Johannes Altenbach
Holm Altenbach

TRENDS IN ENGINEERING PLATE THEORIES

The paper summerises some basic trends in modelling and analysis of refined plate theories not primary from a mathematical aspect but from the viewpoint of engineering application and of an increasing reliability of engineering structure analysis. Shear rigid and shear deformable plate models including shear correction factors or kinematical warping effects can be formulated. An extended two-dimensional theory for transverse shear stress analysis yields improved results, the influence of singularities on the global structure behaviour must be considered and the kinematical degree of freedom of plate models is an important criterion in structural analysis of folded plate structures. Composite materials have become an increasing importance in engineering structures. Multilayered, laminated and sandwich plates are used in aerospace and many other industries. Thin-walled structures composed of composite material have generally a moderate thickness and low transverse stiffness. An adequate modelling of structural plate elements should be given by refined plate theories.

1. Introduction

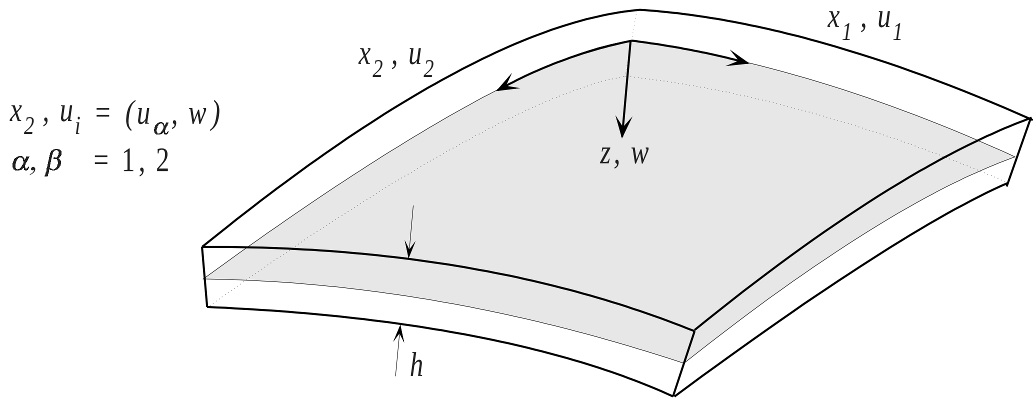
Modelling and analysis of plates and plate structures are basic problems of mechanical structure analysis in civil and mechanical engineering. The classical Kirchhoff's plate theory was for a long period the predominant model in engineering applications, e.g. in bridge-building, shipbuilding, etc. The structural elements of these engineering constructions were in general single layered plates composed of metal or reinforced concrete. Their mechanical properties were on a macroscopic level homogeneously through the thickness and the structural response was isotropic or orthotropic.

A suitable extension of the geometrically linear Kirchhoff's plate theory has been presented by von Kármán. In selected thin-walled structures the values of the displacements and the plate thickness are of the same order. The nonlinear terms in the kinematical plate equations affect the stress and the deformation states of the thin plate and must be partly considered. The necessity to apply the von Kármán's plate equations is connected with the loading level and the plate geometry. Note that loading conditions and geometrical relations which are typically for the use of the Kirchhoff's plate theory in dependence on the material model (elastic, plastic or time-dependent) in some cases demand the introduction of the von Kármán's plate equations.

Nowadays there is a growing use of composite materials and modern lightweight structures employ plate elements consisting of three or a much greater number of layers. The material of the layers can be either homogeneous or inhomogeneous. Typical three-layered structure elements are sandwich plates. The outer layers are made of high strength material, e.g. homogeneous metal faces or fibre reinforced multilayered faces. They cover the inner layer consisting of a soft and rather homogeneous material, e.g. a foam, or of an inhomogeneous material, e.g. a cellular filler. The first serious applications of sandwich structures concerned the aircraft industry, because sandwich structural elements created lightweight design and improved strength properties and corresponded functional requirements and mechanical characteristics.

At present, sandwich and multilayered, laminated plates are widely used in various fields of engineering, for instance in aircraft and rocket constructions, machine building, automotive industry, oil production, civil engineering, energetics, sports industry, etc. Modelling and analysis of sandwich and laminated plates is a more complicated problem of engineering plate modelling and yield an increasing significance of extended, refined plate theories. First steps in this direction were presented by Reissner and Mindlin for homogeneous moderately thick plates.

