilde{p}_j(t)$ for t=180	$\tilde{p}_j(t)$ for t=240	$\tilde{p}_j(t)$ for t=300	$\tilde{p}_j(t)$ for t=360
j	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$
1	(0.720279,0.724095, 0.731181,0.734988)	(0.652165,0.655343, 0.66104,0.664203)	(0.603616,0.606389, 0.611232,0.613988)	(0.568691,0.571173, 0.5754,0.577863)	(0.543538,0.54602, 0.550247,0.55271)	(0.525421,0.527484, 0.530833,0.532877)
2	(0.071153,0.072092, 0.074107,0.075045)	(0.065846,0.066938, 0.069151,0.070288)	(0.062073,0.063193, 0.065337,0.066453)	(0.059359,0.060453, 0.06345,0.063539)	(0.057404,0.058498, 0.060495,0.061584)	(0.055996,0.056988, 0.058685,0.059671)
3	(0.044381,0.045198, 0.046916,0.047741)	(0.040352,0.041218, 0.042925,0.043805)	(0.037486,0.038314, 0.039853,0.040364)	(0.035423,0.036189, 0.037551,0.038333)	(0.033938,0.034704, 0.036066,0.036848)	(0.032869,0.033514, 0.034591,0.03525)
4	(0.096162,0.096893, 0.098549,0.099281)	(0.169323,0.172501, 0.174963,0.175998)	(0.221951,0.223285, 0.22648,0.227818)	(0.259851,0.261463, 0.265271,0.266889)	(0.287149,0.28731, 0.291118,0.292736)	(0.306812,0.308887, 0.313591,0.315675)
5	(0.000738,0.000761, 0.000807,0.00083)	(0.000688,0.00069, 0.000729,0.00075)	(0.000618,0.000637, 0.00067,0.000689)	(0.000582,0.000599, 0.000627,0.000643)	(0.000556,0.000573, 0.000601,0.000617)	(0.000537,0.000551, 0.000573,0.000586)
6	(0.007563,0.007585, 0.007641,0.007663)	(0.007414,0.007452, 0.007546,0.007584)	(0.006856,0.006907, 0.007028,0.007079)	(0.006417,0.006478, 0.006618,0.006679)	(0.006097,0.006158, 0.006298,0.006359)	(0.005867,0.005941, 0.006103,0.006177)
7	(0.001027,0.001049, 0.0011,0.001122)	(0.000929,0.000956, 0.00101,0.001037)	(0.000859,0.000885, 0.000933,0.000959)	(0.000809,0.000833, 0.000875,0.000899)	(0.000773,0.000797, 0.000839,0.000863)	(0.000747,0.000768, 0.000801,0.000821)
8	(0.001802,0.001829, 0.001915,0.001942)	(0.001629,0.001667, 0.001813,0.00185)	(0.001505,0.001546, 0.001737,0.001778)	(0.001417,0.001458, 0.001684,0.001725)	(0.001353,0.001394, 0.00162,0.001661)	(0.001307,0.001344, 0.001616,0.001652)
9	(0.000072,0.000076, 0.000087,0.000091)	(0.000067,0.000073, 0.000086,0.000092)	(0.000063,0.00007, 0.000084,0.000091)	(0.000061,0.000068, 0.000081,0.000088)	(0.000058,0.000065, 0.000078,0.000085)	(0.000057,0.000063, 0.000074,0.00008)
10	(0.000719,0.000722, 0.000733,0.000736)	(0.00074,0.000749, 0.000773,0.000781)	(0.000699,0.000713, 0.00075,0.000764)	(0.000665,0.000683, 0.00073,0.000748)	(0.000641,0.000659, 0.000706,0.000724)	(0.000623,0.000649, 0.000711,0.000737)
11	(0.000101,0.000105, 0.000115,0.000119)	(0.000093,0.0001, 0.000116,0.000123)	(0.000088,0.000096, 0.000114,0.000122)	(0.000084,0.000093, 0.000111,0.00012)	(0.000081,0.00009, 0.000108,0.000117)	(0.000079,0.000088, 0.000104,0.000109)
12	(0.000177,0.000181, 0.000197,0.000201)	(0.000164,0.000173, 0.000211,0.00022)	(0.000154,0.000166, 0.000224,0.000236)	(0.000147,0.000161, 0.000237,0.000251)	(0.000142,0.000156, 0.000232,0.000246)	(0.000139,0.000154, 0.000257,0.000272)
