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RISK AVERSION IN MAINTENANCE

The concept of risk averse maintenance is introduced. It is formulated in terms of seeking to minimize a disutility rather than a cost per unit time. A general formalism is given, followed by an example, the application to age-based replacement. The problem of overmaintenance caused by undue risk aversion on the part of engineers is briefly discussed.

Keywords*:* *risk-aversion, age-based replacement, maintenance, utility-function approach*

1. Introduction

The concept of risk-aversion is central to economic and financial thought. In this context, the word 'risk' denotes variability in cash flows. In general, both individuals and organizations are risk averse, and this has implications for maintenance practice. Risk averse policies require more frequent replacement or maintenance. The higher spend on maintenance can be thought of as an insurance against unexpected losses.

There seems to be no existing work on risk-averse maintenance policies, and very little on risk-averse operational research in general. Exceptions are the papers of Padmanabhan and Rao (1993) and Chun and Tang (1995), who have studied risk-averse warranty policies.

In this paper, risk-averse maintenance policies are modelled using the methodology of utility functions developed in economics. Rather than seeking to minimize a mean cost per unit time, a rational risk-averse individual would seek to maximize a concave utility function.

A degree of risk aversion is entirely rational. A conflict of interest can however arise when a maintenance engineer carries out maintenance policies on behalf of Management. If the engineer is more risk averse than the manager, from the viewpoint of management, the equipment is being overmaintained. This is an example of what within principal-agent theory (e.g. Laffont and Martimort 2002) is called the moral hazard problem. A solution is to use incentives to induce maintenance engineers not to overmaintain. There is not space to discuss this topic further.

2. Risk-averse maintenance

Risk aversion can be modelled via a concave utility function. The utility of a sum of money y is $U(y)$, where $U' > 0$, $U'' < 0$, the primes denoting differentiation.

We use here only the exponential utility function, defined as

$$U = \frac{1 - \exp(-\eta y)}{\eta} \qquad (1)$$

where $\eta > 0$ is a measure of risk aversion. An expenditure $x = -y$ has disutility

$$-U = \frac{\exp(\eta x) - 1}{\eta} \qquad (2)$$

and this form is used from now on.

The *certainty-equivalent* of a policy is the sum of money that if definitely gained or lost would have the same expected utility as the variable cashflows of the policy. Here we use the certainty equivalent sum D per unit time. Hence if a policy is carried out for time T, we have for the exponential utility function that

$$\frac{\exp(\eta DT) - 1}{\eta} = \frac{E\exp(\eta X) - 1}{\eta}$$

or

$$D = \frac{\log\{E\exp(\eta X)\}}{\eta T} \qquad (3)$$

Using the exponential utility function given in equation 2, consider a general maintenance or replacement policy in which cycles, which can be of fixed or variable length, end in replacement, inspection, or some regenerating event. During the i^{th} cycle, a random number of failures N_i occurs, at cost c_f each, and the regenerating event has cost c_s. More generally, a random cashflow F_i occurs during the cycle. Consider the certainty-equivalent expenditure per unit time D, when the cycles continue to some very large time T. We consider first the case where cycles are of fixed length t.

The certainty-equivalent expenditure per unit time is then

$$D = \frac{\log E\{\exp(\eta\sum_{i=1}^{T/t}(F_i))\}}{\eta T}$$

Hence as the cycles are independent,

$$D = \frac{\log E\exp(\eta F)}{\eta t} \quad (4)$$

dropping the cycle subscript *i*. This is the criterion to be minimized in place of the cost per unit time.

The expression $E\exp(\eta F)$ is the moment generating function of the random variate *F*, with parameter η. If $F_i = c_f N_i + c_s$, then

$$E\exp(\eta F) = \exp(\eta\, c_s)\, E(\eta\, c_f N)$$

i.e. proportional to the mgf. of the number of failures, with parameter $c_f\,\eta$.

For optimization problems the task of finding the mean of a random variate has been replaced by the task of finding its moment generating function. In general, $\log E\exp(\eta F)$ is the cumulant generating function, so that

$$D = \frac{\sum_{j=1}^{\infty} \eta^{j-1}\kappa_j / j!}{t}$$

where κ_j is the j^{th} cumulant of the cost per cycle. As risk aversion increases, the function *D* to be minimised puts increasing weight on the higher cumulants, such as skewness and kurtosis.

When cycles have variable length, such as for age-based replacement, equation 4 becomes

$$\exp(\eta DT) = \sum_{N=0}^{\infty} (\log E\exp(\eta F))^N P_N = G(\log E\exp(\eta F))$$

where P_N is the probability that *N* cycles of the embedded renewal process have occurred by time *T*, and hence *G* is the moment generating function for the number of cycles. Thus,

$$D = \frac{\log\{G(\log E\exp(\eta F))\}}{\eta T} \quad (5)$$

where log *G* is the cumulant generating function for the number of cycles.

