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PLANNING INSPECTIONS IN SERVICE OF FATIGUE-SENSITIVE AIRCRAFT STRUCTURE COMPONENTS FOR INITIAL CRACK DETECTION

Based on a random sample from the Weibull distribution with unknown shape and scale parameters, lower and upper prediction limits on a set of m future observations from the same distribution are constructed. The procedures, which arise from considering the distribution of future observations given the observed value of an ancillary statistic, do not require the construction of any tables, and are applicable whether the data are complete or Type II censored. The results have direct application in reliability theory, where the time until the first failure in a group of m items in service provides a measure regarding the operation of the items, as well as in service of fatigue-sensitive aircraft structures to construct strategies of inspections of these structures; examples of applications are given. Keywords: Aircraft structure, fatigue crack, Weibull distribution, prediction limit, inspection strategy

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1. Introduction

The Weibull distribution is a powerful modelling tool used in reliability analyses to predict failure rates and to provide a description of the failure of parts and equipment. The Weibull distribution has been widely used in the empirical modelling of economic models. Applications include the modelling of unemployment spells, strike durations, income distributions, the length of a firm's innovation period, and the size of research and development budgets. Depending on the particular problem, the variable under consideration may not be fully observed, requiring censoring procedures for estimation.

Based on engineering and macroscopic viewpoints, the mechanical properties of metallic materials are often considered homogeneous. However, a considerable amount of scatter has been observed in fatigue data even under the same loading condition. It may be attributed to the inhomogeneous material properties. As a result, probabilistic approaches for the fatigue crack growth have received great attention in recent years. Along with the development of fracture mechanics for the past three decades and the need of reliability or risk assessment for some important structures or components such as:

- Transportation Systems and Vehicles – aircraft, space vehicles, trains, ships;
- Civil Structures – bridges, dams, tunnels;
- Power Generation – nuclear, fossil fuel and hydroelectric plants;
- High-Value Manufactured Products – launch systems, satellites, semiconductor and electronic equipment;
- Industrial Equipment – oil and gas exploration, production and processing equipment, chemical process facilities, pulp and paper;

the so-called 'probabilistic fracture mechanics' has thus arisen [1]. One of the important issues in the probabilistic fracture mechanics analysis lies in the probabilistic modeling of fatigue crack growth phenomenon. Many probabilistic models have been proposed to capture the scatter of the fatigue crack growth data. Some of these models are based on the two-parameter Weibull distribution. It exhibits a wide range of shapes for the density and hazard functions that makes this distribution suitable for modelling complex failure data sets. Many authors have considered the problem of constructing prediction limits for the extreme value and Weibull distributions. References [3] and [4] contain good discussions of available procedures. As a rule, the better procedures involve the use of tables generated by Monte Carlo methods.

In this paper our focus is on prediction limits for future samples of observations from the two-parameter Weibull distribution and the purpose is to present a technique for constructing the prediction limits which can be used very generally, for Type II censored as well as complete data. The procedures should in particular be useful in situations not handled by the tables in the aforementioned references.

The proposed technique may be useful when we consider, for example, the reliability problem associated with fatigue damage that arises from the initiation of fatigue cracks originating from rivet holes along the top longitudinal row of the outer skin of the fuselage (Fig. 1).

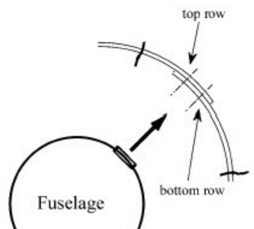

Fig. 1. Rivet row in consideration

It is assumed that a fatigue crack can initiate randomly at either side of a hole with diameter d. Experiments show that the number of flight cycles at which an initial crack will appear at one side with respect to a particular rivet follows the two-parameter Weibull distribution.

A post-failure photograph of one of the F-16 479 bulkhead test components (Fig. 2) indicates the location of fatigue crack initiation at the radius between the bulkhead and one of the two vertical tail attach pads.