13	(0.000045,0.000049, 0.000058,0.000062)	(0.000041,0.000046, 0.000057,0.000062)	(0.000038,0.000043, 0.000053,0.000058)	(0.000036,0.000041, 0.00005,0.000055)	(0.000034,0.000039, 0.000048,0.000053)	(0.000033,0.000037, 0.000044,0.000048)
14	(0.000457,0.00046, 0.00047,0.000473)	(0.000457,0.000464, 0.000485,0.000492)	(0.000425,0.000436, 0.000466,0.000477)	(0.000399,0.000413, 0.00045,0.000465)	(0.00038,0.000394, 0.000431,0.000446)	(0.000366,0.000385, 0.00043,0.00045)
15	(0.000063,0.000066, 0.000075,0.000078)	(0.000057,0.000063, 0.000076,0.000082)	(0.000053,0.00006, 0.000074,0.000081)	(0.00005,0.000057, 0.00007,0.000077)	(0.000048,0.000055, 0.000068,0.000075)	(0.000046,0.000052, 0.000062,0.000068)
16	(0.000111,0.000115, 0.00013,0.000134)	(0.000101,0.000109, 0.000141,0.000149)	(0.000093,0.000103, 0.00015,0.00016)	(0.000088,0.000099, 0.000158,0.000169)	(0.000084,0.000095, 0.000154,0.000167)	(0.000081,0.000092, 0.000168,0.000179)
17	(0.000835,0.00084, 0.000856,0.000861)	(0.000809,0.000822, 0.000857,0.00087)	(0.00076,0.00078, 0.000833,0.000853)	(0.000723,0.000749, 0.000816,0.000842)	(0.000697,0.000705, 0.000772,0.000798)	(0.000679,0.000715, 0.0008,0.000836)
18	(0.001591,0.001613, 0.001675,0.001697)	(0.001453,0.0015, 0.001624,0.001671)	(0.001345,0.001412, 0.001581,0.001649)	(0.001266,0.00135, 0.001548,0.001633)	(0.001211,0.001295, 0.001493,0.001578)	(0.00117,0.001273, 0.001495,0.0016)
19	(0.000835,0.000839, 0.000852,0.000865)	(0.001794,0.001804, 0.001836,0.001846)	(0.002496,0.002514, 0.00257,0.002589)	(0.003002,0.00303, 0.003114,0.003142)	(0.003366,0.003394, 0.003478,0.003506)	(0.003628,0.003678, 0.003822,0.003872)
20	(0.002753,0.002772, 0.002827,0.002846)	(0.00552,0.00597, 0.00729,0.00774)	(0.007515,0.007592, 0.007819,0.007896)	(0.008951,0.009065, 0.009398,0.009512)	(0.009986,0.0101, 0.010433,0.010547)	(0.010731,0.010927, 0.011477,0.011673)
21	(0.000096,0.000099, 0.000107,0.00011)	(0.000172,0.000177, 0.00019,0.000195)	(0.000226,0.000233, 0.000251,0.000258)	(0.000265,0.000274, 0.000297,0.000306)	(0.000293,0.000302, 0.000325,0.000334)	(0.000313,0.000325, 0.000354,0.000366)
22	(0.000672,0.000675, 0.000683,0.000686)	(0.001549,0.001556, 0.001578,0.001585)	(0.002214,0.002227, 0.002266,0.002279)	(0.002696,0.002945, 0.003004,0.003023)	(0.003043,0.003062, 0.003121,0.00314)	(0.003293,0.003328, 0.003432,0.003467)
23	(0.000132,0.000135, 0.000143,0.000146)	(0.000237,0.000243, 0.000259,0.000265)	(0.000313,0.000321, 0.000344,0.000352)	(0.000367,0.000378, 0.000407,0.000418)	(0.000406,0.000417, 0.000446,0.000457)	(0.000435,0.000451, 0.00049,0.000506)
24	(0.000225,0.000228, 0.000241,0.000244)	(0.000411,0.000418, 0.000448,0.000455)	(0.000544,0.000556, 0.000617,0.000629)	(0.000641,0.000657, 0.000751,0.000767)	(0.000709,0.000725, 0.000819,0.000835)	(0.000759,0.000784, 0.000952,0.000977)