All engineering plate concepts reduce the three-dimensional equilibrium, kinematical and constitutive equations of the mechanics of deformable bodies and describe approximately the mechanical response of plate structures by a two-dimensional model. These engineering concepts based in general on hypotheses of the deformation or stress states. Figure 1 summerises some linear plate models with different approximations of the in-plane and the transverse displacements and or the in-plane and the transverse stresses, respectively. Also mixed approximations of displacements and stresses are possible and can be useful. In the equations of the displacement approximations $u_\alpha(x_\beta,z)$ are the in-plane displacements and $w(x_\beta,z)$ the transverse displacement or the plate deflection. $u_\alpha^0(x_\beta)$ are membrane displacements, $w(x_\beta)$ the deflection of the middle surface of the plate, $\varphi_\alpha(x_\beta)$ are cross-sectional rotations. and are sets of functions in product series expansions for the in-plane displacements and the transverse displacements. The second term in the product series expansion for in-plane displacements with a set of functions $\psi^q(z)$ introduces a coupling between the in-plane and the transverse displacements and allowing to impose restrictions on the distributions of the transverse shear stresses of a plate. Most applied classical and refined plate theories can be incorporated in the product series approximation as special series of the general expansions, e.g. Kirchhoff's, Hencky's, Preußer's or Touratier's plate models [2, 3]. Plate modelling that incorporate conditions on the transverse normal stresses are not included in these approximations. It demands additional terms in the series approximation for $w(x_\beta,z)$ which contain second derivatives of u^q_α and w^q. Asymptotic integration methods or concepts of the



- Displacement approximations

$u_\alpha(x_\beta,z) = u_\alpha^0(x_\beta) - zw,_\alpha(x_\beta)$ — Kirchoff (1850)
$w(x_\beta,z) = w(x_\beta)$

$u_\alpha(x_\beta,z) = u_\alpha^0(x_\beta) + z\varphi_\alpha(x_\beta)$ — Hencky, Bollé (1974)
$w(x_\beta,z) = w(x_\beta)$ — Mindlin (1951)

$u_\alpha(x_\beta,z) = u_\alpha^0(x_\beta) - [w,_\alpha(x_\beta) + \varphi_\alpha(x_\beta)]4z^3/3h^2$ — Levinson (1981)
$w(x_\beta,z) = w(x_\beta)$ — Reddy (1989)

- Displacement approximations - Product series

$u_\alpha(x_\beta,z)=\sum_{q=0}^{K_1} u^q_\alpha(x_\beta)\phi^q(z)+\sum_{q=0}^{K_2} w^q_\alpha(x_\beta)\psi^q(z)$ — Meenen,Altenbach (1999)
$w(x_\beta,z) = w(x_\beta)$

- Stress approximations

$$(\sigma_\alpha,\sigma_\beta,\tau_{\alpha\beta}) = \frac{12z}{h^3}(M_\alpha,M_\beta,M_{\alpha\beta}),$$ — E. Reissner (1944), (1950)

$$\tau_{\alpha 3} = \frac{3Q_\alpha}{2h}\left[1-\left(\frac{2z}{h}\right)^2\right]$$

$$\sigma_{33} = \frac{3q}{4}\left[\frac{2z}{h}-\frac{1}{3}\left(\frac{2z}{h}\right)^3\right]$$

- Mixed approximations — E. Reissner (1984), (19870)

Fig. 1. Two-dimensional modelling of thin and moderately thick plates, linear plate theories

introduction of deformable two-dimensional directed surfaces are more mathematical techniques and are not considered in this paper. For more information see, for example, [4].

2. Engineering Theories for Isotropic Plates

A classification of engineering plate theories can be given by

- small deflection linear theory for thin plates,
- small deflection linear theory for moderately thick plates, and
- large deflection nonlinear theory for thin plates.

These classical plate models for homogeneous isotropic plates are connected with the fundamental papers of Kirchhoff, Reissner, Mindlin and v. Kármán [1]. Figures 2 and 3 show the kinematical hypotheses and the stress distributions for these classical models.