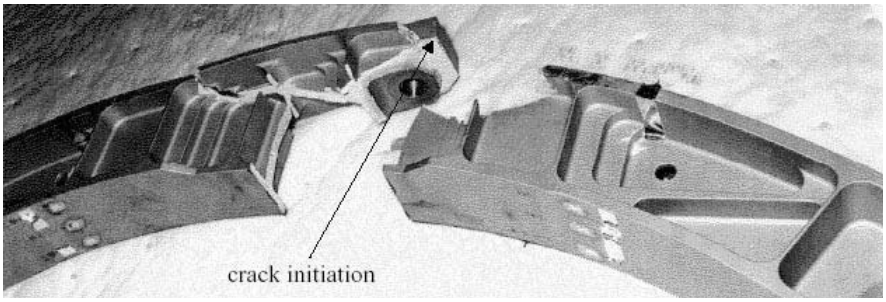

Fig. 2. F-16 479-bulkhead test specimen number -7B, post-failure crack initiation

The probability density function for the random variable X of the two-parameter Weibull distribution is given by:

$$f(x;\sigma,\delta) = \frac{\delta}{\beta}\left(\frac{x}{\beta}\right)^{\delta-1} exp\left[-\left(\frac{x}{\beta}\right)^{\delta}\right] \quad (x > 0) \qquad (1)$$

where $\delta>0$ and $\beta>0$ are the shape and scale parameters, respectively. Writing:

$$S = \mu + \sigma Z \qquad (2)$$

where Z is a random variable with standardized extreme value density,

$$f(z) = exp(z - e^{z}), \quad -\infty < z < \infty \qquad (3)$$

then the density of S can be obtained as

$$f(u;\mu,\sigma) = \frac{1}{\sigma} exp\left(\frac{s-\mu}{\sigma} - exp\left(\frac{s-\mu}{\sigma}\right)\right), \quad -\infty < s < \infty \qquad (4)$$

The distribution of S is known as the smallest extreme value distribution (SEV). If $S=lnX$, so that, $X=e^{S}$, then:

$$f(x;\mu,\sigma) = \frac{1}{x}\frac{1}{\sigma}(xe^{-\mu})^{1/\sigma} exp[-(xe^{-\mu})^{1/\sigma}] \qquad (5)$$

With $\sigma=1/\delta$ and $\mu=ln\beta$, X is distributed as Weibull with shape parameter δ and scale parameter β. Given this, for analytical and computational convenience, this paper works in the $S=lnX$ scale, the results, however, are reported directly for the Weibull observations.

2. Lower prediction limit

Theorem 1. Let $X_1 < \ldots < X_r$ be the first r ordered past observations from a sample of size n from the distribution (1). Then a lower conditional $(1-\alpha)$ prediction limit y_1 on the minimum Y_1 of a set of m future ordered observations $Y_1 < \ldots < Y_m$ is given by:

$$Pr\{Y_1 > y_l;\mathbf{z}\} = Pr\left\{\hat{\delta}\, ln\left(\frac{Y_1}{\hat{\beta}}\right) > \hat{\delta}\, ln\left(\frac{y_l}{\hat{\beta}}\right);\mathbf{z}\right\} = Pr\{W_1 > w_l;\mathbf{z}\}$$

$$= \frac{\int_0^\infty v^{r-2} e^{v\hat{\delta}\sum_{i=1}^{r} ln(x_i/\hat{\beta})} \left(me^{vw_l} + \sum_{i=1}^{r} e^{v\hat{\delta}\, ln(x_i/\hat{\beta})} + (n-r)e^{v\hat{\delta}\, ln(x_r/\hat{\beta})}\right)^{-r} dv}{\int_0^\infty v^{r-2} e^{v\hat{\delta}\sum_{i=1}^{r} ln(x_i/\hat{\beta})} \left(\sum_{i=1}^{r} e^{v\hat{\delta}\, ln(x_i/\hat{\beta})} + (n-r)e^{v\hat{\delta}\, ln(x_r/\hat{\beta})}\right)^{-r} dv} = 1-\alpha \qquad (6)$$

where $\hat{\beta}$ and $\hat{\delta}$ are the maximum likelihood estimators of β and δ based on the first r ordered past observations $(X_1, \ldots, X_r)$ from a sample of size n from the Weibull distribution, which can be found from solution of

$$\hat{\beta} = \left(\frac{\sum_{i=1}^{r} x_i^{\hat{\delta}} + (n-r)x_r^{\hat{\delta}}}{r}\right)^{1/\hat{\delta}} \qquad (7)$$

