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MULTI-STATE RELIABILITY: MODELS AND APPLICATIONS

This paper focuses on customer-centered reliability models and measures for multi-state systems with multi-state components. A review of general models which capture the customer's experience with the product is presented. An approach is given to develop the system structure function using equivalent classes and develop reliability bonds. In addition to measures and models, ideas are given for potential applications of these models to infrastructure problems such as transportation, computer network, supply chain, communication systems and network reliability.

***Keywords**: Multi-state reliability, multi-state systems and components, general structure functions, reliability bounds*

1. Introduction

In the traditional reliability methods (Kapur et al. 1997), the system and all of its components are assumed to have only two states of working efficiency which are working perfectly and total failure. Although this assumption simplifies the complicated problems for reliability evaluation, it losses the ability to reflect the reality that most systems actually degrade gradually and have a wide range of working efficiency (Barlow et al. 1978, Boedigheimer et al. 1994, Kapur 1986, Lisnianski et al. 2003, and Natvig 1982). In the literature most of the work on multi-state reliability research makes the assumption that the system and all of its components have same number of states. This assumption is not realistic because in reality the system and components have different numbers of states (Lisnianski et al. 2003, Boedigheimer et al. 1994, Brunelle et al. 1997 and 1999, Hudson et al., 1983 and 1985). The main focus of this paper is to make sure that the reliability measures capture the reality of multiple states for the systems and the components, and assure that they can capture the total experience of the customer with the system. Then these general measures can be applied to broad problems in engineering systems, supply chain and logistics, general networks for transportation and distribution, and computer and communication systems.

2. Development of general structure function

For a multi-state system with n components, let each component i have (m_i+1) different and distinct states or levels of working efficiency. Also the system has M+1 different levels of working efficiency.

Let S = [0, 1, …, m_1] × [0, 1, …, m_2] × … × [0, 1, …, m_n] be the components state space, and s = [0, 1, …, M] be the set of all possible states of the system. Then we can express the relationship between the components and system at time t by

$$\phi(x,t) : S \rightarrow s$$

For any state of working efficiency $k \in (0, 1, \ldots, M)$ of a system, we define

$$S_k = \{x \mid \phi(x) = k\}, \forall k \in (0,\ 1, \cdots,\ M)$$

where $x = (x_1, x_2, \ldots, x_n)$

S_k is known as the Equivalent Class, the collection of all combination of the n components with different states that make the system to be in state k. S_k's are mutually exclusive and $\bigcup_{k=0}^{M} S_k = S$, the component state space. Let Θ_k be the number of elements in each S_k. Of those Θ_k different elements, L_k of them are called the "Lower Boundary Points" and U_k of them are called the "Upper Boundary Points".

Definition 1 (Lower Boundary Points):

$x = (x_1, x_2, \ldots, x_n) \in S_k$ is called a lower boundary point if only if for any $y = (y_1, y_2, \ldots, y_n) < x$, then $\phi(y) < k$.

Definition 2 (Upper Boundary Points):

$x = (x_1, x_2, \ldots, x_n) \in S_k$ is called an upper boundary point if only if for any $y = (y_1, y_2, \ldots, y_n) > x$, then $\phi(y) > k$.

$LB(k) = \left(\hat{x}_{(1,k)}, \hat{x}_{(2,k)}, \cdots, \hat{x}_{(L_k,k)}\right) \subseteq S_k$ is called the "Lower Boundary Points Set" and $\hat{x}_{(i,k)}$ is the i_{th} lower boundary point for S_k, $\forall i \in (1, 2, \cdots, U_k)$. $UB(k) = \left(\breve{x}_{(1,k)}, \breve{x}_{(2,k)}, \cdots, \breve{x}_{(U_k,k)}\right) \subseteq S_k$ is called the "Upper Boundary Points Set" and $\breve{x}_{(i,k)}$ is the i_{th} upper boundary point for S_k, $\forall i \in (1, 2, \cdots, U_k)$.

2.1. Generation of the generic structure function with lower (upper) boundary points

With the lower boundary points ($\hat{x}_{(i,k)}$), Liu et al. (2004) have developed the generic structure function for the system. From the definition of the lower boundary point, we know that a system is in the state k or higher if x is greater than or equal to at least one lower boundary point in the lower boundary points set *LB(k)*. We can formulate this as

$$\hat{I}(k) = 1 - \prod_{i=1}^{L_k}\left[1 - I(x \geq \hat{x}_{(i,k)})\right]$$

If $\hat{I}(k+1) > \hat{I}(k)$ then let $\hat{I}(k) = \hat{I}(k+1)$. When $k = 0$ means that the system is totally failed, and we let $\hat{I}(0) = 1$. Then the structure function is

$$\phi(x) = \sum_{k=0}^{M} \hat{I}(k) - 1$$

Similarly, with the upper boundary points ($\breve{x}_{(i,k)}$), we can formulate

$$\breve{I}(k) = \prod_{i=1}^{U_k}\left[1 - I(x \leq \breve{x}_{(i,k)})\right]$$

If $\breve{I}(k+1) > \breve{I}(k)$ then let $\breve{I}(k+1) = \breve{I}(k)$. When $k = M$ means that the system is perfectly working, and we let $\breve{I}(M) = 0$. Then the structure function is

$$\phi(x) = \sum_{k=0}^{M} \breve{I}(k)$$

Using the above structure functions, we can find the expected values of the state of the system as below:

$$E[\phi(X)] = E\left[\sum_{k=0}^{M} \hat{I}(k) - 1\right] = \sum_{k=0}^{M} E\left[\hat{I}(k)\right] - 1$$

and

$$E[\phi(X)] = E\left[\sum_{k=0}^{M} \breve{I}(k)\right] = \sum_{k=0}^{M} E\left[\breve{I}(k)\right]$$

2.2. Reliability bounds

In addition, we can find bounds on the expected values of state of the system as below (for details see Liu et al. (2004)).

