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ON STATISTICAL MODELLING IN ACCELERATED LIFE TESTING

The aim of this paper is to present some models used in accelerated life testing. The AFT model, the Sedyakin model, the Power Generalized Weibull model and the CHSS model are discussed. Many recent references are given in order to help readers in there choices.

***Keywords**: Accelerated life testing, (ALT), stress, AFT model, Sedyakin model, Power Generalized Weibull model, CHSS model, Weibull family.*

Introduction

Accelerated life models involve as soon as once wants to model lifetime data. In biomedical research, these modelling are grouped under the generic term of survival analysis and study the lifetime under the influence of covariables. In industrial research, these modelling are gathered under the name of reliability and study the lifetime under influence of stresses. The most traditional model is the Accelerated Failure Time model (AFT), see for example, Nelson (1990), Lawless (2003), Meeker and Escobar (1998), Bagdonavicius and Nikulin (2002). After some preliminary definition, we introduce AFT model, Sedyakin model, Power Generalized Weibull model and CHSS model.

Preliminary definitions

Let $\mathcal{E}$ be the set of admissible (possible) stresses or covariables (deterministic or stochastic, time dependant), define from $\Re^+$ to $\Re^m$:

$$\mathcal{E} = \left\{ \mathbf{x}(.) = \left(x_1(.),...,x_m(.)\right)^T : \left[0,\infty\right[\to B \subset \Re^m \right\}$$

We note $\mathcal{E}_1$ the set of all deterministic constant in time stresses, $\mathcal{E}_1 \subset \mathcal{E}$.

In accelerated life testing (ALT) the most used types of stresses are: constant in time stresses, step-stresses, progressive (monotone) stresses, cyclic stresses and random stresses (see, for example, Bagdonavicius and Nikulin (1995,1998,2002), Duchesne (2000, 2004), Duchesne and Lawless (2000,2002), Duchesne and Rosenthal (2003), Elsayed and Liao (2004), Lawless (2003), LuValle (2000), Meeker and Escobar (1998), Nelson (1990), Shaked & Singpurwalla (1983), etc.

The mostly used time-varying stresses in ALT are step-stresses: units are placed on test at an initial low stress and if they do not fail in a predetermined time t_1, the stress is increased. If they do not fail in a predetermined time $t_2 > t_1$, the stress is increased once more, and so on. Thus step-stresses have the form

$$\mathbf{x}(u) = \begin{cases} \mathbf{x}_1, & 0 \le u \le t_1, \\ \mathbf{x}_2, & t_1 \le u \le t_2, \\ \dots & \dots \\ \mathbf{x}_k, & t_{k-1} \le u \le t_k \le \infty, \end{cases} \qquad (1)$$

where $\mathbf{x}_1,...,\mathbf{x}_k$ are from $\mathcal{E}_1$. These sets of step-stresses will be denoted by $\mathcal{E}_k$, $\mathcal{E}_k \subset \mathcal{E}$.

Denote $T_{\mathbf{x}(.)}$ the positive random variable of the time to failure under the stress $\mathbf{x}(.)$, $S_{\mathbf{x}(.)}(t)$ (respectively $f_{\mathbf{x}(.)}(t)$) the survival function (respectively the density) associated to $T_{\mathbf{x}(.)}$. In the deterministic case, we have:

$$S_{\mathbf{x}(.)}(t) = \mathbf{P}\left\{ T_{\mathbf{x}(.)} > t \right\}, \quad \mathbf{x}(.) \in \mathcal{E}.$$

In the stochastic case, we define:

$$S_{\mathbf{x}(.)}(t) = \mathbf{P}\left\{ T_{\mathbf{x}(.)} > t \mid \mathbf{x}(s),\ 0 \le s \le t \right\}, \quad \mathbf{x}(.) \in \mathcal{E}.$$

We can define also the hazard rate $\lambda_{\mathbf{x}(.)}(t)$ and the cumulative hazard rate $\Lambda_{\mathbf{x}(.)}(t)$ such that:

$$f_{\mathbf{x}(.)}(t) = -S'_{\mathbf{x}(.)}(t), \quad \lambda_{\mathbf{x}(.)}(t) = -\frac{S'_{\mathbf{x}(.)}(t)}{S_{\mathbf{x}(.)}(t)}$$

and $\Lambda_{\mathbf{x}(.)}(t) = \int_0^t \lambda_{\mathbf{x}(.)}(u)du = -\log\left(S_{\mathbf{x}(.)}(t)\right)$ for $t \ge 0$.

