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A WARRANTY COST MODEL WITH INTERMITTENT AND HETEROGENEOUS USAGE

Estimation of warranty servicing costs during the product life cycle is of great importance to the manufacturers. Earlier research has usually assumed that the product is in continuous use and the usage intensity is the same for all buyers. This paper deals with the problem of estimating the expected warranty cost for the case where the item usage is intermittent and of heterogeneous usage intensity over the product life cycle when sales occur continuously. The failure of the item is dependent on the number of times, the duration the unit has been used and the usage intensity. Also, the product sales depend on product price and design quality. We consider repairable and nonrepairable items and obtain results for the free-replacement warranty (FRW) and Pro-rata Warranty (PRW) policy. Furthermore, the models consider the influences of price level, investment growth and warranty execution effects for the expected warranty costs. It also incorporates the cash flows of warranty reserve costs at any time intervals during the product life cycle. A numerical example is given to illustrate the application of the models.

***Keywords**: Warranty, intermittent use, usage intensity, product sales.*

1. Introduction

A warranty is a seller's assurance to a buyer that a product is or shall be as represented. It may be considered to be a contractual agreement between buyer and seller who are entered into upon sale of the product [1]. The warranty is considered to be the representation of the product quality. It can also be used as a very important marketing tool. Servicing warranty involves additional cost to the manufacturers and greatly influences their profit. So the manufacturer needs to create a warranty reserve fund before the product sale. If the manufacturer is too conservative and sets aside too much reserve fund, he will lose more investment repay opportunity. And if he is too risky, he will reduce the profit and even go bankrupt. Thus, an efficient warranty cost analysis is important to a company's production management and profitability.

Because of its importance, warranty cost analysis has received a lot of attention of many researchers. The handbook by Blischke and Murthy [1] is a collection of research papers dealing with warranty. A general treatment of warranty cost analysis can be found in [2-6] and the references cited therein. Dimtrov [7] modeled the virtual failure rate by considering the repair as the age-reducing or age-accelerating repair factor in the warranty cost analysis. Ja [8] estimated the warranty costs during the life cycle of a product under nonrenewable minimal repair warranty policy, based on a selected level of confidence. The model assumes the repair costs depend on the product age. Chukova evaluated related expected costs using alternating renewal process to model renewing free replacement warranties and non-renewing free replacement warranties in [9] and [10]. The models both allowed for non-zero repair time and associated cost with it. Mitra [11] investigated warranty programs that offer customers the option to renew warranty, after an initial period, for a certain premium. The paper explored the effect of such programs on market share and warranty costs. Reference [12] obtained the probability distributions of the manufacturer's rebate, cost, revenue and profit during a product cycle, under a combination free-replacement/pro-rata warranty policy, with the incorporation of the customer repurchase behavior under warranty. Balcer [13] derived moments of the user's replacement cost over time under renewing pro-rata and non-renewing free-replacement policies. Jun [14] presented discounted warranty cost models for repairable series systems under free repair policy and pro-rata warranty policy. Dimtrov [15] modeled warranty claim as a marked point process and obtained particular results for non-stationary Poisson purchase process with periodic intensity function. Balachandran [16] dealt with warranty cost estimation of the product consisting of several components using Markovian approach. Markovian states are defined dependent on the number of failures of each component. The model assumes the failure rate constant. These examine a variety of warranty policies for both repairable and nonrepairable items. A review and summary of warranty analysis can also be found in [17-18]. Murthy [19] deals with the administration aspects of warranty.

In the models studied so far, it is implicitly assumed that the item is in continuous use. However, this is normally not the case and many items are used intermittently over the warranty period and the life of the item, such as television, rice cooker, microwave oven etc. The failure rate of an item when in use can be different from that when idle. In order to evaluate the warranty costs from a realistic viewpoint, we should study the failure models under various usage patterns. Murthy [20] studied the unit expected warranty cost during the warranty period for the item used intermittently and the duration of usage each time is often very small in relation to the time between usages. The item usage is modeled as a point process and the item failure is characterized by a discrete distribution and relates to the number of times the current unit has been used. They characterized the life of an item by usage number at failures. Murthy [21] further assumed that the item used intermittently can be either in use (U) or idle (I) and transitions between U and I in a random manner which is modeled by a continuous-time Markov chain. They obtained the expected warranty cost and assumed the transitions rate between the two states to be constant. Kim [22] assumed that the usage intensity varies across the buyer population. The failure rate was modeled as a function of the usage intensity and the expected warranty cost during the warranty period for the unit item was obtained. They considered the usage intensity was modeled as continuous and discrete random variables.