and

$$\hat{\delta} = \left[\left(\sum_{i=1}^{r} x_i^{\hat{\delta}}\, ln\, x_i + (n-r)x_r^{\hat{\delta}}\, ln\, x_r\right)\left(\sum_{i=1}^{r} x_i^{\hat{\delta}} + (n-r)x_r^{\hat{\delta}}\right)^{-1} - \frac{1}{r}\sum_{i=1}^{r} ln\, x_i\right]^{-1} \qquad (8)$$

$$\mathbf{z} = (z_1, z_2, \ldots, z_{r-2}) \qquad (9)$$

$$Z_i = \hat{\delta}\, ln\left(\frac{X_i}{\hat{\beta}}\right), \quad i = 1, \ldots, r-2 \qquad (10)$$

$$W_1 = \hat{\delta}\, ln\left(\frac{Y_1}{\hat{\beta}}\right), \quad w_l = \hat{\delta}\, ln\left(\frac{y_l}{\hat{\beta}}\right) \qquad (11)$$

Proof. The joint density of $S_1=ln(X_1), \ldots, S_r=ln(X_r)$ is given by:

$$f(s_1, \ldots, s_r;\mu,\sigma) = \frac{n!}{(n-r)!}\prod_{i=1}^{r}\frac{1}{\sigma} exp\left(\frac{s_i-\mu}{\sigma} - exp\left(\frac{s_i-\mu}{\sigma}\right)\right) exp\left(-(n-r)exp\left(\frac{s_r-\mu}{\sigma}\right)\right) \qquad (12)$$

Let $\hat{\mu}$, $\hat{\sigma}$ be the maximum likelihood estimators (estimates) of μ, σ based on $S_1, \ldots, S_r$ and let:

$$V_1 = \frac{\hat{\mu}-\mu}{\hat{\sigma}} \qquad (13)$$

$$V = \frac{\hat{\sigma}}{\sigma} \qquad (14)$$

and

$$Z_i = \frac{S_i - \hat{\mu}}{\hat{\sigma}} \qquad i = 1(1)r \qquad (15)$$

Parameters μ and σ in (12) are location and scale parameters, respectively, and it is well known that if $\hat{\mu}$ and $\hat{\sigma}$ are estimates of μ and σ, possessing certain invariance properties, then the quantities V_1 and V are parameter-free. Most, if not all, proposed estimates of μ and σ possess the necessary properties; these include the maximum likelihood estimates and various linear estimates. Z_i, $i=1(1)r$, are ancillary statistics, any r-2 of which form a functionally independent set. We then find in a straightforward manner that the joint density of V_1, V, conditional on fixed $z=(z_1, z_2, \ldots, z_{r-2})$, is:

$$v_1 \in (-\infty, \infty), \quad v \in (0, \infty) \qquad (16)$$

where:

$$\vartheta(\mathbf{z}) = \left(\int_0^{\infty}\int_{-\infty}^{\infty} v^{r-1} exp\left(\sum_{i=1}^{r}(z_i + v_1)v - \sum_{i=1}^{r} exp[(z_i + v_1)v] - (n-r)exp[(z_r + v_1)v] \right) dv_1 dv \right)^{-1} \quad (17)$$

is the normalizing constant. For notational convenience we include all of $z_1, \ldots, z_r$ in (15); z_{r-1} and z_r can be expressed as function of $z_1, \ldots, z_r$ only.

Writing:

$$W = \frac{ln Y_1 - \mu}{\sigma} \quad (18)$$

where Y_1 is the smallest observation from an independent second sample of m observations also from the distribution (1), and noting that exp(W) is the smallest observation in a sample of m observations from the standard exponential distribution, we have the density of W as:

$$f(w) = me^{w} exp(-me^{w}) \quad w \in (-\infty, \infty) \quad (19)$$

Since W is distributed independently of v_1, v we find the joint density of w, v_1, v, conditional on z, as the product of (16) and (19),

$$f(w, v_1, v; \mathbf{z}) = f(w) f(v_1, v; \mathbf{z}) \quad (20)$$

Note that $$W_1 = \frac{ln Y_1 - \hat{\mu}}{\hat{\sigma}} = \frac{W - V_1 V}{V} \quad (21)$$