One can interpret $\mathbf{x}(s)$, $0 \le s \le t$, as the « history » until the time t.

Accelerated Failure Time model

The AFT model, sometime named Additive Accumulative of Damages model, is verified on $\mathcal{E}$ if there exist a basic survival function S_0 and a positive function $r : \mathcal{E} \to \Re^+$ such that:

$$S_{\mathbf{x}(.)}(t) = S_0\left(\int_0^t r\left\{\mathbf{x}(\tau)\right\} d\tau\right), \ \mathbf{x}(.) \in \mathcal{E}.$$

If the stress is constant, i.e. $\mathbf{x}(.) \equiv \mathbf{x} \in \mathcal{E}_1$, we have:

$$S_{\mathbf{x}}(t) = S_0\left(r\left\{\mathbf{x}(t)\right\}\right)$$

Often it is reasonable to take the baseline function S_0 from a parametric family, for example from the Power Generalized Weibull (PGW) family of distributions, which was suggested by accelerated life models in survival analysis and reliability, describing dependence of the lifetime distributions on the explanatory variables. (see, Bagdonavicius and Nikulin (2002)). In terms of the survival function the PGW family is given by the next formula:

$$S(t,\sigma,\nu,\gamma) = \exp\left\{1-\left[1+\left(\frac{t}{\sigma}\right)^{\nu}\right]^{\frac{1}{\gamma}}\right\},$$
$$t>0,\ \gamma>0,\ \nu>0,\ \sigma>0.$$

If $\gamma = 1$ we have the Weibull family of distributions. If $\gamma = 1$ and $\nu = 1$, we have the exponential family of distributions. This class of distributions has very nice probability properties. All moments of this distribution are finite. In dependence of parameter values the hazard rate can be constant, monotone (increasing or decreasing), unimodal or $\cap$-shaped, and bathtube or $\cup$-shaped. Another interesting family, the Exponentiated Weibull Family of distributions, was proposed Mudholkar and Srivastava (1995).

The AFT model can be parametric, non-parametric and semi-parametric, according our knowledge about the model. The model is complicated and estimation is not trivial. It is important to think about the plans in ALT.

To estimate the unknown parameters of this model was proposed several plans within the framework of the following experimental design:

Two groups of items are used: the first group of size n_1 is used under an one-dimensional constant in time accelerated stress $x_1 \in \mathcal{E}_1$ and all failures are observe during the time of experiment T (or noted t_2). The second group of size n_2 is used under an one-dimensional step stress $x_2(.) \in \mathcal{E}_2$ which consist in the accelerated stress x_1 until the moment $t_1 < T$ and then under the normal stress x_0 until the end of the experiment T (see figure 1).

In the case of accelerated experiment $x_2(.)$, we have:

$$S_{x_2(.)}(t) = \begin{cases} S_{x_1}(t),\ 0 \le t \le t_1, \\ S_{x_0}(t - t_1 + t_1^*),\ t \ge t_1, \end{cases}$$

where the moment t_1^* is determined by the equality $S_{x_1}(t_1) = S_{x_0}(t_1^*)$.

If the functions $r(.)$ and $S_0(.)$ are unknown we have a nonparametric model. If the function $r(.)$ is parameterized and the baseline function S_0 is completely unknown, we have a semiparametric model. Very often the baseline survival function S_0 is also taken from some class of parametric distributions, such as Weibull, lognormal, loglogistic, etc. In this case we have a parametric model and the maximum likelihood estimators of the parameters are obtained by almost standard way for any plans. Parametric case was studied by many people, see, for example, Bagdonavicius, Gerville-Réache and Nikulin (2002), Nelson (1990), Meeker & Escobar (1998), Kahle and Lehmann (1998) , Kahle and Wendt (2000), Sethuraman and Singpurwalla (1982), Shaked and Singpurwalla (1983), Viertl (1988), etc. Nonparametric and semiparametric analysis of AFT model was considered by Lin and Ying (1995), Duchesne & Lawless (2000, 2002), Bagdonavicius and Nikulin (1997, 2002, 2004), etc.

For example, numerical studies on the finite sample proprieties of those estimators show in particular that the variability of the nonparametric estimator of the survival function is closed to the variability of the semi-parametric estimator (Bagdonavicius, Gerville-Réache, Nikoulina and Nikulin (2000)).