The Kirchhoff's plate model yields, from viewpoint of engineering applications, reasonable accuracy in the analysis of various global response parameters like deflection, vibration, buckling, etc. This simplest and most widely used plate model in classical structural mechanics based on the following assumptions:

- Points on a normal to the undeflected midplane of the plate lie on a normal to the midplane of the deflected plate, i.e. a lineal element of the plate extending through the plate thickness, normal to the midplane in the unstressed state, undergoes at most a translation and a rotation and mains normal to the deformed middle surface. The lineal element through the thickness does not elongate or contract and the plate behaviour in thickness direction is shear rigid.
- The slope of the deflected plate in any direction is small, so that its square may be neglected in comparision with unity.
- The midplane of the plate is a "neutral plane", i.e. any midplane stresses arising from the deflection of the plate into a non-developable surface may be ignored.
- The stresses normal to the midplane, arising from applied loading, are negligible in comparision with the normal stresses in the plane of the plate.

Experimental tests and analytical or numerical reference results of three-dimensional plate problems qualify the Kirchhoff's assumptions if an elastic isotropic plate is sufficiently thin and its transverse deflection is small compared to its thickness. The contradiction between the number of physically motivated boundary conditions and the order of the governing differential equation can be overcome by introducing "Kirchhoff's edge forces". From a theoretical point of view the Kirchhoff's plate theory underestimates the transverse deflection and overestimates natural frequencies and buckling loads.

The von Kármán's plate model is in combination with the Kirchhoff's assumptions based on the fundamental additional assumption that geometrical nonlinearity can not be neglected, but that the strains and the square of rotations are small compared to unity. So, in a more correct view, the von Kármán's plate theory, which is restricted to small strains, but moderately large rotations must be termed as a "moderately large deflection theory" but, customarily, we find in literature the technical term "large deflection theory". The von Kármán plate theory predicts the deflection and stresses in thin plates with reasonable accuracy for deflections having the order of the plate thickness.

The Kirchhoff and the von Kármán plate theory are based on three independent kinematical degrees of freedom that means three independent translations for all material points of the middle surface, if there are transverse and in-plane loadings. If the plate loading is restricted to transverse loading only, the independent kinematical degree of freedom reduces for Kirchhoff plates to one. The Reissner or the Mindlin plate theories include approximately the effect of transverse shear strains and yields a simple refined plate theory by introducing assumptions on the displacement distribution over the plate thickness, which is less restrictive than the Kirchhoff assumption. The improved plate model has five kinematical degrees of freedom: three translations and two independent rotations. These shear deformable linear plate models of Reissner and Mindlin can be simple extended to a nonlinear shear deformable plate model for moderate large deflections by including the von Kármán's geometrical nonlinear relations [5].

The shear rigid von Kármán plate model has been used in creep damage analysis. This problem is mathematical formulated by a non-linear initial-boundary problem which can be solved numerically. Numerical examples demonstrate the significance of the geometrical nonlinear plate theory in the creep-damage analysis even for moderate deflections [6].

3. Classical and Refined Theories for Multilayered Plates

The modelling of multilayered plates, i.e. sandwich and laminated plates, based on an analogy with the modelling of isotropic, single-layer plates, section 2. But there are two principal different approaches for a two-dimensional plate modelling of multilayered plates [7]:

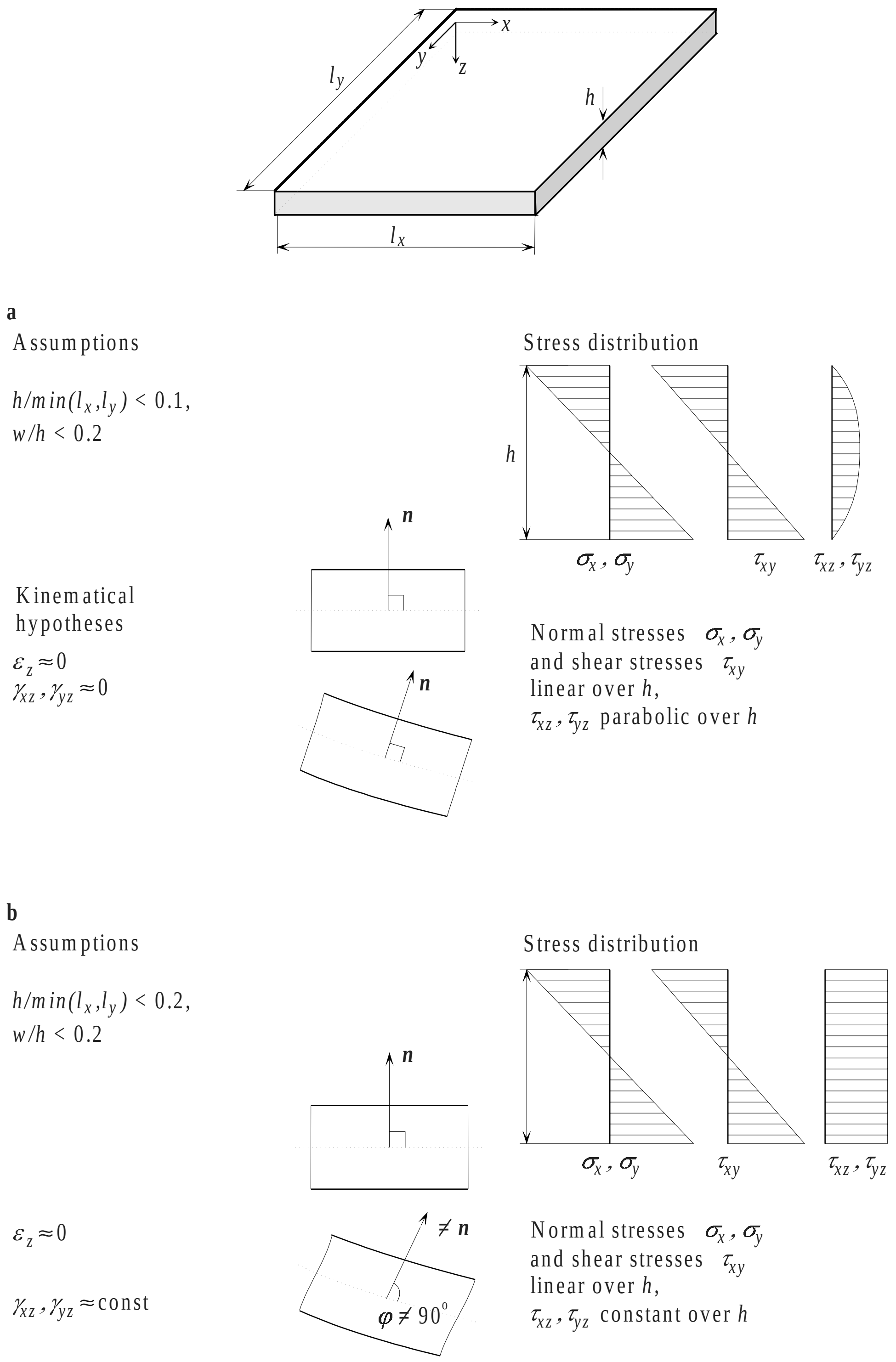


Fig. 2. Kirchoff's, Reissner's and Mindlin's plate models:
***a)** Shear rigid Kirchoff model,*
***b)** Shear deformable Reissner's and Mindlin's model*