Tab. 6. Solution of fuzzy Kolmogorov's differential equations for sub-system R_2 obtained by using Mehar's method

	$\tilde{p}_j(t)$ for t=60	$\tilde{p}_j(t)$ for t=120	$\tilde{p}_j(t)$ for t=180	$\tilde{p}_j(t)$ for t=240	$\tilde{p}_j(t)$ for t=300	$\tilde{p}_j(t)$ for t=360
j	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$
1	(0.878882,0.883263, 0.891744,0.896118)	(0.878179,0.882218, 0.890005,0.894036)	(0.878084,0.889381, 0.889381,0.893216)	(0.878072,0.881804, 0.889083,0.892803)	(0.878071,0.881734, 0.888916,0.892565)	(0.87807,0.881689, 0.888812,0.892416)
2	(0.064096,0.065113, 0.067364,0.06838)	(0.064046,0.065329, 0.068091,0.069372)	(0.064039,0.065457, 0.068421,0.069835)	(0.064038,0.065522, 0.068559,0.070038)	(0.064038,0.065554, 0.068611,0.070121)	(0.064038,0.065567, 0.068624,0.070146)
3	(0.002943,0.002986, 0.00309,0.003133)	(0.002941,0.003004, 0.003145,0.003208)	(0.002941,0.003013, 0.003164,0.003236)	(0.002941,0.003016, 0.003169,0.003244)	(0.002941,0.003018, 0.003171,0.003247)	(0.002941,0.003018, 0.00317,0.003246)
4	(0.000425,0.000432, 0.00045,0.000459)	(0.000425,0.000434, 0.000456,0.000467)	(0.000425,0.000435, 0.000457,0.000469)	(0.000425,0.000435, 0.000457,0.000469)	(0.000425,0.000434, 0.000455,0.000467)	(0.000425,0.000434, 0.000455,0.000467)
5	(0.00011,0.000111, 0.000116,0.000119)	(0.00011,0.000111, 0.000116,0.000119)	(0.00011,0.000111, 0.000116,0.000119)	(0.00011,0.000111, 0.000116,0.000119)	(0.00011,0.000111, 0.000116,0.000119)	(0.00011,0.000111, 0.000116,0.000119)
6	(0.005269,0.00528, 0.005311,0.005324)	(0.005966,0.005985, 0.006041,0.006065)	(0.006059,0.006086, 0.006164,0.006198)	(0.006071,0.006105, 0.006202,0.006206)	(0.006073,0.006114, 0.006228,0.006279)	(0.006073,0.00612, 0.00625,0.006309)
7	(0.000024,0.000025, 0.000026,0.000027)	(0.000024,0.0000244, 0.0000254,0.000026)	(0.000024,0.0000244, 0.0000245,0.0000253)	(0.000024,0.0000244, 0.0000254,0.0000262)	(0.000024,0.0000244, 0.0000254,0.000026)	(0.000024,0.0000243, 0.000025,0.000026)
8	(0.000214,0.000221, 0.000242,0.000249)	(0.000214,0.000229, 0.000268,0.000283)	(0.000214,0.000235, 0.000286,0.000307)	(0.000214,0.000239, 0.000297,0.000322)	(0.000214,0.000242, 0.000303,0.000331)	(0.000214,0.000244, 0.000307,0.000337)
9	(0.000031,0.000032, 0.000035,0.000036)	(0.000031,0.000033, 0.000039,0.000042)	(0.000031,0.000034, 0.000042,0.000045)	(0.000031,0.000034, 0.000042,0.000046)	(0.000031,0.000034, 0.000043,0.000047)	(0.000031,0.000034, 0.000043,0.000047)
10	(0.000008,0.0000082, 0.0000093,0.0000098)	(0.000008,0.0000085, 0.00001,0.000011)	(0.000008,0.0000086, 0.0000106,0.0000116)	(0.000008,0.0000086, 0.0000106,0.0000116)	(0.000008,0.000009, 0.000011,0.000012)	(0.000008,0.000009, 0.00003,0.000032)
11	(0.000373,0.000374, 0.000379,0.000381)	(0.000433,0.000437, 0.000451,0.000456)	(0.000441,0.000448, 0.000471,0.00048)	(0.000442,0.000452, 0.000483,0.000495)	(0.000442,0.000455, 0.000494,0.00051)	(0.000442,0.000457, 0.000503,0.000522)
12	(0.000001,0.0000021, 0.0000024,0.0000026)	(0.000001,0.0000011, 0.0000015,0.0000017)	(0.000001,0.0000011, 0.0000015,0.0000018)	(0.000001,0.0000011, 0.0000016,0.0000019)	(0.000001,0.0000011, 0.0000016,0.0000019)	(0.000001,0.0000011, 0.0000016,0.0000019)
13	(0.001617,0.001642, 0.001714,0.001739)	(0.001618,0.001676, 0.001835,0.001951)	(0.001617,0.001707, 0.001944,0.002034)	(0.001617,0.001736, 0.002037,0.002155)	(0.001617,0.00176, 0.002109,0.002252)	(0.001617,0.00178, 0.002165,0.002328)

