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BAYESOWSKI MODEL WZROSTU NIEZAWODNOŚCI OPARTY NA DYNAMICZNYCH PARAMETRACH ROZKŁADU

BAYESIAN RELIABILITY GROWTH MODEL BASED ON DYNAMIC DISTRIBUTION PARAMETERS

W artykule przestudiowano metody analizy statystycznej na różnych etapach wzrostu niezawodności w oparciu o model monotoniczny. Zamodelowano zmiany jakim dynamiczne parametry rozkładu podlegają podczas badań. Podano bayesowskie modele wzrostu niezawodności dla licznych etapów wzrostu niezawodności. Na koniec metodę zweryfikowano w oparciu o przykład praktyczny.

Słowa kluczowe: *Statystyka populacji niejednorodnej, model monotoniczny, Bayes, model przyrostu niezawodności, rozkład wykładniczy.*

In this paper we study the statistical analysis methods at different stages of reliability growth based on the monotone model. The changes of dynamic distribution parameters during test are modeled. Bayesian reliability growth models for multiple stages of reliability growth are given. Finally the method is validated by a practical example.

Keywords: *Non-Homogeneous population statistics, monotone model, Bayes, reliability growth model, exponential distribution.*

1. Introduction

With failures having been removed and design having been perfected during reliability growth tests, the reliability of the product will keep growing [2, 9, 14]. In development of small sample weapon test policy, the product state is usually different at each test stage because changes are made to the design of the product. Therefore the quality and reliability indexes of the product are not a constant because of reliability improvement [11]. The assessment results are not accurate or effective when traditional reliability growth models and non-changeable population hypotheses are used. The probability distribution models for mature product life distributions, such as the exponential distribution, the lognormal distribution, and the Weibull distribution [3-6], do not correctly depict the changing process of product life during the design and test stages. In this paper, we analyze the reliability growth test data based on the monotone model [7], study the data analysis methods of product at various stages, model the trend of dynamic distribution parameters during design and test, and provide a multi-stage Bayesian reliability growth model.

2. Model hypothesis

(1) Assume that a product's reliability growth process includes h stages. The tests are either fixed time censored (a test is terminated after a pre-determined time duration) or fixed number of failures censored (a test is terminated after a fixed number of failures have been observed). The tests at different stages are independent. The sample sizes for the h stages are denoted by $n_1,n_2,\cdots n_h$, the numbers of failures observed are $z_1,z_2,\cdots z_h$, and the total test times are $\tau_1,\tau_2,\cdots\tau_h$.

(2) The product life T_k for the k^{th} stage test is assumed to follow the exponential distribution with probability distribution function (PDF) of $f(t)=\lambda_k e^{-\lambda_k t}, t>0$. If the mission time is T_0, the product reliability of the k^{th} stage product for the mission is:

$$R_k = \Pr(T_k \ge T_0) = e^{-\lambda_k T_0}, k=1,\cdots h \tag{1}$$

(3) The product state is assumed to be steady within each of the $k(1\le k\le h)$ stages. The product reliability increases from stage to stage because of removing defects, thus:

$$0 \le R_1 \le R_2 \le \ldots \le R_{h-1} \le R_h \le 1 \tag{2}$$

Obviously, the assumption in equation (2) is equivalent to the following monotone model when the exponential distribution is used:

$$\infty \ge \lambda_1 \ge \lambda_2 \cdots \ge \lambda_h > 0 \tag{3}$$

Therefore, exponential distribution reliability growth is modeled by the monotone model, i.e. expression (3).

3. Statistical method of reliability data

The monotone model is easy to use when there is little reliability data [12]. Therefore, this paper deals with product reliability data by the monotone model in order to get point estimation and confidence lower limit of reliability index. The confidence limit is used to determine values of hyper-parameters of Bayesian prior distribution.