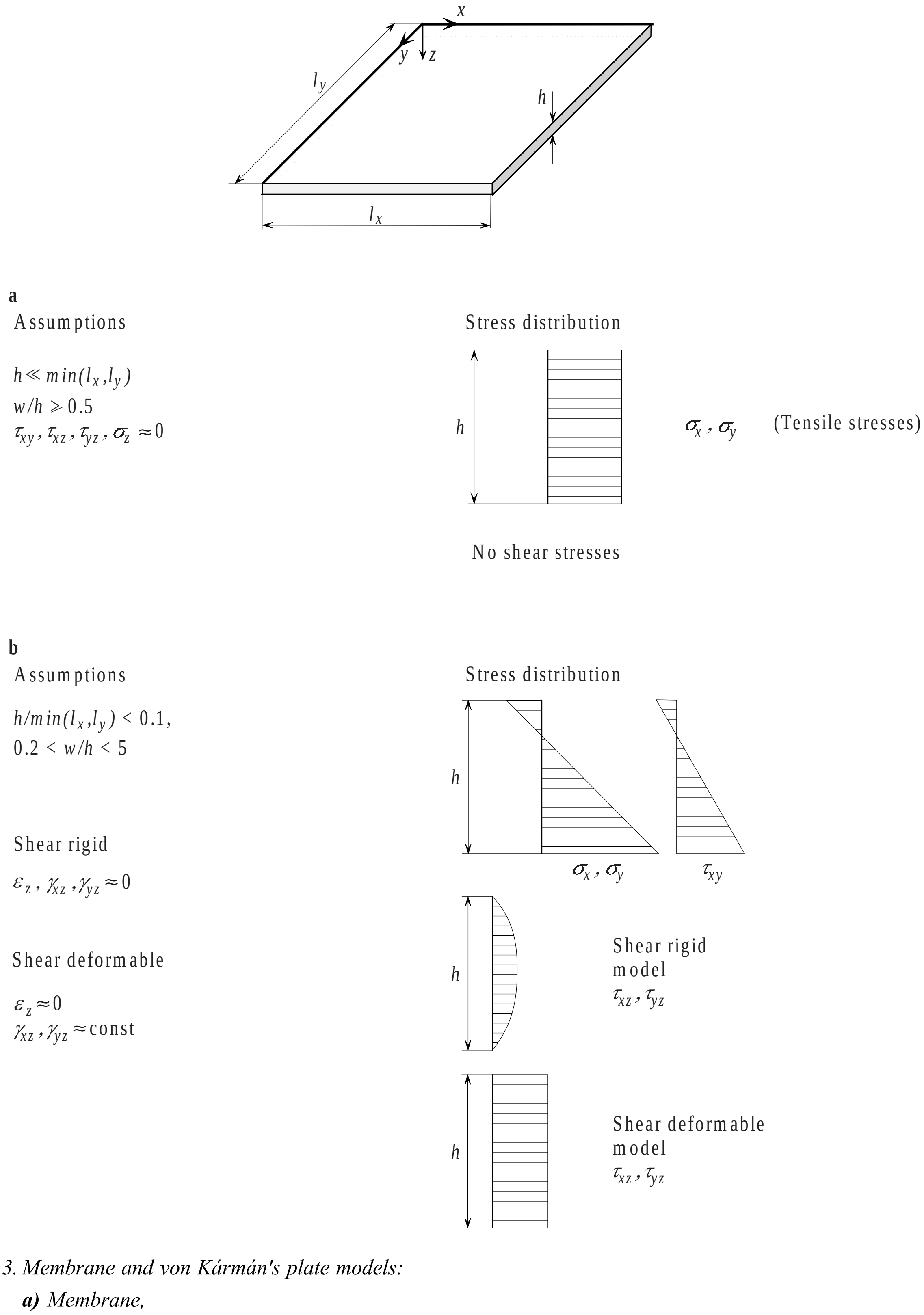


Fig. 3. Membrane and von Kármán's plate models:

***a)** Membrane,*

***b)** Shear rigid and deformable von Kármán models*

1. *The hypothesis of a broken normal*. The plate theory is deduced using a layerwise description. The so-called partial layerwise plate theories assume layerwise expansions for the in-plane displacements only, the full layerwise theories use expansions of all three displacement components.
2. *The hypothesis of a mechanical equivalent single layer*. The plate theories are formulated for the entire layer package and not for separate layers.

Theories of multilayered plates have to take into account the inhomogeneous distribution of the mechanical properties over the plate thickness and the anisotropic response of any single layer and of the equivalent layer. For the analysis of local plate responses, e.g. delamination, or for sandwich and multilayered plates with great differences in the layer thicknesses and the mechanical layer properties, the layerwise theories should be used. The description of the cross-sectional deformations of each layer is also in the layerwise theory not correct, but the error is much less than the error of a equivalent single layer model. If the layer thicknesses and the layerwise elastic moduli vary not so much, the equivalent single layer plate models estimate the global response parameters, e.g. deflection, eigen-vibration, buckling, with a sufficient accuracy. An alternative approach if the layer thicknesses and the layerwise elastic moduli are significantly varying is demonstrated in [8]. In this case an improved determination of the shear correction can be useful and the global estimates show a good agreement with results based on improved theories.