Tab. 7. Solution of fuzzy Kolmogorov's differential equations for sub-system R_1 obtained by using Mehar's method with existing representation of trapezoidal fuzzy numbers

	$\tilde{p}_j(t)$ for t=60	$\tilde{p}_j(t)$ for t=120	$\tilde{p}_j(t)$ for t=180	$\tilde{p}_j(t)$ for t=240	$\tilde{p}_j(t)$ for t=300	$\tilde{p}_j(t)$ for t=360
j	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$
1	(0.720279,0.681444, 0.635114,0.600719)	(0.652165,0.600728, 0.541703,0.501324)	(0.603616,0.547756, 0.487123,0.448286)	(0.568691,0.512688, 0.454965,0.41975)	(0.543538,0.48945, 0.436,0.404381)	(0.525421,0.474049, 0.424813,0.396102)
2	(0.071153,0.077829, 0.084789,0.088344)	(0.065846,0.070633, 0.075201,0.077243)	(0.062073,0.065921, 0.069611,0.071331)	(0.059359,0.062802, 0.066318,0.06815)	(0.057404,0.060736, 0.064376,0.066437)	(0.055996,0.059366, 0.063231,0.065514)
3	(0.044381,0.048441, 0.052512,0.054488)	(0.040352,0.042938, 0.045117,0.045866)	(0.037486,0.039332, 0.040802,0.041272)	(0.035423,0.036946, 0.038259,0.038801)	(0.033938,0.035364, 0.036761,0.037469)	(0.032869,0.034316, 0.035876,0.036752)
4	(0.096162,0.125721, 0.166747,0.196529)	(0.169323,0.213389, 0.26945,0.306637)	(0.221951,0.271397, 0.329903,0.365802)	(0.259851,0.309835, 0.365552,0.397663)	(0.287149,0.335309, 0.386578,0.414824)	(0.306812,0.352192, 0.398981,0.424067)
5	(0.000738,0.000813, 0.00089,0.00093)	(0.000688,0.000716, 0.000758,0.000775)	(0.000618,0.000652, 0.000681,0.000692)	(0.000582,0.000611, 0.000636,0.000648)	(0.000556,0.000583, 0.000609,0.000624)	(0.000537,0.000564, 0.000594,0.000611)
6	(0.007563,0.008481, 0.009508,0.010101)	(0.007414,0.008024, 0.008591,0.008833)	(0.006856,0.007273, 0.007621,0.007747)	(0.006417,0.006741, 0.007021,0.007136)	(0.006097,0.006385, 0.006663,0.006805)	(0.005867,0.00615, 0.006453,0.006627)
7	(0.001027,0.001132, 0.001241,0.001296)	(0.000929,0.000996, 0.001056,0.001079)	(0.000859,0.000907, 0.000948,0.000963)	(0.000809,0.000849, 0.000884,0.000901)	(0.000773,0.00081, 0.000847,0.000867)	(0.000747,0.000784, 0.000825,0.000849)
8	(0.001802,0.001987, 0.002179,0.002279)	(0.001629,0.001747, 0.001852,0.001893)	(0.001505,0.001591, 0.001661,0.001687)	(0.001417,0.001486, 0.001548,0.001576)	(0.001353,0.001417, 0.001482,0.001517)	(0.001307,0.001372, 0.001443,0.001485)
9	(0.000072,0.000092, 0.000118,0.000136)	(0.000067,0.000084, 0.000105,0.000119)	(0.000063,0.000078, 0.000097,0.00011)	(0.000061,0.000074, 0.000092,0.000105)	(0.000058,0.000072, 0.00009,0.000102)	(0.000057,0.000071, 0.000088,0.000101)
10	(0.000719,0.000932, 0.001221,0.001426)	(0.00074,0.000931, 0.001173,0.001335)	(0.000699,0.000867, 0.001077,0.001218)	(0.000665,0.000819, 0.001015,0.00115)	(0.000641,0.000788, 0.000979,0.001113)	(0.000623,0.000767, 0.000957,0.001093)
11	(0.000101,0.000129, 0.0001651,0.00019)	(0.000093,0.000117, 0.000146,0.000166)	(0.000088,0.000109, 0.000135,0.000153)	(0.000084,0.000103, 0.000128,0.000146)	(0.000081,0.0001, 0.000125,0.000142)	(0.000079,0.000098, 0.000122,0.00014)