This paper develops the total warranty cost model during the product life cycle for continuous sales process of items used intermittently. The product sales are assumed to be dependent on product price and design quality. The item can be either in use or idle and the failure depends on the number of times, the duration and the usage intensity the unit has been used. The usage intensity varies across the population of users and is modeled as a continuous random variable. This model considers the influences of price level, investment growth and warranty execution effects and obtains the cash flows of warranty costs at any time intervals during the product life cycle. The outline of the paper is as follows. Section 2 deals with the model formulation of the usage pattern and failure. In section 3, we model the expected warranty cost sold for FRW and PRW policy during the product life cycle. A numerical example is given in Section 4 to illustrate the proposed models and make a contrast with continuous usage model. Finally, in Section 5, we conclude with a brief discussion of some extensions for future investigation.

2. Model Development

Notations:

- c Unit product cost, not including warranty cost
- p Unit product sale price, including warranty cost
- w Duration of warranty period
- U Usage intensity (random variable)
- $G(u)$ Usage distribution function
- $g(u)$ Usage intensity function
- $c(t)$ Refund amount when the item fails under Pro-rata Warranty Policy (PRW)
- $F(t)$ Distribution function for the first time to failure
- $f(t)$ Probability density function associated with $F(t)$
- $r(t)$ Failure rate function associated with $F(t)$
- $p_i(t,u)$ Probability that the Markov chain $X(t)$ is in state i at time t conditional on the usage rate u, $0 \le i \le 1$
- $\omega(w)$ Expected warranty cost per unit for a warranty period w
- θ Investment growth rate
- ϕ Expected change rate in the general price level
- L Product life cycle
- $g(t,w)$ Warranty execution function for a warranty period w at time t
- c_r Expected minimal repair cost per failure for repairable product
- $q(t)$ Sale rate at time t, $0 \le t \le L$
- $v(\tau)$ Warranty return rate at time t, $0 \le \tau \le L + w$
- $Q(t)$ Accumulated sales volume in $[0,t]$
- $Y(t)$ Age of the unit at time t, $0 \le Y(t) \le t$
- $\tau(Y(t))$ Duration for which the consumer has used the current unit, $0 \le \tau(Y(t)) \le Y$
- $N(Y(t))$ Number of times the current unit has been used over the interval $[t - Y(t), t]$
- $E[\tau(t)]$ Expected using duration during the interval $[0,t]$
- $E[N(t)]$ Expected number of the item used during the interval $[0,t]$

2.1. Product Warranty Strategy

Many types of warranty policies have been used because of their importance. Blischke and Murthy [1] introduced a classification and definitions of various warranty strategies. In this paper, we consider the free replace-repair policy (FRW) and pro-rata warranty policy (PRW) which are defined as follows:

(1) *Free Replacement-repair Policy* (FRW): Under the policy, the seller agrees to repair or provide replacements or repair for failed items free of charge up to a time w from the time of the initial purchase. Typical applications of FRW are consumer products, ranging from inexpensive to relatively expensive items such as automobiles, refrigerators, TVs, electronic components, and so forth.

(2) *Pro-rata Warranty Policy* (PRW): Under the policy, the seller agrees to refund a fraction of the purchase price if the item fails before time w from the time of the initial purchase. The refund can be either a linear or nonlinear function of w-t, which defines the linear PRW and nonlinear PRW. The linear PRW applies to relatively inexpensive nonrepairable products such as batteries, tires, ceramics, and so on. The nonlinear PRW usually uses quadratic rebate function. The refund amount can be given as

$$c_1(t) = \begin{cases} \alpha(1-t/w)p, & 0 \le t < w \\ 0, & otherwise \end{cases} \quad 0 < \alpha \le 1 \tag{1a}$$

or

$$c_2(t) = \begin{cases} \alpha(1-t/w)^2 p, & 0 \le t < w \\ 0, & otherwise \end{cases} \quad 0 < \alpha \le 1 \tag{1b}$$

where $c_1(t)$, $c_2(t)$ are the refund amount when the item fails.

2.2. Product usage model

Different consumers have heterogonous usage intensity for their different own characteristics to a product. For example, the usage intensity (in terms of load and frequency of usage per unit time) of a domestic washing machine varies depending on the size of the family and being used in various situations, such as hospital or at home. So the usage intensity across the buyer population is different. This is also true for many other domestic and industrial products. The product failure depends on the usage intensity and this in turn has an important influence on the expected warranty cost.

In this paper, we refer to the models presented by Kim and Djamaludin [22]. The usage intensity is modeled as a random variable with a distribution function $G(u)$ and density function $g(u)$ which characterizes the different usages across the user population. Conditional on the usage intensity $U=u$, the product failure distribution is given as $F(t,u)$. And $r(t,u)$ is the failure rate function associated with $F(t,u)$, which is given by

$$r(t,u) = k\delta(u)r_0(t,u) \tag{2}$$

where $k(>0)$ is a scale factor to reflect the usage intensity influence. $r_0(t,u)$ is failure rate for a initial design, which may be continuous case as reference [22]. We consider the case of intermittent usage in this paper, which will be obtained in the section 2.3, and $\delta(u)$ defines the effect of the usage intensity to the product which is modeled as

$$\delta(u) = \begin{cases} 1 & u_{min} \le u \le u_0 \\ (\frac{u}{u_0})^{\varepsilon} & u_0 < u \le u_{max} \end{cases} \tag{3}$$

with $\varepsilon \ge 1$ and u_0 as the additional design parameters which represents the product quality.