In general, layered composite plates are slender structures and their displacements and in-plane stresses are obtained quite accurately by means of two-dimensional plate theories. These theories based on assumptions concerning the distribution of the in-plane displacements. Restricting the modelling on laminated plates with a great number of fibre reinforced layers, the most simple theories based on the equivalent single layer hypothesis assume linear distributions of the in-plane displacements. The general used models in engineering applications are the classical laminate theory (CLT) and the first-order-shear-deformation theory (FOSDT) 9.

$$
\begin{aligned}
u_1(x_1, x_2, x_3) &= u(x_1, x_2) + x_3\psi_1(x_1, x_2),\\
u_2(x_1, x_2, x_3) &= v(x_1, x_2) + x_3\psi_2(x_1, x_2),\\
u_3(x_1, x_2, x_3) &= w(x_1, x_2)
\end{aligned}
\tag{1}
$$

In Eqs (1) u, v are the in-plane displacements in the middle surface of the plate and w is the out-of-plane displacement, i.e. the deflection. In the CLT we have $\psi_\alpha=-\delta w/\delta x_\alpha$, $\alpha = 1, 2$ and in the FOSDT the ψ_α are independent kinematical functions. In contrast to the CLT or the FOSDT modelling of isotropic plates, multilayered plates with an unsymmetrical stacking sequence of the layers have a coupling of in-plane and out-of-plane response for tranxzsverse loading and the kinematical degree of freedom is 3 (CLT) or 5 (FOSDT). For plates with strong changes in geometry or folded plate structures it can be necessary to expand the FOSDT by including transverse shear deformations and thickness effects.

$$
\begin{aligned}
u_1(x_1, x_2, x_3) &= u(x_1, x_2) + x_3\psi_1(x_1, x_2),\\
u_2(x_1, x_2, x_3) &= v(x_1, x_2) + x_3 y_2(x_1, x_2),\\
u_3(x_1, x_2, x_3) &= w(x_1, x_2) + x_3\psi_3(x_1, x_2)
\end{aligned}
\tag{2}
$$

In this extended sense the FOSDT-6 is characterised by 6 independent degrees of freedom, Eq. (2). Numerical tests demonstrate that for standard problems with a simple geometry the results are very similar for a modelling with 5 or 6 degrees of freedom and therefore mostly in engineering applications the FOSDT-5 version with 5 degrees of freedom is used [10]. The reason for such limitation is that the magnitude of the assumed drilling ψ_3 of the normal is much smaller in comparison with the other two rotation, because the deformability of a thin-walled structure even in the case of moderate thickness is very small. Counter-example are presented, fort example, in [11].

The Kirchhoff's plate theory is unable to describe the mechanical behaviour of structural elements for which the transverse shear strains are not negligible. Nevertheless the CLT is used in many engineering applications and enable a simple structural design and optimisation of laminate stacking, etc. The FOSDT proposed an extended kinematical model with additional degrees of freedom and included transverse shear deformations. But the FOSDT yields a linear distribution of the in-plane displacements and a constant distribution of the transverse shear stresses over the plate thickness which is incompatible with the real structural response. A simple way to improve the determination of the transverse shear stiffness can be the introduction of a shear correction factor [8, 12, 13].

Recently by several authors so-called high-order-shear deformation theories (HOSDT) [14, 15] are developed:

$$
\begin{aligned}
u_1(x_1, x_2, x_3) &= \sum_{i=0}^{\infty} a_i f_i(u_1, x_2) x_3\\
u_2(x_1, x_2, x_3) &= \sum_{i=0}^{\infty} b_i g_i(x_1, x_2) x_3
\end{aligned}
\tag{3}
$$

$$u_2(x_1,x_2,x_3) \;=\; \sum_{i=0}^{\infty} b_i g_i(x_1,x_2)x_3$$

These series must be truncated so that the number of functional degrees of freedom is limited, e.g. FOSDT-5