12	(0.000177,0.000226, 0.00029,0.000334)	(0.000164,0.000205, 0.000256,0.000291)	(0.000154,0.000191, 0.000237,0.000268)	(0.000147,0.000181, 0.000225,0.000255)	(0.000142,0.000175, 0.000218,0.000249)	(0.000139,0.000171, 0.000214,0.000245)
13	(0.000045,0.000057, 0.000073,0.000084)	(0.000041,0.000051, 0.000063,0.000071)	(0.000038,0.000046, 0.000057,0.000063)	(0.000036,0.000044, 0.000053,0.000059)	(0.000034,0.000042, 0.000051,0.000057)	(0.000033,0.000041, 0.00005,0.000056)
14	(0.000457,0.000592, 0.000772,0.000901)	(0.000457,0.000571, 0.000712,0.000804)	(0.000425,0.000521, 0.000636,0.000711)	(0.000399,0.000485, 0.000589,0.000658)	(0.00038,0.000461, 0.000561,0.00063)	(0.000366,0.000444, 0.000544,0.000614)
15	(0.000063,0.00008, 0.000102,0.000117)	(0.000057,0.000071, 0.000087,0.000098)	(0.000053,0.000065, 0.000079,0.000088)	(0.00005,0.000061, 0.000074,0.000083)	(0.000048,0.000058, 0.000071,0.00008)	(0.000046,0.000056, 0.000069,0.000078)
16	(0.000111,0.000141, 0.00018,0.000206)	(0.000101,0.000124, 0.000154,0.000173)	(0.000093,0.000114, 0.000139,0.000155)	(0.000088,0.000107, 0.00013,0.000145)	(0.000084,0.000102, 0.000124,0.000141)	(0.000081,0.000099, 0.000121,0.000137)
17	(0.000835,0.001076, 0.001398,0.001626)	(0.000809,0.001014, 0.001274,0.001447)	(0.00076,0.000941, 0.001167,0.00132)	(0.000723,0.000891, 0.001104,0.001251)	(0.000697,0.000858, 0.001066,0.001213)	(0.000679,0.000836, 0.001044,0.001193)
18	(0.001591,0.002036, 0.002617,0.003017)	(0.001453,0.001806, 0.002237,0.002515)	(0.001345,0.001644, 0.002005,0.002238)	(0.001266,0.001537, 0.001869,0.00209)	(0.001211,0.001467, 0.001788,0.00201)	(0.00117,0.00142, 0.001741,0.001967)
19	(0.000835,0.001296, 0.002071,0.002741)	(0.001794,0.002665, 0.004017,0.005103)	(0.002496,0.003587, 0.005182,0.006392)	(0.003002,0.004198, 0.005869,0.007086)	(0.003366,0.004603, 0.006274,0.007461)	(0.003628,0.004871, 0.006513,0.007661)
20	(0.002753,0.004246, 0.006735,0.008871)	(0.00552,0.008165, 0.012241,0.015492)	(0.007515,0.010762, 0.015486,0.019052)	(0.008951,0.012483, 0.017399,0.020969)	(0.009986,0.013623, 0.018527,0.022001)	(0.010731,0.014379, 0.019193,0.022557)
21	(0.000096,0.000147, 0.00023,0.000299)	(0.000172,0.000252, 0.000375,0.000471)	(0.000226,0.000322, 0.00046,0.000563)	(0.000265,0.000368, 0.00051,0.000613)	(0.000293,0.000398, 0.00054,0.000639)	(0.000313,0.000419, 0.000557,0.000654)
22	(0.000672,0.001047, 0.001684,0.00224)	(0.001549,0.002312, 0.003506,0.004472)	(0.002214,0.003194, 0.004636,0.005737)	(0.002696,0.003782, 0.005307,0.006422)	(0.003043,0.004172, 0.005703,0.006792)	(0.003293,0.004431, 0.005936,0.006991)
23	(0.000132,0.000201, 0.000315,0.000411)	(0.000237,0.000348, 0.000518,0.000651)	(0.000313,0.000445, 0.000637,0.000781)	(0.000367,0.000510, 0.000707,0.000851)	(0.000406,0.000553, 0.000749,0.000888)	(0.000435,0.000581, 0.000773,0.000908)
24	(0.000225,0.000344, 0.000539,0.000704)	(0.000411,0.000604, 0.000898,0.001131)	(0.000544,0.000776, 0.00111,0.001361)	(0.000641,0.000889, 0.001235,0.001484)	(0.000709,0.000965, 0.001308,0.001551)	(0.000759,0.001015, 0.001352,0.001587)