The product is assumed to be used intermittently. As a result, at time t, $0 \le t \le w$, the product can be either in use (U) or idle (I). The transitions from I to U and form U to I occur in a random manner [21]. So we model the transitions by a two-state continuous time Markov chain formulation $X(t)$. Here

$X(t)=1$ if the item is in use at time t and $X(t)=0$ if the item is idle. Conditional on the usage rate $U=u$, the probabilities

$$\{X(t+\delta t)=j \mid X(t)=i\}, 0 \le i, j \le 1$$

are given by the following matrix:

$$X(t)\begin{cases} 1 & 1-\lambda_1(u)\delta t \quad \lambda_1(u)\delta t \\ 0 & \lambda_0(u)\delta t \quad 1-\lambda_0(u)\delta t \end{cases} \qquad \overbrace{\quad 1 \qquad\qquad 0 \quad}^{X(t+\delta t)}$$

We assume that the consumer uses the unit soon after purchase, i.e., $X(0)=1$.

2.3. Product failure model

We assume that the item is new at $t = 0$, i.e., $Y(t) = t$. The item failure rate is dependent on the item historical usage condition. We refer to the models presented by Murthy [21]. Given the usage intensity u, we assume the failure rate is constant when the item is idle and the failure rate depends on the usage history of the current unit when the unit is in use [21]. Conditional on the usage rate u, the failure rate function G when the unit is in use is a linear function of the form

$$r_0(t,u \mid \tau(t), N(t), X(t)=1) = G(Y(t), \tau(Y(t)), N(Y(t))) = \\ = \theta_0 + \theta_1 t + \theta_2 \tau(t) + \theta_3 N(t) \tag{4}$$

where $\theta_i, 0 \le i \le 3$ are nonnegative constants.

On removing the conditioning, we have

$$r_0(t,u \mid X(t)=1) = \theta_0 + \theta_1 t + \theta_2 E[\tau(t)] + \theta_3 E[N(t)] \tag{5}$$

From the theory of Markov chains [25], we have

$$E[\tau(t)] = \frac{\lambda_0(u)}{\lambda_0(u)+\lambda_1(u)} t + \frac{\lambda_1(u)}{(\lambda_0(u)+\lambda_1(u))^2}(1-e^{-(\lambda_0(u)+\lambda_1(u))t}) \tag{6}$$

Similarly [21], we have

$$E[N(t)] = \frac{\lambda_0(u)\lambda_1(u)}{\lambda_0(u)+\lambda_1(u)} t + \frac{\lambda_0(u)\lambda_1(u)}{(\lambda_0(u)+\lambda_1(u))^2}(1-e^{-(\lambda_0(u)+\lambda_1(u))t}) \tag{7}$$

The failure rate when the item is idle is given by

$$r_0(t,u \mid X(t)=0) = \varphi, \quad (\varphi > 0) \tag{8}$$

$\varphi \le \theta_0$, which ensures that the failure rate when idle is always less than the failure rate when in use.

Using Eqs. (5) and (8), we have

$$r_0(t,u) = r_0(t,u \mid X(t)=1) p_1(t,u) + r_0(t,u \mid X(t)=0) p_0(t,u) \tag{9}$$

where $p_i(t,u)$, $0 \le i \le 1$ is the probability that the Markov chain $X(t)$ is in state i at time t. From the theory of Markov chains [21, 25], we have

$$p_1(t,u) = \frac{\lambda_0(u)}{\lambda_0(u)+\lambda_1(u)} + \frac{\lambda_1(u)}{\lambda_0(u)+\lambda_1(u)} e^{-(\lambda_0(u)+\lambda_1(u))t} \tag{10a}$$

$$p_0(t,u) = \frac{\lambda_1(u)}{\lambda_0(u)+\lambda_1(u)} - \frac{\lambda_1(u)}{\lambda_0(u)+\lambda_1(u)} e^{-(\lambda_0(u)+\lambda_1(u))t} \tag{10b}$$

Using Eqs. (5-8) and (10) in (9), we have

$$r_0(t,u) = \frac{1}{\lambda_0(u)+\lambda_1(u)}\{\lambda_0(u)[\theta_0+\theta_1 t+\theta_2 E[\tau(t)]+\theta_3 E[N(t)]] + \lambda_1(u)\varphi\}$$

$$+ \frac{\lambda_1(u)}{\lambda_0(u)+\lambda_1(u)} e^{-(\lambda_0(u)+\lambda_1(u))t}\{\theta_0-\varphi+\theta_1 t+\theta_2 E[\tau(t)]+\theta_3 E[N(t)]\} \tag{11}$$