$$a_0 = b_0 = c_0;\; f_0 = u,\; g_0 = v,\; h_0 = w,$$

$$a_1 = b_1 = 1;\; c_1 = 0,\; f_1 = \psi_1,\; g_1 = \psi_2,$$

$$a_i = b_i = c_i = 0,\; i > 1$$

Higher order approximations lead warping in-plane displacements (Fig. 4) and also non-constant transverse displacements over the thickness. In [16] a special zig-zag-function is proposed for FOSDT-modelling that leads to improved in-plane responses and allows to include thickness effects as well as transverse shear and transverse normal stresses. The use of HOSDT increases the computational effort rapidly whereas the gain in accuracy is small. Thus higher order theories are of advantage from a mathematical research, but in structural analysis only within a relatively narrow band-width of problems. The FOSDT approach is sufficient for determining in-plane stresses even if the plate slenderness is not very high [17].

For isotropic plates the in-plane stresses are often fully sufficient in structural strength analysis. Multilayered composite plates are however inhomogeneous and anisotropic and to specify delamination or other failure modes transverse shear stresses and normal out-of-plane stresses must be known too. Using local applications of equilibrium conditions and integrating the derivatives of the in-plane stresses over the plate thickness lead to a so-called extended two-dimensional theory and yield excellent results for transverse shear and normal stresses if a mechanical loading applied on laminated plates with cross-ply or angle-ply stacking [18, 19]. In some cases of non-linear temperature distribution higher order approximation theories or three-dimensional modelling should be used [20].

4. Singularities and Plate Responses

In structure analysis of plates different kinds of singularities can greatly influence the local and global structure behaviour [21]. It is well-known that concentrated forces or moments and sharp notches or cracks yield singular stress fields. But there are some other problems in modelling plates with singularities:

- Concentrated forces acting upon a very thin plate with a membrane response or concentrated moments acting upon a Kirchhoff plate are quite improper models. In the first case the membrane has no resistance to a concentrated force and already the smallest concentrated force results in infinitive deflection. Point supports to membranes of any shape will have for the reason of no resistance no influence to their vibration behaviour, i.e they will not change the frequencies. In the second case a Kirchhoff plate has no rotational resistance to a concentrated moment and consequently influence coefficients or stiffness matrices according to the classical plate theory are meaning full for concentrated forces but meaning less for concentrated moments. Furthermore point constraints can not resist rotations, they are only capable of supplying transverse concentrated forces to constrain the displacement. The calculation of static deflections or bending moments, of buckling or vibration models of pointwise clamped plates became unacceptable.
- Interior sharp corners will produce infinite stresses, i.e singularities in classical plate theories. The singularities can also have significant effects upon the global behaviour of the plate. Stress singulari-

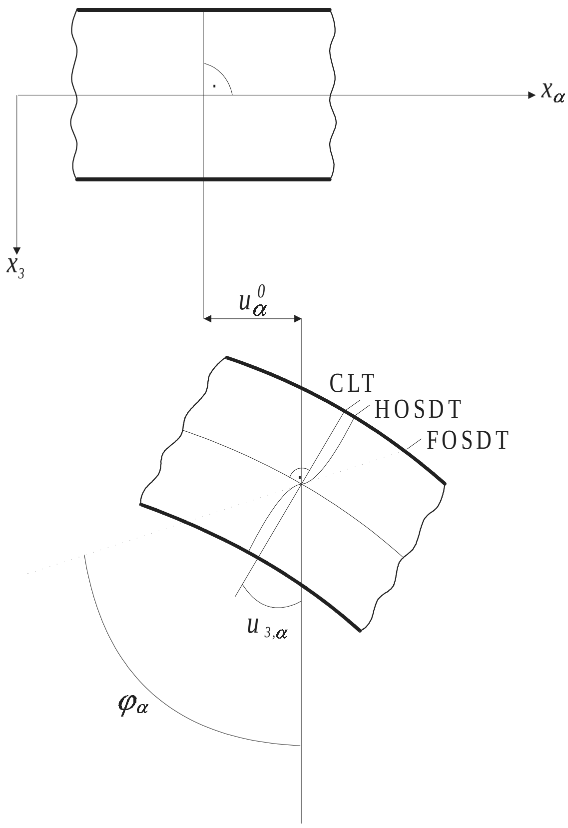


Fig. 4. Geometry of a undeformed lineal element. CLT: rotation $u_{3,\alpha}$, FOSDT: independent rotations φ_α, HOSDT: independent rotations and warping