Tab. 8. Solution of fuzzy Kolmogorov's differential equations for sub-system R_2 obtained by using Mehar's method with existing representation of trapezoidal fuzzy numbers

	$\tilde{p}_j(t)$ for t=60	$\tilde{p}_j(t)$ for t=120	$\tilde{p}_j(t)$ for t=180	$\tilde{p}_j(t)$ for t=240	$\tilde{p}_j(t)$ for t=300	$\tilde{p}_j(t)$ for t=360
j	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$	$(p_{j1}(t), p_{j2}(t), p_{j3}(t), p_{j4}(t))$
1	(0.878882,0.872595, 0.868478,0.863537)	(0.878178,0.871845, 0.867666,0.862687)	(0.878084,0.871751, 0.867573,0.862596)	(0.878072,0.871739, 0.867563,0.862586)	(0.878071,0.871737, 0.867561,0.862585)	(0.87807,0.871737, 0.867561,0.862585)
2	(0.064096,0.073858, 0.086077,0.094293)	(0.064046,0.073796, 0.085999,0.094203)	(0.064039,0.073788, 0.085991,0.094193)	(0.064038,0.073788, 0.085989,0.094192)	(0.064038,0.073787, 0.085989,0.094192)	(0.064038,0.073787, 0.085989,0.094192)
3	(0.002943,0.0034, 0.003973,0.004361)	(0.002941,0.003396, 0.003969,0.004357)	(0.002941,0.003396, 0.003969,0.004356)	(0.002941,0.003396, 0.003969,0.004356)	(0.002941,0.003396, 0.003969,0.004356)	(0.002941,0.003396, 0.003969,0.004356)
4	(0.000425,0.000481, 0.000562,0.000625)	(0.000425,0.00048, 0.000561,0.000624)	(0.000425,0.00048, 0.000561,0.000624)	(0.000425,0.00048, 0.000561,0.000624)	(0.000425,0.00048, 0.000561,0.000624)	(0.000425,0.00048, 0.000561,0.000624)
5	(0.00011,0.000119, 0.000139,0.000159)	(0.00011,0.000119, 0.000139,0.000159)	(0.00011,0.000119, 0.000139,0.000159)	(0.00011,0.000119, 0.000139,0.000159)	(0.00011,0.000119, 0.000139,0.000159)	(0.00011,0.000119, 0.000139,0.000159)
6	(0.005369,0.006013, 0.007112,0.007977)	(0.005966,0.006755, 0.007915,0.008817)	(0.006059,0.006848, 0.008007,0.008908)	(0.006071,0.00686, 0.008018,0.008918)	(0.006073,0.006861, 0.008019,0.008919)	(0.006073,0.006862, 0.008019,0.008919)
7	(0.000024,0.000026, 0.000031,0.000035)	(0.000024,0.000026, 0.000031,0.000035)	(0.000024,0.000026, 0.000031,0.000035)	(0.000024,0.000026, 0.000031,0.000035)	(0.000024,0.000026, 0.000031,0.000035)	(0.000024,0.000026, 0.000031,0.000035)
8	(0.000214,0.000287, 0.000393,0.000476)	(0.000214,0.000287, 0.000393,0.000475)	(0.000214,0.000287, 0.000393,0.000475)	(0.000214,0.000287, 0.000393,0.000475)	(0.000214,0.000287, 0.000393,0.000475)	(0.000214,0.000287, 0.000393,0.000475)
9	(0.000031,0.000041, 0.000055,0.000068)	(0.000031,0.000041, 0.000055,0.000068)	(0.000031,0.000041, 0.000055,0.000068)	(0.000031,0.000041, 0.000055,0.000068)	(0.000031,0.000041, 0.000055,0.000068)	(0.000031,0.000041, 0.000055,0.000068)
10	(0.000008,0.00001, 0.000013,0.000017)	(0.000008,0.00001, 0.000013,0.000017)	(0.000008,0.00001, 0.000013,0.000017)	(0.000008,0.00001, 0.000013,0.000017)	(0.000008,0.00001, 0.000013,0.000017)	(0.000008,0.00001, 0.000013,0.000017)
11	(0.000373,0.000496, 0.000689,0.000852)	(0.000433,0.00057, 0.000782,0.000961)	(0.000441,0.000579, 0.000793,0.000972)	(0.000442,0.000581, 0.000794,0.000973)	(0.000442,0.000581, 0.000794,0.000973)	(0.000442,0.000581, 0.000794,0.000973)
12	(0.000001,0.000002, 0.000003,0.000004)	(0.000001,0.000002, 0.000003,0.000004)	(0.000001,0.000002, 0.000003,0.000004)	(0.000001,0.000002, 0.000003,0.000004)	(0.000001,0.000002, 0.000003,0.000004)	(0.000001,0.000002, 0.000003,0.000004)
13	(0.001617,0.002168, 0.002969,0.003591)	(0.001618,0.002168, 0.002967,0.003588)	(0.001617,0.002168, 0.002967,0.003588)	(0.001617,0.002168, 0.002967,0.003588)	(0.001617,0.002168, 0.002967,0.003588)	(0.001617,0.002168, 0.002967,0.003588)