Using (11) in (2), we have $r(t,u)$. On removing the conditioning, the failure rate is given by

$$r(t) = \int_{u_{\min}}^{u_{\max}} r(t,u)\, dG(u) \tag{12}$$

Finally we can obtain $F(t,u)$ and $f(t,u)$ using the relationship

$$F(t,u) = 1-\exp\left\{-\int_0^t r(t,u)dt\right\}$$

and

$$f(t,u) = r(t,u)\exp\left\{-\int_0^t r(t,u)dt\right\} \tag{13}$$

2.4. Warranty Execution Function

In warranty cost analysis, it is usually assumed that the warranty is fully claimed at the time of product failure, which is within the warranty period. In practice, the assumption is not always valid. For example, a consumer may develop dissatisfaction for the product and prefer to change brands rather than to exercise warranty. A customer may purchase some other product cheaper than the cost of repurchase of the same product using the warranty right [23].

Many factors influence customer behavior in exercising warranties such as the warranty time, warranty attrition due to costs of executing the warranty, the product class, the form of reimbursement, change in product preference, the consumer's geography position, and so on. The form of the weight function describes warranty not full execution factors. The execution function is usually to be a decreasing function of time. Patankar and Mitra [28] examined two examples of the conditional warranty execution weight function. They modeled the heterogeneity in consumer behavior in warranty execution with random variables and investigated its impact towards expected warranty cost. Liu [29] obtained the estimating warranty costs model for continuous sales process of nonrepairable products under pro-rate warranty policy, which modeled the warranty execution with deterministic and random variables. In this paper, we refer to the models presented by Liu [29], which is given by Eqs. (14) and is shown in Fig.1 when $k = 0.5$.

$$g(t,w) = -\frac{k}{w^2}t^2 + 1 \quad 0 \le t \le w \tag{14}$$

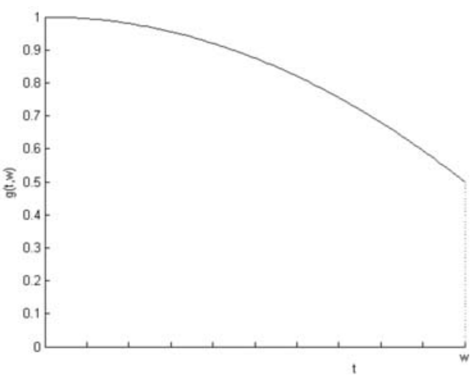

Fig.1. Warranty execution functions

2.5. Product Sale Model

In order to estimate the total expected warranty cost (and, ultimately, total profit) for a product, it is necessary to model the product sales as well. Many factors are involved in the product sale, such as the product class, quality, price, post-sale service, and the rival circumstance and so on. Many models expressing sales through time as a function of these factors have been developed. A demand model that explicitly considers warranty as a factor has been proposed by Glickman and Berger [26]. Demand is assumed to be a decreasing function of price and an increasing function of w. Bass and Bruce [27] presented an epidemic model which has been used to explain the penetration of many consumer durables in the American market. The Bass model assumes that there are two basic kinds of purchasers, innovators and consumers who are sensitive to the actions of their peers. In this paper, we assume the demand is a decreasing function of price and an increasing function of the product quality parameter u_0, and the sales model is given by

$$q(t) = \frac{dQ(t)}{dt} = ku_0^{\ a} P(t)^{-b}\left[1 - \frac{Q(t)}{Q_M}\right]\left[\psi + \frac{Q(t)}{Q_M}\right] \quad (15)$$
$$Q(0) = Q_0$$

where $q(t)$ is the demand rate function, and $a,b,k>0$. The interpretation of the parameters of this model is as follows: k is a scale factor to reflect the competitor and other environmental influence, such as the number of potential consumers, the consumer purchasing power, etc. a represents the design usage intensity elasticity; and b is price elasticity. $P(t)$ denotes unit sale price at time t (Marketing variable). The square brackets reflect the concept of sales as a diffusion process involving innovators and imitators as in the Bass model (see reference [27]). The parameter ψ reflects the relative influence of innovators.

The total sales during the product life cycle $Q(L)$ is given by

$$Q(L) = Q_0 + \int_0^L q(\tau)d\tau \quad (16)$$

with Q_0 is a parameter which captures the past experience at $t = 0$, from research and development and pilot plant operation.

3. Warranty cost models during the life cycle

When an item is returned for rectification under warranty, the manufacturer incurs many costs, such as transportation cost, handling costs of warranty, material cost and labor cost, etc. We aggregate all of these costs into a single cost termed "warranty cost" for each claim. Because some of the costs are uncertain, this cost is a random variable [1]. The number of claims over the warranty period depends on the product quality, warranty policy and the type of rectification action used and these in turn determine the warranty costs. This paper considers repairable and nonrepairable items under FRW and PRW policy and minimal repair action for repairable items.