ties in sharp corners can be represented exactly [22, 23]. For various types of corners are different sets of functions which satisfy the boundary conditions exactly. These are the so-called corner functions. Only if these corner functions are added to another global set of displacement functions, the corner stresses can be accounted properly. If the Ritz method is utilised e.g. for free vibration analysis of plates, mathematically complete sets of admissible displacement functions, which only satisfy geometric boundary conditions, yield upper bound approximations to the frequencies and the exact values may be approach as closely as desired if sufficient terms are taken. But if a set of corner functions will be added the convergence will be accelerated rapidly. The same situation is for finite element analysis.

- For a shear deformable plate, e.g. the Mindlin plate, already a concentrated force acting transverse to the middle surface results in infinitive deflection and so the Mindlin plate has a different response in comparison to the Kirchhoff plate, and it is a quite improper mathematical model for singular forces too. A real concentrated loading is distributed over a small area and the Mindlin plate model yields a correct modelling.

5. Solution Procedures

Analytical and numerical solution procedures for the different plate models are discussed and summerised in various monographes and textbooks, e.g. [1, 24, 25, 26, 27, 28, 29].Analytical solutions are limited to plates with simple geometry, e.g. rectangular or circular plates, and plates with simple elastic response, e.g. isotropic, orthotropic or quasi-orthotropic, etc. In most cases analytical solutions can be formulated in terms of infinite series and the effect of truncating the series on the accuracy of the solution have to be estimated. The few closed analytical plate solutions are limited to very special problems.

The validity of approximate plate theories, i.e. of the displacement or the stress approximations can be assessed by comparing their predictions with analytical solutions of the three-dimensional equations of elasticity. This problem is discussed in various research papers, e.g. in [30] for the anisotropic equations.

Approximate plate solutions can be given as numerical solutions or approximate series solutions with sets of trial functions. The most general solution procedures are numerical methods, i.e. the finite element method and the boundary element method, based on variational or energy methods, on weak formulations of boundary problems or on the theory of singular integral equations.

Applying the finite element methods available in a great number of research and commercial finite element programs, all problems can be solved, but dependent on the selected plate theory the numerical effort can be increasing rapidly, especially for multilayered or three-dimensional plate modelling. But for the simple Mindlin's plate model numerical problems may arise too. A first problem, termed as shear locking in finite element solutions, due to vanishing shear deformation terms in the variational displacement based formulation. A second problem of domain discretisation based on numerical methods is the increasing effort to find a suitable discretisation, i.e. a suitable element mesh which allows to represent the boundary-layer solution arising in the case of several types of boundary conditions. Various techniques are available to overcome these problems, e.g. [31], however, it is difficult to make general conclusions on the influence of the shear correction for different boundary conditions and loading types if only numerical results are given.

For plates with simple geometry, e.g. with straight line boundaries, the variational methods of Ritz and Galerkin can be recommended to construct accurate approximate solutions. In the last years it could be demonstrated that the variational iteration method, based on the Vlasov-Kantorovich approach, yields good approximate solutions for various plate theories and optimal one-term approximate analytical solutions for general boundary conditions [32, 33, 34, 35, 36].

6. Conclusions

Engineering plate theories got many impacts to improve the structure analysis of two-dimensional structures. The increasing importance of composite materials required refined theories for sandwich and laminate plates. Stress analysis and life-time predictions of thin-walled structures operating at elevated temperatures require to include the modelling of creep-damage behaviour and a generalisation of the solution procedures. The paper summerises some selected aspects of trends in engineering plate theories. Further ideas and overviews on relevant research paper are given, e.g., in [2, 37, 38, 39].

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Prof. dr. - ing. habil. dr. h.c. Johannes Altenbach

Förderstedter Str. 28
D-39112 Magdeburg, Germany

Prof. dr. - ing. habil. Holm Altenbach

Lehrstuhl für Technische Mechanik
Fachbereich Ingenieurwissenschaften
Martin - Luther - Universität Halle - Wittenberg
D-06099 Halle, Germany
e-mail: holm.altenbach@iw.uni-halle.de