The lower bound is

$$\sum_{l=0}^{M-1} \prod_{k=0}^{l} \prod_{i=1}^{U_k}\left[1 - \prod_{j=1}^{n} Prob\left(X_j \leq \breve{x}_{[(i,k),j]}\right)\right]$$

and the upper bound is

$$M - \sum_{l=0}^{M-1} \prod_{k=M-l}^{M} \prod_{i=1}^{L_k}\left[1 - \prod_{j=1}^{n} Prob\left(X_j \geq \hat{x}_{[(i,k),j]}\right)\right]$$

2.3. Example

Consider a system with two components with $m_1 = 3$, $m_2 = 2$ and $M = 3$. The information on the boundary points is given in Table1, and information for the component state probabilities is given in Table2.

For this system, we get $E[\phi(X)] = 1.65$ using either the lower boundary points or upper boundary points. Also, bounds on system reliability are $1.54 \leq E[\phi(X)] \leq 1.75$.

3. Customer-centered reliability measures

One proposed measure for customer-centered reliability for a target life t_0 is

$$\int_0^{t_0} E[\Phi(t)]\, dt = \int_0^{t_0} \sum_{k=0}^{M} k\, P(\Phi(t) = k)\, dt =$$
$$= \sum_{k=0}^{M} \int_0^{t_0} k\, P(\Phi(t) = k)\, dt$$

With customer's utility as a function of the state of the system, we can calculate the customer's expected total utility for experience (ETUE) with the system from time 0 to time t_0. This is given by:

$$ETUE = \int_0^{t_0} E\left[U(\Phi(t))\right] dt =$$
$$\int_0^{t_0} = \sum_{k=0}^{M} U(k) P(\Phi(t) = k)\, dt =$$
$$= \sum_{k=0}^{M} \int_0^{t_0} U(k) P(\Phi(t) = k)\, dt$$

The greater the ETUE, the better the system is for the customer.

For details and applications of these measures, see [Liu et al. 2005].

Table 1. Lower/Upper Boundary Points

k	S_k	Lower Boundary Point	Upper Boundary Point
0	(0,0)		(0,0)
1	(3,0),(2,0),(0,2),(1,0),(0,1)	(1,0),(0,1)	(3,0),(0,2)
2	(2,2),(3,1),(2,1),(1,2),(1,1)	(1,1)	(2,2),(3,1)
3	(3,2)	(3,2)	

Table 2. Component-state Probability

Component	Component State (x)			
	0	1	2	3
1	0.2	0.4	0.1	0.3
2	0.3	0.2	0.5	

4. Infrastructure applications

Modern society increasingly relies on infrastructure networks such as supply chain and logistics, transportation networks, commodity distribution networks (oil/ water/ gas distribution networks), computer and communication networks, etc. Network and its components can provide several levels of performance and thus the performance of the network and its components can be considered as a range from perfect functioning to complete failure.

A network consists of two classes of components: nodes and arcs (or edges). A topology of a network model can be represented by a graph, $\boldsymbol{G} = (\boldsymbol{N}, \boldsymbol{A})$ where $\boldsymbol{N} = \{s, 1, 2, \ldots, n, t\}$ is the set of nodes with s as the source node and t as the sink node and $\boldsymbol{A} = \{a_i | 1 \le i \le n\}$ is the set of arcs where an arc a_i joins an ordered pairs of nodes $(i, i') \in \boldsymbol{N} \times \boldsymbol{N}$ such that $i \neq i'$. Let $m = \{m_1, m_2, \ldots, m_n\}$ be a vector of maximum capacities for the arcs. Assume all the nodes in the network are perfectly reliable. Based on the maximum capacity, we can easily find the maximum flow in the network from node s to node t. This maximum value of flow is equivalent to state M of the system for the development of the structure function in section 2, and $0 \le k \le M$. The actual capacity at any time of the arc degrades from m_i, $i = 1, \ldots, n$, to 0. Let x_i be the actual capacity of the arc a_i, $0 \le x_i \le m_i$, and x_i integer.

We can solve the following optimization problem:

Max f
subject to
$Ex^t = (e_s - e_t) f$, E is the node-arc incidence matrix
$x^t \le m^t$
$x^t \ge 0$ and integer.

Thus, $S_M = \{x \mid f(x) = M\}$, the equivalent class for the highest value M of the state of the system.

Research is under way to generate all the equivalent classes and their boundary points. Then we can apply the methods discussed in sections 2 and 3 to evaluate reliability of the infrastructure networks.

5. References

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