3.1. Warranty cost model under Free Replacement Warranty Policy (FRW)

3.1.1. Non-repairable product

For non-repairable product, let $M(t,u)$ denote the expected number of failures during the interval $[0,t]$, $0 \le t \le w$ conditional on $U = u$. From the renewal theory [25], we have

$$M(t,u) = F(t,u) + \int_0^t M(t,u)dF(t,u) \quad (17)$$

Removing the conditioning

$$M(t) = \int_{u_{\min}}^{u_{\max}} M(t,u)dG(u) \quad (18)$$

The warranty execution weight function that reflects not full execution factors is given by Eqs. (14). The warranty return rate $v_1(\tau)$ at time τ is given by

$$v_1(\tau) = \int_a^b q(\tau - t)g(t,w)dM(t) \quad (19)$$

where the lower and upper limits of the integral are as given in Table 1.

Thus the total expected warranty reserve costs in $[\tau,\ \tau+d\tau]$ can be evaluated by

$$h_1(\tau) = c\int_a^b q(\tau - t)g(t,w)e^{-(\theta+\phi)\tau}dM(t) \quad (20)$$

where the limits of the integral are the same as those given in Table 1.

Tab. 1. Lower and upper limits ***a, b*** for the integral

	a	b	Interval
$L \le w$	0	τ	$0 \le \tau \le L$
	τ - L	τ	$L < \tau \le w$
	τ - L	w	$w < \tau \le L + w$
$L > w$	0	τ	$0 \le \tau \le w$
	0	w	$w \le \tau \le L$
	τ - L	w	$L < \tau \le L + w$

From Eqs. (20), the expected warranty costs in $[\tau_0,\tau_1]$ is given by

$$\int_{\tau_0}^{\tau_1} h_1(\tau)d\tau \quad (21)$$

3.1.2. Repairable product

For a repairable item, we consider the failed item is repaired minimally. For other rectification action, we can also obtain the models according to the models presented by Blischke and Murthy [1]. Under such a repair, the failure rate of the product after repair is the same as that just before the failure. Let $S(t,u)$ denote the expected number of failures during the interval $[0,t]$, $0 \le t \le w$ conditional on $U = u$. $S(t,u)$ is given by

$$S(t,u) = \int_0^t r(t,u)dt \qquad 0 \le t \le w \quad (22)$$

By removing the conditioning

$$S(t) = \int_{u_{\min}}^{u_{\max}} dG(u)\int_0^t r(t,u)dt \qquad 0 \le t \le w \quad (23)$$

The warranty execution weight function, which reflects not full execution factors, is given by Eqs. (14). The warranty return rate $v_2(\tau)$ at time τ is given by

$$v_2(\tau) = \int_a^b q(\tau - t)g(t,w)dS(t) \quad (24)$$

where the limits of the integral are the same as those given in Table 1.

Thus the total expected warranty reserve costs $h_2(\tau)$ in $[\tau,\ \tau+d\tau]$ can be evaluated by

$$h_2(\tau) = c_r\int_a^b q(\tau - t)g(t,w)e^{-(\theta+\phi)\tau}dS(t) \quad (25)$$

where c_r is expected minimal repair cost per failure for repairable product and the limits of the integral are the same as those given in Table 1.

The expected warranty costs in $[\tau_0,\tau_1]$ is given by

$$\int_{\tau_0}^{\tau_1} h_2(\tau)d\tau \quad (26)$$

3.2. Warranty cost model under the Pro-rata Warranty Policy (PRW)

Under PRW policy, the fraction refunded is a function which is given by Eqs. (1a) or (1b).

The product failure probability is given by

$$f(t) = \int_{u_{min}}^{u_{max}} f(t,u)dG(u) \quad (27)$$

The warranty execution weight function that reflects not full execution factors is given by Eqs. (14). The warranty return rate $v_3(\tau)$ at time τ is given by

$$v_3(\tau) = \int_a^b q(\tau - t) f(t) g(t,w) dt \quad (28)$$

The lower and upper limits for the integral $v_3(\tau)$ and $h_3(\tau)$ are given at Table 1. Thus the total expected warranty reserve costs $h_3(\tau)$ in $[\tau, \tau+d\tau]$ can be evaluated by

$$h_3(\tau) = \int_a^b q(\tau - t) f(t) g(t,w) c(t) e^{-(\theta+\phi)\tau} dt \quad (29)$$

From Eqs. (29), the expected warranty costs in $[\tau_0, \tau_1]$ is given by

$$\int_{\tau_0}^{\tau_1} h_3(\tau) d\tau \quad (30)$$

3.3. The unit product's expected warranty cost

The total warranty cost during the product life cycle is the sum of warranty cost for $Q(L)$ units. Since $Q(L)$ is large, according to the central limit theorem the total warranty cost can be approximated as being normally distributed with mean $Q(L)\omega(w)$[25]. Thus the unit product's expected warranty cost during the life cycle can be evaluated by

$$\frac{\int_0^{L+w} h(\tau) d\tau}{Q(L)} \quad (31)$$

In general, it is not possible to derive analytical expressions for $M(t)$, $S(t)$ and the complex nature of the integrand for other equation. In this case, numerical integration methods can be used to evaluate the expected warranty reserve costs.