For stage k, the prior distribution of the failure rate λ_k of the exponential distribution is expressed in terms of a gamma distribution. The form of the gamma distribution with shape parameter a_k and scale parameter b_k is:

$$\pi(\lambda_k|a_k,b_k)=\frac{b_k^{a_k}}{\Gamma(a_k)}\lambda_k^{a_k-1}\exp(-b_k\lambda_k) \tag{4}$$

Here $\pi(\lambda_k|a_k,b_k)$ is the prior PDF of λ_k, $\lambda_k>0$, $a_k>0$, $b_k>0$, and the hyper-parameters (a_k,b_k) of the Bayesian prior distribution can be obtained from prior experimental results.

Assuming that the number of failures from the prior experimental results of the product is c_k, the upper confidence limit λ_{kU} can be obtained from the following equation [15]:

$$uI_{T_k\lambda_{kU}}(c_k+1)+(1-u)I_{T_k\lambda_{kU}}(c_k)=\gamma \tag{5}$$

where T_k is the prior equivalent test time duration, $I_x(\alpha)$ is the incomplete gamma function, $I_x(\alpha)=\frac{1}{\Gamma(\alpha)}\int_0^x e^{-t}t^{\alpha-1}\mathrm{dt}$, γ is the confidence level, u is a parameter taking values in the [0,1] interval (in practice we often take $u = 0.5$). When choosing different confidence levels γ_1 and γ_2, the lower confidence limits of failure rate λ_1 and λ_2 are quantiles of the prior distribution (γ_1, γ_2) [1]:

$$\begin{cases}\int_0^{\lambda_1}\pi(\lambda_k|a_k,b_k)\mathrm{d}\lambda_k=\gamma_1\\ \int_0^{\lambda_2}\pi(\lambda_k|a_k,b_k)\mathrm{d}\lambda_k=\gamma_2\end{cases} \tag{6}$$

The hyper-parameters (a_k,b_k) of the Bayesian prior distribution can be obtained from expression (6).

The test data are obtained from fixed time censored test with replacement. The censor time is t' and the total test time is τ_k. Furthermore the total number of failures, denoted by $N(\tau_k)$, within interval $[0, t']$ follows the following Poisson distribution.

$$\mathrm{P}(N(\tau_k)=z_k)=\frac{(\tau_k\lambda_k)^{z_k}}{z_k!}\exp(-\lambda_k\tau_k),\quad k=0,1,2,... \tag{7}$$

The likelihood function of the fixed time censored test with replacement (τ_k, z_k) is expressed as following.

$$L(\lambda_k)=\frac{(\tau_k\lambda_k)^{z_k}}{z_k!}\exp(-\lambda_k\tau_k) \tag{8}$$

According to the Bayes formula, the posterior distribution can be described by following expression.

$$\begin{aligned}\pi_\rho(\lambda_k|D_k)&=\frac{L(\lambda_k)\pi(\lambda_k|a_k,b_k)}{\int_{\Theta\lambda_k}L(\lambda_k)\pi(\lambda_k|a_k,b_k)\mathrm{d}\lambda_k}=\\&=\frac{(b_k+\tau_k)^{a_k+z_k}}{\Gamma(a_k+z_k)}\cdot\lambda^{a_k+z_k-1}\exp(-\lambda_k(b_k+\tau_k))\end{aligned} \tag{9}$$

where D_k denotes the test result of the stage k test as described in assumption (1), $\pi_\rho(\lambda_k|D_k)$ is the posterior distribution of λ_k, obtained from the prior $\pi(\lambda_k|a_k,b_k)$ and the likelihood function obtained from the test result D_k. Thus the upper confidence limit $\lambda_U(k)$ of the failure rate satisfies the following expression (10).

$$\int_0^{\lambda_U(k)}\pi_\rho(\lambda_k|D_k)\mathrm{d}\lambda=\gamma \tag{10}$$

where γ is the confidence level. The point estimate of the failure rate is described by the following expression.

$$\hat{\lambda}(k)=\int_0^{\infty}\lambda_k\pi_\rho(\lambda_k|D_k)\mathrm{d}\lambda_k \tag{11}$$

After the k^{th} test the lower confidence limit of MTBF and the point estimate of MTBF are given by formula (12) and formula (13), respectively.

$$MTBF_L(k)={}^{1}\!/_{\lambda_U(k)} \tag{12}$$

$$MTBF_{Avg}(k)={}^{1}\!/_{\hat{\lambda}(k)} \tag{13}$$

4. Reliability growth test

In order to test whether the product reliability is growing or not, we need to check if the test data confirms the monotone relationship, i.e. expression (2) or (3), via hypothesis testing [14].