There is an elegant exact solution for *D*, from applying the Wald identity to a renewal process (Cox 1962). This identity yields the asymptotic result

$$\log E\exp(-\log M(s) N(T)) = sT \quad (6)$$

where $M(s) = E(\exp(-st))$, and the expectation is of the distribution of cycle length. The left hand side of equation 6 is the cumulant generating function of the number of cycles $N(T)$. Hence equation 5 yields simply

$$D = s^*/\eta \quad (7)$$

where s^* is the value of s for which the coefficient of $N(T)$ in equation 6, equals $\log E\exp(\eta F)$, i.e.

$$M(s^*) = (E\exp(\eta F))^{-1} \quad (8)$$

The exact calculation of *D* from equation 7 requires only the solution of equation 8 for s^*, or explicitly

$$1 - s^* \int_0^t S(u)\exp(-s^* u)\,\mathrm{d}u - (E\exp(\eta F))^{-1} = 0 \quad (9)$$

where $S(u)$ is the survival function of the cycle length. This equation can be solved by Newton-Raphson iteration.

As an example, in age-based replacement an item is replaced at age t or on failure. The cost per unit time is

$$C = \frac{c_f + (c_s - c_f)S(t)}{\int_0^t S(u)\,\mathrm{d}u}$$

where $S(t)$ is the survival function of the failure-time distribution (Jardine, 1973).

Let *X* be a random (indicator) variable, where $X = 1$ denotes failure in $(0, t]$ and $X = 0$ denotes survival to time *t* without failure. Then

$$D = \frac{\log\{E\exp(\eta(c_f X + c_s(1-X))\}}{\eta\mu} + \ldots$$

where $\mu = \int_0^t S(u)\,\mathrm{d}u$.

Expanding the exponential, since *X* is idempotent and $E(X) = 1 - S(t)$,

$$E\exp((c_f - c_s)\eta X) = S(t) + \exp((c_f - c_s)\eta)(1 - S(t))$$

and rearranging

$$D = c_f/\mu + \frac{\log\{1 + S(t)(\exp(\eta(c_s - c_f)) - 1)\}}{\eta\mu} + \ldots \quad (10)$$

As a concrete example, consider a Weibull distribution of time to failure, with scale parameter 1 and shape parameter 3, and let $c_s = 1$, $c_f = 5$. The replacement age decreases with increasing risk aversion and the survival function increases (figure 1).

3. Conclusions

The utility-function approach to risk-aversion can be generally applied to maintenance and replacement problems. On choosing an exponential utility function, a mathematically elegant scheme for deriving the disutility of a policy results. The mathematics now requires computation of higher moments than simply the mean cost.

Overmaintenance or undermaintenance of equipment by engineers can be regarded as an example of a principal-agent problem. It can be shown how the use of incentives may reduce net cost to management by reducing overmaintenance. Note that the same approach would also correct undermaintenance.

This general approach to risk aversion could be used throughout OR, wherever a minimum cost per unit time policy is considered. As human beings are undoubtedly risk-averse, it is a little surprising that OR routinely ignores this fact. Modifying standard OR solutions to include risk aversion gives a wide application area indeed.

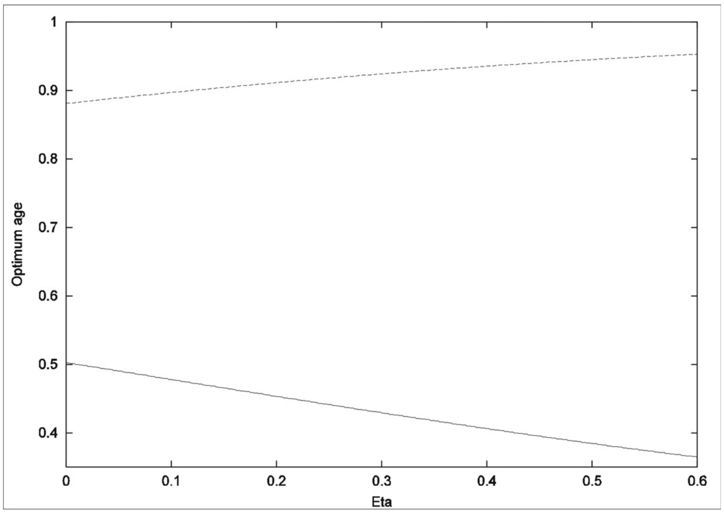

Fig. 1. Age based replacement with a Weibull distribution of scale parameter 1, shape parameter 3, replacement cost $c_s = 1$, failure cost $c_f = 5$. The solid line shows the variation of optimum age at replacement with the risk-aversion parameter η, and the dotted line the survival function

4. References

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