4. Illustrative example

4.1. Cost analysis for product used intermittently under FRW policy

First, we consider the warranty cost analysis for the repairable product under the free replacement policy as an example. We assume the usage intensity is given by a Gamma distribution with parameter α, i.e.

$$g(u) = \begin{cases} \frac{1}{\Gamma(\alpha)} u^{\alpha-1} e^{-u} & u > 0 \\ 0 & u \le 0 \end{cases} \quad (31)$$

where

$$\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$$

Let $\alpha = 2$. That is, $\Gamma(\alpha) = 1$, $g(u) = ue^{-u}$. We assume $\lambda_0(u) = \lambda_1(u) = u$, $u_0 = 1$, $k = 1$, $\varepsilon = 1$, $u_{min} = 0$, $u_{max} = 3$. So

$$\delta(u) = \begin{cases} 1 & 0 \le u \le 1 \\ u & 1 \le u < 3 \end{cases}$$

For the sales rate model given by (15), let $\alpha = 0.5$, $b = 2$, $k = 2 \times 10^8$, $\psi = 0$, $Q_0 = 200$, $Q_M = 6000$, $L = 5$ years. We consider the case that the price is constant, i.e. $p(t) = p = 200$. So the sales model during the product life cycle is given as

$$q(t) = 5 \times 10^3 u_0^{0.5} (1 - Q(t)/6000) Q(t)/6000$$

The sales rate function is shown in Fig.2 when u_0=1.

For failure model, let $\theta_i = 0.1$, $0 \le i \le 3$, $\varphi = 0.05$.

Using (6) and (7) in (11), we have $r_0(t,u)$. Using (2), (22) and (23), we have $S(t)$.

Thus from (24), we can obtain the expected warranty return rate during the product life cycle. The sales function and the corresponding warranty return rate functions are shown in Fig.2 for w = 1 year, u_0 = 1. From Fig. 2, the peak of the warranty return rate function lags the peak of the sales function, since failed products are returned in some periods after sales. From the warranty return rate, the expected number of failed unit returned for repair in any time periods can be evaluated.

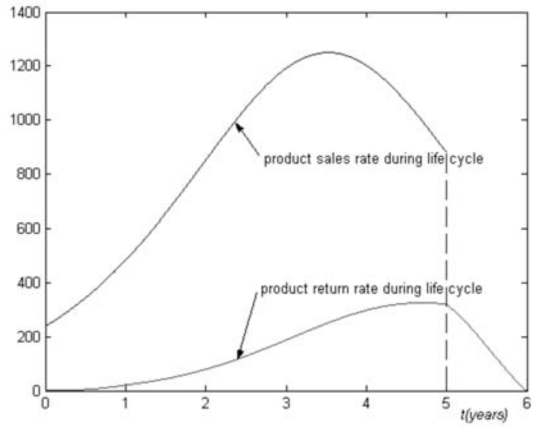

Fig. 2. Sales rate and warranty return rate during the product life cycle (w=1, u_0=1)

From the market experimental data [5], we let θ + φ = 0.1. From (25), (26) and (31), we can obtain the expected warranty cost results. Table 2 shows the unit discounted expected warranty cost $\omega(w)$ and the total discounted warranty costs for a product with a life cycle of $L = 5$ for various values of w and u_0. Management may use information in Table 2 to select the warranty period w and the design parameter u_0. Given the life cycle of the product, the amount to be needed for warranty costs may be determined for a selected value of w. If the budged warranty costs are identified, a corresponding warranty period and the design parameter u_0 could be estimated. For example, for a product with a life cycle of $L = 5$ years, if a budget for expected warranty costs not to exceed $1500c_r$, the selected warranty period could be 1 year for various design parameter u_0.

To study the sensitivity of the design parameter u_0 on the amount of required warranty costs, several values of u_0 are selected. Table 2 shows warranty costs for values of u_0 = 1.0, 1.5, 2.0, 2.5 respectively for different values of w. For example, for a product with a life cycle of $L = 5$ years, the expected warranty cost per unit for a warranty period $w = 1$ reduces from $0.3804c_r$ for $u_0 = 1$ to $0.1673c_r$ for $u_0 = 2$, which represents an decrease of about 56.0%. Higher values for u_0 are the result of better design and will cost more design expenses, so this may help us choose more better design plan when we know design expenses for various u_0.