For stage i and i+1, calculate the following test statistics based on the test data obtained following assumption (1):

$$F^*=\begin{cases}\dfrac{\tau_{i+1}z_i}{\tau_i z_{i+1}}, & \text{for fixed number of failures censored test}\\ \dfrac{\tau_{i+1}(2z_i+1)}{\tau_i(2z_{i+1}+1)}, & \text{for fixed time censored test}\end{cases} \tag{14}$$

Then the F^* follows the F distribution with degrees of freedom of $(2r_{i+1},2r_i)$ for the fixed time censored case, and with degrees of freedom of $(2r_{i+1},2r_i+1)$ for the fixed number of failures censored case. If

$$F^*\geq\begin{cases}F_{1-\alpha}(2r_{i+1},2r_i) & \text{for fixed number of failures censored test}\\ F_{1-\alpha}(2r_{i+1},2r_i+1) & \text{for fixed time censored test}\end{cases} \tag{15}$$

we conclude that reliability grows from stage i to stage i+1. The condition (15) is not satisfied, we conclude that the data from these two stages are not different. In this case, we merge the data from these two stages and conduct a reliability growth test with the data from the next stage, that is, stage i+2. In condition (15), $F_{1-\alpha}(2r_{i+1},2r_i)$ is the upper 100(1-α)% point of the F distribution with degrees of freedom of $(2r_{i+1},2r_i)$ for the fixed time censored test, α is the significance level (α is usually chosen to be 0.2). If we are convinced that the product reliability has been improved, we can set a tighter requirement by choosing the value of α to be 0.3, 0.4 or more. However, the idea of merging the data from two stages that are not significantly different and compare them with the next stage is difficult to operate in practice. The reason is that if design flaws have been removed during product development, then the data from the improved design cannot be merged with the data from the design without the improvement. To solve this problem, we apply the inheritance factor method [13] in this merge process. Furthermore the inheritance factor must be determined before merging data from adjacent stages and then reliability growth test can be conducted. We discuss the calculation of the inheritance factor in the next section.

5. Inheritance factor calculation

Expert assessment method and goodness of fit test are affected by many subjective factors. In order to decrease the subjective influence， this paper determines the inheritance factor by an information fusion method, i.e., Kullback method [8].

The definition of Kullback information between cumulative distribution function P and cumulative distribution function Q in the metric space (Ω,F) is described by the following expression:

$$I(Q:P)=\begin{cases} E_Q(\ln(\frac{dQ}{\mathrm{d}P})) & \text{if } Q<<P \\ \infty & \text{others} \end{cases} \quad (16)$$

where Ω is the definition domain, F is σ algebra in Ω, $FQ(\bullet)$ is an expectation relative to Q. $\frac{dQ}{\mathrm{d}P}$ is Radon-Nikodym derivative of Q about P. When Q is much smaller than P, it means that Q is absolutely continuous about P.

If Ω is a real space, then P and Q can be described by the following expression [10]:

$$P(dx) = p(x)dx \; Q(dx) = q(x)dx \quad (17)$$

In this case, the definition of the Kullback information is equivalent to formula (18).

$$I(Q:P)=I(q:p)=\int_R \ln[\frac{q(x)}{p(x)}]q(x)dx \quad (18)$$

If Ω is a finite discrete space, i.e. Ω $=\{x_1,...,x_m\}$, following $P(\{x_i\})=p_i$, $Q(\{x_i\})=q_i$ $(i=1,...,m)$, the Kullback information between P and Q is [10]:

$$I(Q:P)=\sum_{i=1}^{m} q_i \ln\left[\frac{q_i}{p_i}\right] \quad (19)$$

Furthermore, $I(Q:P)\geq 0$ and the necessary and sufficient condition is $P=Q$ when the expression is equal to zero.