making the transformation $w_1=(w-v_1v)/v$, $v_1=v_1$, $v=v$ we find the joint density of w_1, v_1, v, conditional on z, as:

$$f(w_1, v_1, v; \mathbf{z}) = m\vartheta(\mathbf{z}) v^{r} exp\left((r+1)v_1 v + \left(w_1 + \sum_{i=1}^{r} z_i \right) v \right)$$
$$\text{x}\, exp\left(-m\, exp[(w_1 + v_1)v] \right)$$
$$\text{x}\, exp\left(-exp[v_1 v] \left(\sum_{i=1}^{r} exp[z_i v] + (n-r) exp[z_r v] \right) \right)$$

$$w_1 \in (-\infty, \infty), \quad v_1 \in (-\infty, \infty), \quad v \in (0, \infty) \quad (22)$$

Now v_1 can be integrated out of (22) in a straightforward way to give:

$$f(w_1, v; \mathbf{z}) = \frac{\Gamma(r+1)\vartheta(\mathbf{z}) v^{r-2} exp\left(v \sum_{i=1}^{r} z_i \right) mv\, exp[w_1 v]}{\left(m\, exp[w_1 v] + \sum_{i=1}^{r} exp[z_i v] + (n-r) exp[z_r v] \right)^{r+1}} =$$

$$= \frac{\dfrac{r v^{r-2} exp\left(v \sum_{i=1}^{r} z_i \right) mv\, exp[w_1 v]}{\left(m\, exp[w_1 v] + \sum_{i=1}^{r} exp[z_i v] + (n-r) exp[z_r v] \right)^{r+1}}}{\int_0^{\infty} v^{r-2} exp\left(v \sum_{i=1}^{r} z_i \right) \left(\sum_{i=1}^{r} exp[z_i v] + (n-r) exp[z_r v] \right)^{-r} dv}$$

$$w_1 \in (-\infty, \infty), \quad v \in (0, \infty) \quad (23)$$

Thus, for fixed w_1 ($-\infty < w_1 < \infty$),

$$Pr\{W_1 > w_l; \mathbf{z}\} = \int_0^{\infty}\int_{w_l}^{\infty} f(w_1, v; \mathbf{z}) dw_1 dv$$

$$= \frac{\int_0^{\infty} v^{r-2} exp\left(v \sum_{i=1}^{r} z_i \right) \left(m\, exp[w_l v] + \sum_{i=1}^{r} exp[z_i v] + (n-r) exp[z_r v] \right)^{-r} dv}{\int_0^{\infty} v^{r-2} exp\left(v \sum_{i=1}^{r} z_i \right) \left(\sum_{i=1}^{r} exp[z_i v] + (n-r) exp[z_r v] \right)^{-r} dv}$$

$$= \frac{\int_0^{\infty} v^{r-2} e^{v\hat{\delta} \sum_{i=1}^{r} ln(x_i/\hat{\beta})} \left(me^{vw_l} + \sum_{i=1}^{r} e^{v\hat{\delta}\, ln(x_i/\hat{\beta})} + (n-r) e^{v\hat{\delta}\, ln(x_r/\hat{\beta})} \right)^{-r} dv}{\int_0^{\infty} v^{r-2} e^{v\hat{\delta} \sum_{i=1}^{r} ln(x_i/\hat{\beta})} \left(\sum_{i=1}^{r} e^{v\hat{\delta}\, ln(x_i/\hat{\beta})} + (n-r) e^{v\hat{\delta}\, ln(x_r/\hat{\beta})} \right)^{-r} dv} \quad (24)$$

This completes the proof.

Corollary 1.1. If $r=n$ and n is large, (24) should be more or less independent of Z. Also, $Z_1, \ldots, Z_n$ will be nearly independent and approximately distributed as standard extreme values, with pdf

$$f_{\circ}({}^{\bullet}w) = e^{{}^{\bullet}w} exp(-e^{{}^{\bullet}w}) \quad {}^{\bullet}w \in (-\infty, \infty) \quad (25)$$