From (25), (26), the expected total warranty costs in a particular time period can be evaluated. The expected warranty costs in the various one-year intervals for w = 1year, u_0 = 1 are given in Table 3. For example, in the second year of the warranty costs totally $39.0c_r$ will be paid.

Management may use the information in Table 2 and 3 to store appropriate cash for the warranty. As expected with an increase in the warranty period, warranty costs increase, but at different rates depending on the warranty period. For a 50% increase from 1 year to 1.5 years in warranty period, warranty costs would increase by approximately 82.6% for $u_0 = 1.5$. Such information could be used to determine the magnitude of the warranty parameter and the design parameter u_0 in order to maximize the expected profit.

4.2. Cost analysis for product used continuously under FRW policy

Now, we evaluate the expected warranty cost for continuous usage under FRW policy in order to make contrast with the intermittent usage case.

We assume the product initial design failure rate is given by

$$r_0(t,u) = 0.2+0.2t \quad (32)$$

We take the other parameters or equations as Section 4.1.

Using (32), (2), (22) and (23), we have $S(t)$.

From (25), (26) and (31), we can obtain the expected warranty cost results for product used continuously with a life cycle of $L = 5$ for various values of w and u_0. as showed in Table 4.

From Table 2 and Table 4, we can see the expected warranty cost has a large decrease from the continuous usage to intermittent usage. For example, the expected warranty cost per unit for a warranty period $w = 1$ and $u_0 = 1.5$ reduces from $0.4002c_r$ for continuous usage to $0.2281c_r$ for intermittent usage which represents an decrease of about 43.0%. If the manufacturer sets aside reserve fund according to the earlier research estimating warranty cost model for product used intermittently, he will overestimate the warranty cost and lose more investment repay opportunity.

5. Conclusions

Warranty cost models for the case where the item is used intermittently over the product life cycle is considered in this paper. The model assumes that the usage intensity varies across the population of users and the failure of item is dependent on the number of times, the duration and the usage intensity the unit has been used as opposed to earlier models where the usage is continuous and the users are same. We consider both repairable and nonrepairable items under the FRW and PRW policy and the product sales depend on product price and design quality. Also, this model considers the influences of price level, investment growth and warranty execution effects and studies the cash flows of warranty reserve costs at any time intervals during the product life cycle, which is very important for the product management and post-sale service for the manufacturers. The models can be used to compute different sale programs and warranty policies and plan cash budget and service facilities for the product used intermittently.

This paper assumes that the product can be either in use or idle. In fact, the product can be multi-state. For example, this unit can be used either in the normal specified mode of usage or in an abnormal mode. The failure rate in abnormal use is much higher than in normal use. And the failure caused by the use in an abnormal mode is not within the warranty. Thus the warranty for the multi-state product could be a further research topic of interest.

Tab. 2. The expected warranty costs for product intermittently used

	$\omega(w)$				The total expected warranty cost, $Q(L)\omega(w)$			
w \ u_0	1.0	1.5	2.0	2.5	1.0	1.5	2.0	2.5
0.5	$0.1515c_r$	$0.0902c_r$	$0.0657c_r$	$0.0532c_r$	$702.25c_r$	$485.16c_r$	$374.45c_r$	$311.27c_r$
1	$0.3804c_r$	$0.2281c_r$	$0.1673c_r$	$0.1365c_r$	$1762.9c_r$	$1227.1c_r$	$953.63c_r$	$797.85c_r$
1.5	$0.6917c_r$	$0.4165c_r$	$0.3066c_r$	$0.2513c_r$	$3206.2c_r$	$2240.1c_r$	$1748.0c_r$	$1469.2c_r$

Tab. 3. The expected warranty costs in the various one-year intervals during the life cycle (w = 1 year, u_0 = 1, L = 5)

Time interval (Year)	*0-1*	*1-2*	*2-3*	*3-4*	*4-5*	*5-6*
Expected warranty costs	$5.83c_r$	$39.0c_r$	$100.2c_r$	$172.6c_r$	$202.74c_r$	$95.7c_r$

Tab. 4. The expected warranty costs for product continuously used

	$\omega(w)$				The total expected warranty cost, $Q(L)\omega(w)$			
w \ u_0	1.0	1.5	2.0	2.5	1.0	1.5	2.0	2.5
0.5	$0.2674c_r$	$0.1589c_r$	$0.1155c_r$	$0.0936c_r$	$1239.5c_r$	$854.46c_r$	$658.51c_r$	$547.02c_r$
1	$0.6695c_r$	$0.4002c_r$	$0.2930c_r$	$0.2389c_r$	$3103.3c_r$	$2152.6c_r$	$1670.4c_r$	$1397.0c_r$
1.5	$1.2204c_r$	$0.7334c_r$	$0.5402c_r$	$0.4431c_r$	$5656.4c_r$	$3945.1c_r$	$3079.7c_r$	$2590.9c_r$