8. Fuzzy reliability evaluation of piston manufacturing system

In Section 4.6, it is shown that the results of fuzzy Kolmogorov's differential equations, obtained by using the existing method, may or may not define the α -cut of a fuzzy number. Also, the results of fuzzy Kolmogorov's differential equations, obtained by using Mehar's method with existing representation of trapezoidal fuzzy number, shown in Table 7 and Table 8, may or may not be a fuzzy number. Due to which, the obtained results may not be used to analyze the fuzzy reliability of piston manufacturing system. But in Table 5 and Table 6, it is shown that if the same fuzzy Kolmogorov's differential equations are solved by using Mehar's method then the obtained results are fuzzy numbers. In this section, the results of fuzzy Kolmogorov's differential equations, shown in Table 5 and 6, obtained by using Mehar's method, are used to analyze the fuzzy reliability of piston manufacturing system.

Using the fuzzy probabilities for the sub-systems R_1 and R_2, shown in Table 5 and Table 6, the corresponding fuzzy reliabilities $\tilde{R}_1(t) = \sum_{i=1}^{4} \tilde{p}_i(t)$, $\tilde{R}_2(t) = \sum_{i=1}^{2} \tilde{p}_i(t)$ and $\tilde{R}(t) = \tilde{R}_1(t) \otimes \tilde{R}_2(t)$ i.e., $(R_{11}(t), R_{12}(t), R_{13}(t), R_{14}(t))$, $(R_{21}(t), R_{22}(t), R_{23}(t), R_{24}(t))$ and

$$(R_1(t), R_2(t), R_3(t), R_4(t)) = \\ = (R_{11}(t), R_{12}(t), R_{13}(t), R_{14}(t)) \otimes (R_{21}(t), R_{22}(t), R_{23}(t), R_{24}(t))$$

of sub-system R_1, R_2 and the whole system are computed respectively and are shown in Table 9. The variation in reliability of sub-system R_1, R_2 and the whole system corresponding to variation in time is shown in Figure 3 to Figure 5 respectively.

9. Conclusion

The shortcoming of an existing method for finding the exact solution of fuzzy differential equations is pointed out and to overcome the shortcoming a new method, named as Mehar's method, for solving fuzzy differential equations is proposed. Also, it is shown that the solution of fuzzy Kolomorgov's differential equations, obtained by using the existing method, may or may not be fuzzy number. So, the existing method can not be used to analyze the fuzzy reliability of piston manufacturing system, while the solution of fuzzy Kolomorgov's differential equations, obtained by using Mehar's method, are always fuzzy number. So, it is better to use Mehar's method for solving fuzzy differential equations as compared to existing method.

Tab. 9. Fuzzy reliability of sub-system R_1, R_2 and the whole system obtained by using Mehar's method