Our first step is to replace $z_1, \ldots, z_n$ in the numerator of (24) by nE{$^{\bullet}W$}= $-n\gamma$, where γ=0.577215… is the Euler constant. We now suppose that $(1/n) \sum_{i=1}^{n} exp[z_i v]$ will be approximately equal to the moment generating function for (25), with dummy variable v. Since E{exp[$^{\bullet}w\eta$]}=Γ(1+η), we approximate the above sum in the denominator of (24) by nΓ(1+v). We thus arrive at the following approximation to (6), where $r=n$,

$$Pr\{Y_1 > y_l\} = Pr\left\{ ln\left(\frac{Y_1}{\hat{\sigma}} \right)^{\hat{\delta}} > ln\left(\frac{y_l}{\hat{\sigma}} \right)^{\hat{\delta}} \right\} = Pr\{W_1 > w_l\} =$$

$$= \frac{\int_0^{\infty} v^{n-2} e^{-n\gamma v} \left[(m/n) e^{vw_l} + \Gamma(1+v) \right]^{-n} dv}{\int_0^{\infty} v^{n-2} e^{-n\gamma v} \left[\Gamma(1+v) \right]^{-n} dv} = 1 - \alpha \quad (26)$$

3. Upper prediction limit

Theorem 2. Let $X_1 < \ldots < X_r$ be the first r ordered past observations from a sample of size n from the distribution (1). Then an upper conditional (1–α) prediction limit y_u on the maximum Y_m of a set of m future ordered observations $Y_1 < \ldots < Y_m$ is given by:

$$Pr\{Y_m < y_u; \mathbf{z}\} = Pr\left\{ \hat{\delta}\, ln\left(\frac{Y_m}{\hat{\beta}} \right) < \hat{\delta}\, ln\left(\frac{y_u}{\hat{\beta}} \right); \mathbf{z} \right\} = Pr\{W_m < w_u; \mathbf{z}\}$$

$$= 1 - \left[-\sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^j \int_0^{\infty} v^{r-2} e^{v\hat{\delta} \sum_{i=1}^{r} ln(x_i/\hat{\beta})} (j+1)^{-1} \left((j+1) e^{vw_u} + \right.\right.$$
$$\left.\left. + \sum_{i=1}^{r} e^{v\hat{\delta}\, ln(x_i/\hat{\beta})} + (n-r) e^{v\hat{\delta}\, ln(x_r/\hat{\beta})} \right)^{-r} dv \right] \text{x}$$
$$\times \left[\sum_{j=0}^{m-1} \binom{m-1}{j} (-1)^j \int_0^{\infty} v^{r-2} e^{v\hat{\delta} \sum_{i=1}^{r} ln(x_i/\hat{\beta})} (j+1)^{-1} \left(\sum_{i=1}^{r} e^{v\hat{\delta}\, ln(x_i/\hat{\beta})} + \right.\right.$$
$$\left.\left. + (n-r) e^{v\hat{\delta}\, ln(x_r/\hat{\beta})} \right)^{-r} dv \right]^{-1} = 1 - \alpha \quad (27)$$

Prof. The proof is carried out in the same manner as the proof of Theorem 1, with the exception of that:

$$W = \frac{ln Y_m - \mu}{\sigma} \quad (28)$$

where Y_m is the largest observation from an independent second sample of m observations also from the distribution (1), and noting that $\exp(W)$ is the largest observation in a sample of m observations from the standard exponential distribution, we have the density of W as:

$$f(w) = me^{w} \exp(-e^{w})[1-\exp(-e^{w})]^{m-1} =$$
$$= me^{w}\sum_{j=0}^{m-1}\binom{m-1}{j}(-1)^{j}\exp[-(j+1)e^{w}] \quad w\in(-\infty,\infty) \quad (29)$$

Corollary 2.1. If $r=n$ and n is large, in the same manner as in Theorem 1 we arrive at the following approximation to (27):

$$Pr\{Y_m < y_u\} = Pr\left\{ln\left(\frac{Y_m}{\hat{\sigma}}\right)^{\hat{\delta}} < ln\left(\frac{y_u}{\hat{\sigma}}\right)^{\hat{\delta}}\right\} = Pr\{W_m < w_u\}$$
$$=1-\left[\sum_{j=0}^{m-1}\binom{m-1}{j}(-1)^{j}\int_{0}^{\infty}\frac{v^{n-2}e^{-n\gamma v}(j+1)^{-1}}{\left[(j+1)(m/n)e^{vw_u}+\Gamma(1+v)\right]^{n}}dv\right]$$
$$\left[\sum_{j=0}^{m-1}\binom{m-1}{j}(-1)^{j}\int_{0}^{\infty}\frac{v^{n-2}e^{-n\gamma v}(j+1)^{-1}}{\left[\Gamma(1+v)\right]^{n}}dv\right]^{-1} = 1-\alpha \quad (30)$$

4. Examples of applications

4.1. Estimation of warranty period

For instance, consider the data of fatigue tests on a particular type of structural components (stringer) of aircraft IL-86. The data are for a complete sample of size $r = n = 5$, with observations (Table 1).