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6. References

[1] Blischke W.R., MurthyD.N.P.: *Product Warranty Handbook,* Marcel Dekker, Inc, New York, 1996.
[2] Menke W.W.: *Determination of warranty reserves*, Management Science 15(10) (1969) B542-549.
[3] Amato H.N., Anderson E.E.: *Determination of warranty reserves: an extension*, Management Science 22(12) (1976) 1391-1394.
[4] Patankar J.G., Worm G.H.: *Prediction intervals for warranty reserves and cash flows*, Management Science 27(2) (1981) 237-241.
[5] Thomas M.U.: *A prediction model for manufacturer warranty reserves*, Management Science 35(12) (1989) 1515-1519.
[6] Shau-Shiang Ja, Kulkarni V.G., Mitra A., Patankar J.G.: *Warranty reserves for nonstationary sales processes*, Naval Research Logistics 49 (2002) 499-513.
[7] Dimitrov B., Chukova S., Khalil Z.: *Warranty costs: an age-dependent failure/repair model*, Naval Research Logistics 51 (2004) 959-976.
[8] Shau-Shiang Ja, Kulkarni V.G., Mitra A., Patankar J.G.: *A nonrenewable minimal-repair warranty policy with time-dependent costs*, IEEE Transaction on Reliability 50 (4) (2001) 346-352.
[9] Chukova S., Hayakawa Y.: *Warranty cost analysis: non-renewing with repair time*, Applied Stochastic Models in Business and Industry 20 (2004) 59-71.
[10] Chukova S., Hayakawa Y.: *Warranty cost analysis: renewing warranty with non-zero repair time*, International Journal of Reliability, Quality and Safety Engineering 11(2) (2004) 93-112.
[11] Mitra A., Patankar J.G.: *Market share and warranty costs for renewable warranty programs*, International Journal of Production Economics 50 (1997) 155-168.
[12] Polatoglu, Sahin I.: *Probability distributions of cost, revenue and profit over a warranty cycle*, European Journal of Operational Research 108 (1998) 170-183.
[13] Balcer Y., Sahin I.: *Replacement cost under warranty: cost moments and time variability*, Operations Research 34 (1986) 554-559.
[14] Bai Jun, Pham Hoang: *Discounted warranty cost of minimally repaired series systems*, IEEE Transactions on Reliability 53(1) (2004) 37-42.
[15] Dimitrov B., Chukovav S.: *Warranty claims as marked point processes*, Nonlinear Analysis 47(2001) 2145-2150.
[16] Balachandran K.R., Maschmeyer R.A., Livingstone J.L.: *Product warranty period: a Markovian approach to estimation and analysis of repair and replacement costs*, The Accounting Review LV1 (1) (1981) 115-124.
[17] Thomas M.U., . Rao S.S: *Warranty economic decision models: a summary and some suggested directions for future research*, Operations Research 47(6) (1997) 807-820.
[18] Murthy D.N.P., Djamaludin I.: *New product warranty: a literature review*, International of Production Economics 79(2002) 231-260.
[19] Murthy D.N.P., Blischke W.R.: *Strategic warranty management: a life-cycle approach*, IEEE Transaction on Engineering Management 47 (1) (2000) 40-54.
[20] Murthy D.N.P.: *A new warranty costing model*, Mathematical Computer Modeling 13(9) (1990) 59-69.
[21] Murthy D.N.P.: *A usage dependent model of warranty costing*, European Journal of Operational Research 57 (1992) 89-99.
[22] Kim C.S., Djamaludin I., Murthy D.N.P.: *Warranty cost analysis with heterogeneous usage intensity*, International Transactions in Operational Research 8 (2001) 337-347.
[23] Patankar J.G., Mitra A.: *Effects of warranty execution on warranty reserve costs*, Management Science 41(3) (1995) 395-400.
[24] Nguyen D.G., Murthy D.N.P.: *A general model for estimating warranty costs for reparable products*, IIE Transactions 16(4) (1984) 379-386.
[25] Edward P.C. Kao.: *An introduction to stochastic processes*, China Machine Press, Beijing, 2003.
[26] Glickman T. S., Berger P. D.: *Optimal price and protection period decisions for a product under warranty*, Management Science 22(12) (1976) 1381-1390.
[27] Robinson B., Lakhani C.: *Dynamic price models for new product planning*, Management Science, 21(10) (1975) 1113-1122.
[28] Patankar J.G., Mitra A.: *Effects of warranty execution on warranty reserve costs*. Management Science 41(3) (1995) 395-400.
[29] Liu Z. J., Huang H. Z., XIE M.: *Warranty cost analysis of pro-rata policy for continuous sales process considering warranty execution factors*, the Fourth International Conference on Quality and Reliability (ICQR 2005) (2005) 231-237.

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