Fuzzy Reliability →	$\tilde{R}_1(t)$	$\tilde{R}_2(t)$	$\tilde{R}(t)$
Time (t) ↓	$(R_{11}(t), R_{12}(t), R_{13}(t), R_{14}(t))$	$(R_{21}(t), R_{22}(t), R_{23}(t), R_{24}(t))$	$(R_1(t), R_2(t), R_3(t), R_4(t))$
60	(0.931975,0.938278,0.950753,0.957055)	(0.942978,0.948376,0.959108,0.964498)	(0.878831,0.88984,0.911874,0.923077)
120	(0.927686,0.936,0.948079,0.954246)	(0.942224,0.947547,0.958096,0.963408)	(0.874088,0.886903,0.908351,0.919328)
180	(0.925126,0.931181,0.942902,0.948956)	(0.942123,0.947386,0.957802,0.960951)	(0.871582,0.882187,0.903113,0.9119)
240	(0.923384,0.929278,0.940672,0.946624)	(0.94211,0.947326,0.957642,0.962841)	(0.869929,0.880329,0.900827,0.911448)
300	(0.922029,0.926532,0.937926,0.943878)	(0.942109,0.947288,0.957527,0.962686)	(0.868651,0.877692,0.898089,0.908658)
360	(0.921098,0.926873,0.9377,0.943373)	(0.942108,0.947256,0.957436,0.962562)	(0.867773,0.877986,0.897787,0.908055)

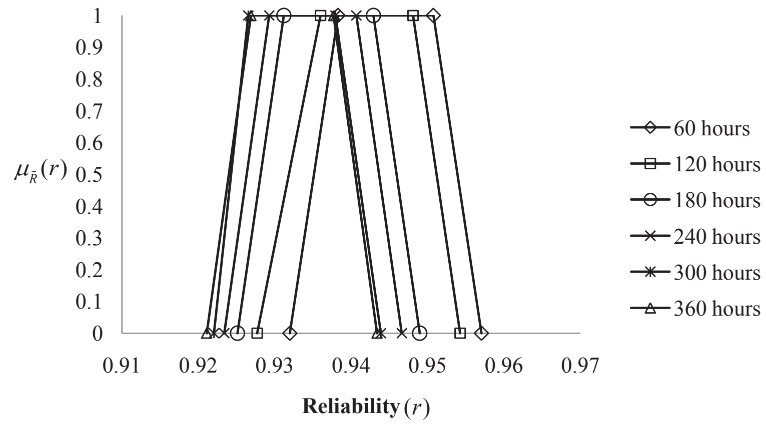

Fig. 3. Trapezoidal fuzzy number representing fuzzy reliability of sub-system R_1

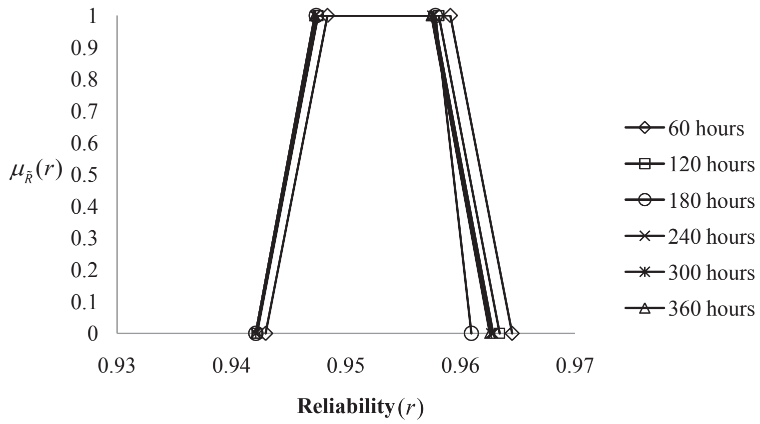

Fig. 4. Trapezoidal fuzzy number representing fuzzy reliability of sub-system R_2

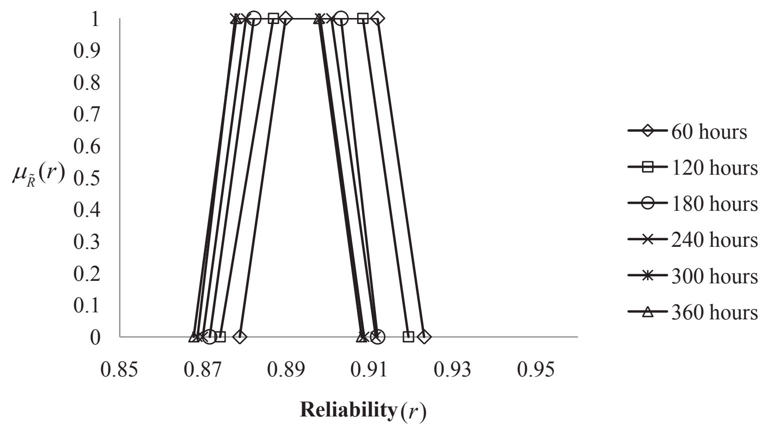

Fig. 5. Trapezoidal fuzzy number representing fuzzy reliability of whole system

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