Tab. 1. The data of fatigue tests

Observations	Time to crack initiation (in number of 10^4 flight hours)
x_1	5
x_2	6.25
x_3	7.5
x_4	7.9
x_5	8.1

and results being expressed here in number of 10^4 flight-hours. On the basis of these data it is wished to estimate a lower 0.95 prediction limit on Y_1 in a group of $m = 5$ identical components (for a fleet of k=1 aircraft IL-86) which are to be put into service.

Goodness-of-fit testing. We assess the statistical significance of departures from the Weibull model by performing empirical distribution function goodness-of-fit test. We use the K statistic [2]. For censoring (or complete) datasets, the K statistic is given by:

$$K = \frac{\sum_{i=[r/2]+1}^{r-1}\left(\frac{ln(x_{i+1}/x_i)}{M_i}\right)}{\sum_{i=1}^{r-1}\left(\frac{ln(x_{i+1}/x_i)}{M_i}\right)} = \frac{\sum_{i=3}^{4}\left(\frac{ln(x_{i+1}/x_i)}{M_i}\right)}{\sum_{i=1}^{4}\left(\frac{ln(x_{i+1}/x_i)}{M_i}\right)} = 0.184 \quad (31)$$

where $[r/2]$ is a largest integer $\le r/2$, the values of M_i are given in Table 13 [2]. The reject region for the α level of significance is $\{K>K_{n;1-\alpha}\}$. The percentage points for $K_{n;1-\alpha}$ were given by Kapur and Lamberson [2]. For this example,

$$K=0.184 < K_{n=5;\,1-\alpha=0.95}=0.86 \quad (32)$$

Thus, there is not evidence to rule out the Weibull model.

The maximum likelihood estimates of unknown parameters β and δ are $\hat{\beta} = 7.42603$ and $\hat{\delta} = 7.9081$, respectively. It follows from (6) that:

$$Pr\{Y_1 > y_l;\mathbf{z}\} = Pr\left\{\hat{\delta}\,ln\left(\frac{Y_1}{\hat{\beta}}\right) > \hat{\delta}\,ln\left(\frac{y_l}{\hat{\beta}}\right);\mathbf{z}\right\} = Pr\{W_1 > w_l;\mathbf{z}\} =$$
$$= Pr\{W_1 > -8.4378;\mathbf{z}\} = \frac{0.0000141389}{0.0000148830} = 0.95 \quad (33)$$

and a lower 0.95 prediction limit for Y_1 is y_1=2.5549 ($\times 10^4$) flight hours, i.e., we have obtained the warranty period (or the time to the first inspection) equal to 25549 flight hours with confidence level $\gamma=1-\alpha=0.95$.

4.2. Planning in-service inspections

Let us assume that in a fleet of k aircraft there are km of the same individual structure components, operating independently. Suppose an inspection is carried out at time τ_j, and this shows that initial crack (which may be detected) has not yet occurred. We now have to schedule the next inspection. Let Y_1 be the minimum time to crack initiation in the above components. In other words, let Y_1 be the smallest observation from an independent second sample of km observations also from the distribution (1). Then the inspection times can be calculated recursively as:

$$\tau_{j+1} = \hat{\beta}\exp(w_{j+1}/\hat{\delta}) \quad j\ge 1, \quad (34)$$

where τ_1 is a time of the first inspection, w_{j+1} is determined from:

$$Pr\{W_1 > w_{j+1}; W_1 > w_j,\mathbf{z}\} = \frac{Pr\{W_1 > w_{j+1};\mathbf{z}\}}{Pr\{W_1 > w_j;\mathbf{z}\}} = \frac{Pr\left\{\hat{\delta}\,ln\left(\frac{Y_1}{\hat{\beta}}\right) > w_{j+1};\mathbf{z}\right\}}{Pr\left\{\hat{\delta}\,ln\left(\frac{Y_1}{\hat{\beta}}\right) > w_j;\mathbf{z}\right\}}$$
$$=\left(\int_0^\infty v^{r-2}e^{v\hat{\delta}\sum_{i=1}^{r}ln(x_i/\hat{\beta})}\left(kme^{vw_{j+1}}+\sum_{i=1}^{r}e^{v\hat{\delta}\,ln(x_i/\hat{\beta})}+(n-r)e^{v\hat{\delta}\,ln(x_r/\hat{\beta})}\right)^{-r}dv\right) \quad (35)$$
$$\times\left(\int_0^\infty v^{r-2}e^{v\hat{\delta}\sum_{i=1}^{r}ln(x_i/\hat{\beta})}\left(kme^{vw_{j}}+\sum_{i=1}^{r}e^{v\hat{\delta}\,ln(x_i/\hat{\beta})}+(n-r)e^{v\hat{\delta}\,ln(x_r/\hat{\beta})}\right)^{-r}dv\right)^{-1} = 1-\alpha \quad j\ge 1$$

where:

$$Pr\{W_1 > w;\mathbf{z}\} = Pr\left\{\hat{\delta}\,ln\left(\frac{Y_1}{\hat{\beta}}\right) > w;\mathbf{z}\right\}$$
$$=\left(\int_0^\infty v^{r-2}e^{v\hat{\delta}\sum_{i=1}^{r}ln(x_i/\hat{\beta})}\left(kme^{vw}+\sum_{i=1}^{r}e^{v\hat{\delta}\,ln(x_i/\hat{\beta})}+(n-r)e^{v\hat{\delta}\,ln(x_r/\hat{\beta})}\right)^{-r}dv\right)$$
$$\times\left(\int_0^\infty v^{r-2}e^{v\hat{\delta}\sum_{i=1}^{r}ln(x_i/\hat{\beta})}\left(\sum_{i=1}^{r}e^{v\hat{\delta}\,ln(x_i/\hat{\beta})}+(n-r)e^{v\hat{\delta}\,ln(x_r/\hat{\beta})}\right)^{-r}dv\right)^{-1} \quad (36)$$

$\hat{\beta}$ and $\hat{\delta}$ are the MLE's of β and δ, respectively, and can be found from solution of (7) and (8), respectively.

But again, for instance, consider the data of fatigue tests on a particular type of structural components of aircraft IL-86: x_1=5, x_2=6.25, x_3=7.5, x_4=7.9, x_5=8.1 (in number of 10^4 flight hours) given in Table I, where $r=n=5$ and the maximum likelihood estimates of unknown parameters β and δ are $\hat{\beta} = 7.42603$ and

$\hat{\delta}$ = 7.9081, respectively. Thus, using (35) with τ_1=2.5549 (×10^4 flight hours) (the time of the first inspection), we have obtained the following inspection time sequence (see Table 2).

Tab. 2. The inspection time sequence

w_j	Inspection time τ_j (×10^4 flight hours)	Interval $\tau_{j+1}-\tau_j$ (flight hours)
–	τ_0= 0	–
w_1 = -8.4378	τ_1= 2.5549	25549
w_2 = -6.5181	τ_2= 3.2569	7020
w_3 = -5.5145	τ_3= 3.6975	4406
w_4 = -4.8509	τ_4= 4.0212	3237
w_5 = -4.3623	τ_5= 4.2775	2563
w_6 = -3.9793	τ_6= 4.4898	2123
w_7 = -3.6666	τ_7= 4.6708	1810
w_8 = -3.4038	τ_8= 4.8287	1579
w_9 = -3.1780	τ_9= 4.9685	1398
⋮	⋮	⋮

5. Conclusions

The method of constructing prediction limits for future samples from a Weibull distribution introduced in this paper utilizes all the information in a sample, but since it involves the use of numerical integration, many may prefer to use this technique only in situations not readily handled by other of the methods described earlier. With modern computing, however, the conditional prediction limits are not difficult to calculate and should be recommended when the ability to do computations is available.

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