Since Kullback information is the measurement of the diversity of two distributions, the more similar two distributions are, the larger the Kullback information is. Furthermore, while the range of the inheritance factor is [0, 1], the Kullback information is not always in the [0, 1] interval. So the monotone function described in expression (20) is used to transform the Kullback information into a function whose range is [0, 1]:

$$\rho = 1-\frac{|I(Q:P)|}{|I(Q:P)|+1} \quad (20)$$

where ρ is the inheritance factor.

6. Reliability growth test plan formulation

Reliability growth test will be done when the design is capable of performing the intended function so that the designer can find design flaws and improve design. Usually it is impossible to fulfill the specified reliability requirement in the initial development stage of a large complex system. Therefore, it is important to do reliability growth test in the initial prototype development in order to remove the flaws of design, manufacture and operation and improve product reliability. Thus, reliability growth test plan formulation is important for new system and product development.

6.1. How to determine parameters of reliability growth

In order to develop a reliability growth test plan, the parameters of reliability growth must be determined according to the actual situation, which usually include growth objective, reliability growth rate and initial value of growth. The objective value can be easily determined by the reliability requirement of product.

The improvement effect and the total test time can be affected by reliability growth rate. To choose a proper growth rate, we must fully consider the reliability growth of similar products and the assumed improvement measures. Basically the growth rate can be determined by reliability data when there is enough data [12]. If reliability data of product is available, the estimation of the failure rate is λ_i at time t_i.

Assuming that the reliability growth of product follows Duane model, the estimation of growth rate m_i is:

$$m_i = 1-\frac{\lambda_i t_i}{N_i} \quad (21)$$

where N_i is the total number of failures observed up to stage i, $N_i=\sum_{j=1}^{i} z_j$. Therefore the growth rate m at the end of stage h is as given in expression (22).

$$m=\frac{1}{h}\sum_{i=1}^{h} m_i = 1-\frac{1}{h}\sum_{i=1}^{h}\frac{\lambda_i t_i}{N_i} \quad (22)$$

After determining the growth rate m, the initial value of reliability growth parameters, T_1 and M_1, can be given by expression (23):

$$\begin{aligned} T_1 &= t_h \\ M_1 &= -(1-m)MTBF_{Avg}(T_1) \end{aligned} \quad (23)$$

where, $T_1=t_h$ is the accumulated test time of the product to the end of stage h, M_1 is the accumulated MTBF, $MTBF_{Avg}(T_1)$ is a point estimate of the mean time between failures at time t_h.

6.2. How to determine total time of reliability growth test

The total test time directly affects reliability test resources. Thus, it must be controlled carefully. We can use the following expression [9, 14].

$$T \geq T_1(\frac{(1-m)M_{obj}}{M_1})^{\frac{1}{m}} \quad (24)$$

If parameters of (T_1, M_1), m and M_{obj} are known, the total time is fixed.

7. Example and conclusion

Assume that the development of a product includes principle prototype, initial prototype and formal prototype. The product will undergo multiple stage tests in the development process. It is necessary to make a reliability growth plan in order to improve test efficiency. During reliability growth tests, 4 failures were observed in the principle prototype. After failure cause analysis and design improvement the mission time of product was 10 hours and the test was fixed time censored. Tab. 1 shows the data obtained from each test stage.

Tab. 1. The data of each test stage

Test stage	Number of failures	Total test time (hour)
The first stage	4	42.9
The second stage	2	45.4
The third stage	1	62.5

The reliability growth effect of different stages has been tested by the method provided in Section 3 of this paper. When $\alpha = 0.2$ the test statistic between stage 1 and stage 2 was equal to 1.9049, i.e., F^*=1.9049, larger than the critical value of 1.8455. The test statistic between stage 2 and stage 3 was 2.2894, i.e., F^*=2.2894, also larger than the critical value of 2.2530. Thus the reliability of the product obviously improved in the development stage. Furthermore the point estimation of the MTBF can be obtained by expressions (11) and (13), which is 103h.

In summary, this paper provided a statistical analysis method for multiple stages test data based on the monotone model, described the change event of dynamic distribution parameters during testing, modeled the Bayesian reliability growth model of multiple stages exponential distribution product, and considered the influence of non-Homogeneous population information. The result of an example demonstrated the usefulness of the proposed method for reliability growth assessment.

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