Lastly, within the framework of the semi-parametric estimators, three principles of optimization of this experimental design were also worked out (see Gerville-Réache (2004)):

- It is reasonable to choose $n_1 \approx n_2$.
- It is necessary to fix x_1 as large as possible.
- It is necessary to fix t_1 such that the probability of failure under the normal stress x_0 be maximal.

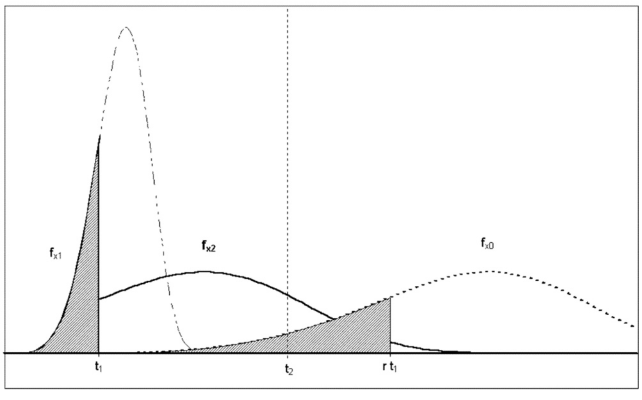

Fig. 1. Densities of the experimental design

Model of Sedyakin

In 1966, Sedyakin (1966) formulated his famous physical principle in reliability which states that for two identical populations of units functioning under different stresses x_1 and x_2, two moments t_1 and t_2 are equivalent if the probabilities of survival until these moments are equal:

$$P\left\{T_{\mathbf{x}_1} \geq t_1\right\} = S_{\mathbf{x}_1}(t_1) = S_{\mathbf{x}_2}(t_2) = P\left\{T_{\mathbf{x}_2} \geq t_2\right\},$$

$$\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{E}$$

With the accelerated experiment (1) we have for all $s>0$

$$\lambda_{\mathbf{x}_1}(t_1 + s) = \lambda_{\mathbf{x}_2}(t_2 + s)$$

In ALT is used the model of Sedyakin on $\mathcal{E}$, based on this idea. Following Bagdonavicius and Nikulin (1995) we give the following definition.

The Sedyakin's model (SM) holds on a set of stresses $\mathcal{E}$ if there exists on $\mathcal{E} \times \Re^+$ a positive function g such that for all $x(.) \in \mathcal{E}$

$$\lambda_{\mathbf{x}(.)}(t) = g\left(x(t), \Lambda_{\mathbf{x}(.)}(t)\right).$$

In the case of accelerated experiment $x_2(.)$, we have:

$$S_{\mathbf{x}_2(.)}(t) = \begin{cases} S_{\mathbf{x}_1}(t), \ 0 \leq t \leq t_1, \\ S_{\mathbf{x}_0}(t - t_1 + t_1^*), \ t \geq t_1, \end{cases}$$

So the AFT model with a stress on $\mathcal{E}_2$ is the Sedyakin model.

Changing shape and scale model

Application of the AFT model in the case of the above considered test plan of experiment is considered in Bagdonavicius and Nikulin (2000). Nevertheless, the AFT model is narrow and not always suitable for applications. Natural generalization of the AFT model for constant stresses (see Nelson (1990), Meeker and Escobar (1998)) is obtained by supposing that under different stresses not only scale but shape parameters are different. In this situation is interesting to apply the so-called changing scale and shape (CHSS) model (Bagdonavicius and Nikulin (2002), Bagdonavicius, Nikulin, Zdorova-Cheminade (2004)). The CHSS model holds in a set $\mathcal{E}$ of time-varying stresses if for any $x(.) \in \mathcal{E}$

$$S_{\mathbf{x}(.)}(t) = S_0\left(\int_0^t r\left\{\mathbf{x}(\tau)\right\} \tau^{\nu(\mathbf{x}(\tau))-1} d\tau\right),$$

where $r, \nu \colon \mathcal{E} \to R_+$.

In terms of the hazard rate the model can be written in the form :

$$\lambda_{\mathbf{x}(.)}(t) = r\left\{\mathbf{x}(t)\right\} q\left(\Lambda_{\mathbf{x}(.)}(t)\right) t^{\nu(\mathbf{x}(t))-1}$$

The hazard rate can be monotone (increasing or decreasing) or ⋃-shaped. For more details one can see in Bagdonavicius, Nikulin, Zdorova-Cheminade (2004).

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